CHAPTER 7

CARTESIAN TENSORS

7.1 INTRODUCTION

Tensor analysis may be regarded as a generalization of vector analysis. It is of great value to a scientist or an engineer in two ways. First, it allows complex mathematical and physical relationships to be expressed in a compact way and simplifies the mechanics of the development of theory. Second, it imparts greater understanding to vector notations and also establishes certain invariance properties with elegance and simplicity. It is of great use in mechanics, fluid dynamics, elasticity, differential geometry, electromagnetic theory, general relativity theory, and numerous other fields of science and engineering. In this chapter, we shall discuss Cartesian tensors i.e. tensors which are expressed in terms of components referred to rectangular Cartesian coordinate systems. At first sight, the notation of Cartesian tensors is somewhat complicated. The aim of this chapter is to provide a familiarity with the notation which will enable the reader to study other texts and applications without difficulty.

Before we start the actual studies of Cartesian tensors, it will be important to give some basic ideas (notations, definitions, transformations, etc.) which are useful in the study of Cartesian tensors.

7.2 SUMMATION CONVENTION

Consider an expression $a_1 x_1 + a_2 x_2 + a_3 x_3$ (i)

which can be written using summation sign as $\sum_{j=1}^{3} a_j x_j$ (ii)

Let us omit the summation sign and write it simply as $a_j x_j$ (iii)

where it is understood that the repeated index (or suffix) j represents the summation from 1 to 3.

Note that the form (iii) is much more convenient than the original form (i). This situation occurs so frequently that it is convenient to adopt a convention which avoids the necessity of writing summation signs. This convention known as the summation convention is as follows:

"Whenever a suffix appears twice in the same expression that expression is to be summed over all values of the suffix namely, 1, 2, 3."

DUMMY AND FREE INDICES

An index which is repeated in a given expression so that the summation convention applies, is called a **dummy index**, while an index occurring only once in a given expression is called a **free index** and does not imply any summation. For example, in the expression $A_k B_{jk}$, k is dummy index while j is a free index.

EXAMPLE (1): Write each of the following using summation convention.

(i) $a_{11}x_1 + a_{12}x_2 + a_{13}x_3$ (ii) $a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13}$

(iii) $(x_1)^2 + (x_2)^2 + (x_3)^2$ (iv) $\frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2 + \frac{\partial\phi}{\partial x_3}dx_3$

SOLUTION: We have

(i) $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = a_{1i}x_i$

(ii) $a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} = a_{1i}b_{1i}$

(iii) $(x_1)^2 + (x_2)^2 + (x_3)^2 = x_1x_1 + x_2x_2 + x_3x_3 = x_ix_i$

(iv) $\frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2 + \frac{\partial\phi}{\partial x_3}dx_3 = \frac{\partial\phi}{\partial x_i}dx_i$

EXAMPLE (2): Write out explicitly the following summations and compare the results :

(i) $a_i(x_i + y_i)$ (ii) $a_jx_j + a_ky_k$

SOLUTION: We have

(i) $a_i(x_i + y_i) = a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3)$

$= a_1x_1 + a_1y_1 + a_2x_2 + a_2y_2 + a_3x_3 + a_3y_3$

(ii) $a_jx_j + a_ky_k = a_1x_1 + a_2x_2 + a_3x_3 + a_1y_1 + a_2y_2 + a_3y_3$

The two summations are identical except for the order in which the terms occur.

NOTE: (i) A repeated suffix may be replaced by any other suitable symbol not already in use. For example, $a_jb_j = a_kb_k = a_\alpha b_\alpha$ since in each expression summation over the repeated suffix is implied.

(ii) No suffix may occur more than twice in an expression. For example, $a_{ii}x_i$ is ambiguous because of the differences in the three quantities :

$$a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$$

$$a_{ii}x_j = (a_{11} + a_{22} + a_{33})x_j$$

$$a_{ij}x_i = a_{1j}x_1 + a_{2j}x_2 + a_{3j}x_3$$

On putting $j = i$ in these equations, we obtain entirely different expressions on the R.H.S.

(iii) An expression of the form $a_i(x_i + y_i)$ is considered well–defined, for it is obtained by composition of the meaningful expressions a_iz_i and $x_i + y_i = z_i$. In other words, the index i is regarded as occurring once in the term $(x_i + y_i)$.

7.3 DOUBLE SUMS

An expression can involve more than one summation index. For example, $a_{ij}x_ix_j$ indicates a summation taking place on both i and j simultaneously. If an expression has two summation (dummy) indices, there will be a total of 3^2 terms in the sum ; if there are three indices, there will be 3^3 terms ; and so on.

EXAMPLE (3): **Write the terms in the expression $a_{ij}x_i x_j$; $i, j = 1, 2, 3$.**

SOLUTION: The given expression represents the double sum and has 9 terms in it. Its expansion can be written logically by first summing over i, and then over j. Since i varies from 1 to 3, therefore holding j fixed, the given expression is the sum of three terms. That is

$$a_{ij}x_i x_j = a_{1j}x_1 x_j + a_{2j}x_2 x_j + a_{3j}x_3 x_j$$

Now each term on the R.H.S. has the repeated index j which implies summation. Hence

$$a_{ij}x_i x_j = a_{11}x_1 x_1 + a_{12}x_1 x_2 + a_{13}x_1 x_3 + a_{21}x_2 x_1 + a_{22}x_2 x_2 + a_{23}x_2 x_3 + a_{31}x_3 x_1 + a_{32}x_3 x_2 + a_{33}x_3 x_3$$

The result is the same if one sums over j first, and then over i.

EXAMPLE (4): **Write the following expression using summation convention.**

$$a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} + a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23} + a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33}$$

SOLUTION: The given expression can be written as

$$(a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13}) + (a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23}) + (a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33})$$

$$= a_{i1}b_{1i} + a_{i2}b_{2i} + a_{i3}b_{3i} = a_{ij}b_{ji}$$

7.4 SUBSTITUTIONS

Suppose it is required to substitute $y_i = a_{ij}x_j$ in the equation $Q = b_{ij}y_i x_j$. Simple substitution would lead to an absurd expression like $Q = b_{ij}a_{ij}x_j x_j$.

The correct procedure is first to identify any dummy indices in the expression to be substituted that coincide with indices occurring in the main expression. Changing these dummy indices to characters not found in the main expression, one may then carry out the substitution in the usual manner as follows :

Step (1) $y_i = a_{ij}x_j$, $Q = b_{ij}y_i x_j$. We see that the dummy index j is duplicated.

Step (2) Change the dummy index from j to r, to get $y_i = a_{ir}x_r$.

Step (3) Substitute and rearrange to get $Q = b_{ij}(a_{ir}x_r)\,x_j = a_{ir}b_{ij}\,x_r x_j$.

EXAMPLE (5): **If $y_i = a_{ij}x_j$, express the quadratic form $Q = g_{ij}y_i y_j$ in terms of x–variables.**

SOLUTION: First write $y_i = a_{ir}x_r$, $y_j = a_{js}x_s$

Then by substitution, $Q = g_{ij}(a_{ir}x_r)(a_{js}x_s) = g_{ij}a_{ir}a_{js}x_r x_s$

7.5 ALGEBRA AND THE SUMMATION CONVENTION

Certain routine algebraic manipulations in tensors can be easily justified by properties of ordinary sums. However, some care should be taken. The following are several valid identities ; they will be used repeatedly from now on .

(1) $a_{ij}(x_j + y_j) \equiv a_{ij}x_j + a_{ij}y_j$

(2) $a_{ij}x_i y_j \equiv a_{ij}y_j x_i$

(3) $a_{ij}x_i x_j \equiv a_{ji}x_i x_j$

(4) $(a_{ij} + a_{ji})x_i x_j \equiv 2a_{ji}x_i x_j$

(5) $(a_{ij} - a_{ji})x_i x_j \equiv 0$

The following non–identities should be carefully noted :

(1) $a_{ij}(x_i + y_j) \not\equiv a_{ij}x_i + a_{ij}y_j$ (2) $a_{ij}x_i y_j \not\equiv a_{ij}y_i x_j$

(3) $(a_{ij} + a_{ji})x_i y_j \not\equiv 2a_{ij}x_i y_j$

EXAMPLE (6): **Show that , generally , $a_{ijk}(x_i + y_j)z_k \neq a_{ijk}x_i z_k + a_{ijk}y_j z_k$**

SOLUTION: Simply observe that on the left side there are no free indices , but on the right , j is free for the first term and i is free for the second .

7.6 THE KRONECKER DELTA δ_{ij}

The Kronecker delta or substitution operator written δ_{ij}, is defined as $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Thus $\delta_{11} = \delta_{22} = \delta_{33} = 1$, and $\delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = \delta_{31} = \delta_{13} = 0$

EXAMPLE (7): **Using the definition of Kronecker delta , calculate $\delta_{ij}x_i x_j$.**

SOLUTION: We have $\delta_{ij}x_i x_j = \delta_{1j}x_1 x_j + \delta_{2j}x_2 x_j + \delta_{3j}x_3 x_j$

$$= \delta_{11}x_1x_1 + \delta_{12}x_1x_2 + \delta_{13}x_1x_3 + \delta_{21}x_2x_1 + \delta_{22}x_2x_2 + \delta_{23}x_2x_3 + \delta_{31}x_3x_1 + \delta_{32}x_3x_2 + \delta_{33}x_3x_3$$

$$= 1\,x_1x_1 + 0\,x_1x_2 + 0\,x_1x_3 + 0\,x_2x_1 + 1\,x_2x_2 + 0\,x_2x_3 + 0\,x_3x_1 + 0\,x_3x_2 + 1\,x_3x_3$$

$$= x_1x_1 + x_2x_2 + x_3x_3 = x_i x_i$$

THEOREM (7.1): **Show that if x_1, x_2, x_3 are independent variables , then $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$**

PROOF: We have , if $i = j$, $\frac{\partial x_i}{\partial x_j} = \frac{\partial x_i}{\partial x_i} = 1$

If $i \neq j$, $\frac{\partial x_i}{\partial x_j} = 0$ since x_i and x_j are independent variables .

Thus $\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

THEOREM (7.2): **Prove that $\delta_{ij}A_j = A_i$.**

PROOF: We know that the index j represents summation , therefore

$$\delta_{ij}A_j = \delta_{i1}A_1 + \delta_{i2}A_2 + \delta_{i3}A_3 \quad (i = 1, 2, 3)$$

when $i = 1$, $\delta_{1j}A_j = \delta_{11}A_1 + \delta_{12}A_2 + \delta_{13}A_3 = A_1$, and

when $i = 2$, $\delta_{2j}A_j = \delta_{21}A_1 + \delta_{22}A_2 + \delta_{23}A_3 = A_2$, and

when $i = 3$, $\delta_{3j}A_j = \delta_{31}A_1 + \delta_{32}A_2 + \delta_{33}A_3 = A_3$.

Thus in all cases : $\delta_{ij}A_j = A_i$

That is , δ_{ij} operating on A_j has substituted the free index i for the index j in A_j which gives a justification of the term **substitution operator** .

NOTE: This result is of fundamental importance and will be used frequently in our later discussion .

THEOREM - 3: Prove that ...

PROOF: We have $\delta_{ik}\delta_{jk} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3}$ $(i, j = 1, 2, 3)$

when $j = 1$, $\delta_{ik}\delta_{1k} = \delta_{i1}$ and

when $j = 2$, $\delta_{ik}\delta_{2k} = \delta_{i2}$ and

when $j = 3$, $\delta_{ik}\delta_{3k} = \delta_{i3}$

It therefore follows that $\delta_{ik}\delta_{jk} = \delta_{ij}$.

EXAMPLE (8): **Show that**

(i) $\delta_{ii} = 3$ (ii) $\delta_{ik}\,\delta_{ik} = 3$ (iii) $\delta_{ij}\delta_{jk}\delta_{ki} = 3$

(iv) $\delta_{ij}\delta_{k\ell}A_{ik} = A_{j\ell}$ (v) $\delta_{ij}\,\delta_{jk}\,A_{ik} = A_{ii}$

SOLUTION: We have (i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

(ii) $\delta_{ik}\delta_{ik} = \delta_{1k}\delta_{1k} + \delta_{2k}\delta_{2k} + \delta_{3k}\delta_{3k}$, $k = 1, 2, 3$

$$= (\delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}) + (\delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}) + (\delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33})$$

$$= \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 3$$

(iii) $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = 3$

$\delta_{i1}A_{i1} + \delta_{i2}A_{i2} + \delta_{i3}A_{i3}$

$\delta_{11}A_{11} + \delta_{22}A_{22} + \delta_{33}A_{33} = A_{ii}$

(iv) $\delta_{ij}\delta_{k\ell}A_{ik} = \delta_{ij}A_{i\ell} = A_{j\ell}$

(v) $\delta_{ij}\,\delta_{jk}\,A_{ik} = \delta_{ik}\,A_{ik} = A_{ii}$.

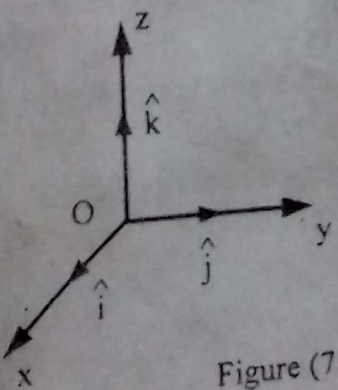

Figure (7.1)

7.7 RECTANGULAR COORDINATE SYSTEM

From vector analysis, we are familiar with the rectangular coordinate system in which we take Ox, Oy, Oz as the coordinate axes and $\hat{i}, \hat{j}, \hat{k}$ the unit vectors along these coordinate axes respectively, as shown in figure (7.1).

(K)

... we take the