Chapter 3

Relations

3.1 INTRODUCTION

The reader is familiar with many relations which are used in mathematics and computer science, e.g., "less than", "is parallel to", "is a subset of", and so on. In a certain sense, these relations consider the existence or nonexistence of certain connections between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs".

There are three kinds of relations which play a major role in our theory: (i) equivalence relations, (ii) order relations, (iii) functions. Equivalence relations are mainly covered in this chapter. Order relations are introduced here, but will also be discussed in Chapter 7. Functions are covered in the next chapter.

The connection between relations on finite sets and matrices are also included here for completeness. These sections, however, can be ignored at a first reading by those with no previous knowledge of matrix theory.

Ordered Pairs

Relations, as noted above, will be defined in terms of ordered pairs (a, b) of elements, where a is designated as the first element and b as the second element. Specifically:

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d$$

In particular, $(a, b) \neq (b, a)$ unless $a = b$. This contrasts with sets studied in Chapter 1 where the order of elements is irrelevant, for example, $\{3, 5\} = \{5, 3\}$.

3.2 PRODUCT SETS

Let A and B be two sets. The *product set* or *cartesian product* of A and B, written $A \times B$ and read "A cross B", is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. Namely:

$$A \times B = \{(a, b) : a \in A,\ b \in B\}$$

One usually writes A^2 instead of $A \times A$.

EXAMPLE 3.1 Recall that $\mathbf{R}$ denotes the set of real numbers, so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader may be familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig. 3-1. Here each point P represents an ordered pair (a, b) of real numbers and vice versa; the vertical line through P meets the (horizontal) x-axis at a, and the horizontal line through P meets the (vertical) y-axis at b. $\mathbf{R}^2$ is frequently called the *cartesian* plane.

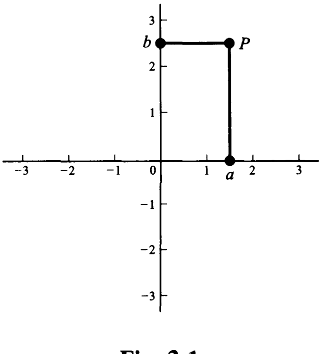


Fig. 3-1

EXAMPLE 3.2 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$
$$B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

Also,

$$A \times A = \{(1,1)(1,2), (2,1), (2,2)\}$$

There are two things worth noting in Example 3.2. First of all, $A \times B \neq B \times A$. The cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly,

$$n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)$$

[where $n(A)$ = number of elements in A]. In fact:

$$\boxed{n(A \times B) = n(A) \cdot n(B)}$$

for any finite sets A and B. This follows from the observation that, for any ordered pair (a, b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

Product of Three or More Sets

The idea of a product of sets can be extended to any finite number of sets. Specifically, for any sets $A_1, A_2, \ldots, A_m$, the set of all m-element lists $(a_1, a_2, \ldots, a_m)$, where each $a_i \in A_i$, is called the *(cartesian) product* of the sets $A_1, A_2, \ldots, A_m$; it is denoted by

$$A_1 \times A_2 \times \cdots \times A_m \quad \text{or equivalently} \quad \prod_{i=1}^{m} A_i$$

Just as we write A^2 instead of $A \times A$, so we write A^n for $A \times A \times \cdots \times A$ where there are n factors. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.

3.3 RELATIONS

We begin with a definition.

Definition: Let A and B be sets. A *binary relation* or, simply, a *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "a is *R-related* to b", written $a\,R\,b$.
(ii) $(a, b) \notin R$; we then say "a is not *R-related* to b", written $a \not{R}\, b$.

The *domain* of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R, and so it is a subset of A; and the *range* of R is the set of all second elements, and so it is a subset of B.

Sometimes R is a relation from a set A to itself, that is, R is a subset of $A^2 = A \times A$. In such a case, we say that R is a relation *on* A.

Although n-ary relations, which involve ordered n-tuples, are introduced in Section 3.11, the term relation shall mean binary relation unless otherwise stated or implied.

EXAMPLE 3.3

(*a*) Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry, \; 1Rz, \; 3Ry, \qquad \text{but} \qquad 1 \not R x, \; 2 \not R x, \; 2 \not R y, \; 2 \not R z, \; 3 \not R x, \; 3 \not R z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(*b*) Suppose we say that two countries are *adjacent* if they have some part of their boundaries in common. Then "is adjacent to" is a relation R on the countries of the earth. Thus:

$$(\text{Italy, Switzerland}) \in R \qquad \text{but} \qquad (\text{Canada, Mexico}) \notin R$$

(*c*) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of sets A and B, either $A \subseteq B$ or $A \not\subseteq B$.

(*d*) A familiar relation on the set $\mathbf{Z}$ of integers is "m divides n". A common notation for this relation is to write $m|n$ when m divides n. Thus $6|30$ but $7 \nmid 25$.

(*e*) Consider the set L of lines in the plane. Perpendicularity, written $\perp$, is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to", written $\|$, is a relation on L since either $a \parallel b$ or $a \nparallel b$.

Universal, Empty, Equality Relations

Let A be any set. Then $A \times A$ and $\varnothing$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively. Thus, for any relation R on A, we have

$$\varnothing \subseteq R \subseteq A \times A$$

An important relation on the set A is that of *equality*, that is, the relation

$$\{(a, a) : a \in A\}$$

which is usually denoted by "=". This relation is also called the *identity* or *diagonal relation* on A, and it may sometimes be denoted by Δ_A or simply Δ.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

For example:

$$\text{If} \quad R = \{(1, y), (1, z), (3, y)\}, \qquad \text{then} \qquad R^{-1} = \{(y, 1), (z, 1), (y, 3)\}.$$

[Here R is the relation from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$ in Example 3.3(*a*).]

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain of R^{-1} is the range of R, and vice versa. Moreover, if R is a relation on A, i.e., R is a subset of $A \times A$, then R^{-1} is also a relation on A.

3.4 PICTORIAL REPRESENTATIONS OF RELATIONS

This section discusses a number of ways of picturing and representing binary relations.

Relations on R

Let S be a relation on the set $\mathbf{R}$ of real numbers; that is, let S be a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. This pictorial representation of S is sometimes called the *graph* of S.

Frequently, the relation S consists of all ordered pairs of real numbers which satisfy some given equation

$$E(x, y) = 0$$

We usually identify the relation with the equation, i.e., we speak of the relation $E(x, y) = 0$.

EXAMPLE 3.4 Consider the relation S defined by the equation

$$x^2 + y^2 = 25 \qquad \text{or equivalently} \qquad x^2 + y^2 - 25 = 0$$

That is, S consists of all ordered pairs (x_0, y_0) which satisfy the given equation. The graph of the equation is a circle having its center at the origin and radius 5, as shown in Fig. 3-2.

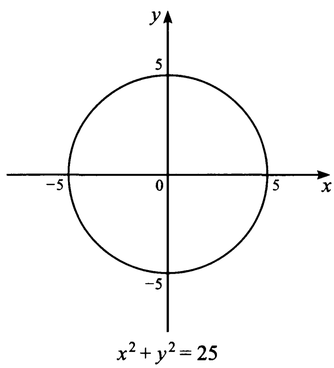


Fig. 3-2

Representation of Relations on Finite Sets

Suppose A and B are finite sets. The following are two ways of picturing a relation R from A to B.

(i) Form a rectangular array whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix* of the relation.

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. This picture will be called the *arrow diagram* of the relation.

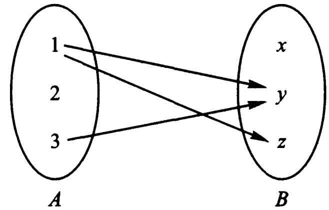
Consider, for example, the following relation R from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$:

$$R = \{(1, y),\ (1, z), (3, y)\}$$

Figure 3-3 pictures this relation R by the above two ways.

		B: x	y	z
A	1	0	1	1
	2	0	0	0
	3	0	1	0

(*i*)



(*ii*)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Fig. 3-3

Directed Graphs of Relations on Sets

There is another way of picturing a relation R when R is a relation from a finite set A to itself. First we write down the elements of the set A, and then we draw an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation R. Figure 3-4, for example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1,2),\ (2,2),\ (2,4),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\}$$

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under R.

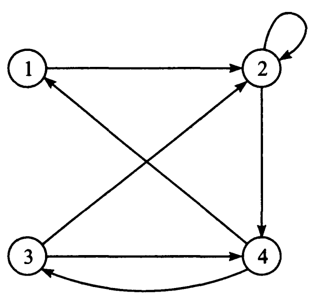


Fig. 3-4

3.5 COMPOSITION OF RELATIONS

Let A, B, C be sets, and let R be a relation from A to B and let S be a relation from B to C. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined as follows:

$$R \circ S = \{(a, c) : \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

That is,

$$a(R \circ S)c \text{ whenever there exists } b \in B \text{ such that } aRb \text{ and } bSc$$

This relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted by RS.

Our first theorem (proved in Problem 3.10) tells us that the composition of relations is associative. Namely:

Theorem 3.1: Let A, B, C, D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

The arrow diagrams of relations give us a geometrical interpretation of the composition $R \circ S$ as seen in the following example.

EXAMPLE 3.5 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z)$ and let

$$R = \{(1, a),\ (2, d),\ (3, a),\ (3, b),\ (3, d)\} \qquad \text{and} \qquad S = \{(b, x),\ (b, z),\ (c, y),\ (d, z)\}$$

Consider the arrow diagrams of R and S as in Fig. 3-5. Observe there is an arrow from 2 to d which is followed by an arrow from d to x. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus

$$2(R \circ S)z \qquad \text{since} \qquad 2Rd \text{ and } dSz$$

Similarly there are paths from 3 to x and from 3 to z. Hence

$$3(R \circ S)x \qquad \text{and} \qquad 3(R \circ S)z$$

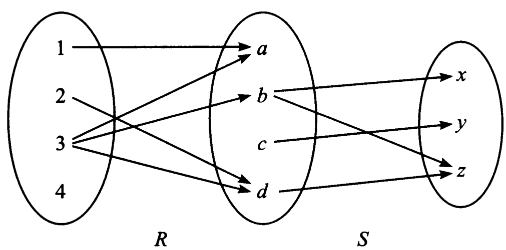


Fig. 3-5

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z),\ (3, x),\ (3, z)\}$$

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself, is always defined, and $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

Warning: Many texts denote the composition of relations R and S by $S \circ R$ rather than $R \circ S$. This is done in order to conform with the usual use of $g \circ f$ to denote the composition of f and g where f and g are functions. Thus the reader may have to adjust his notation when using this text as a supplement with another text. However, when a relation R is composed with itself, then the meaning of $R \circ R$ is unambiguous.

Composition of Relations and Matrices

There is a way of finding the composition $R \circ S$ of relations using matrices. Specifically, let M_R and M_S denote respectively the matrices of the relations R and S in Example 3.5. Then:

$$M_R = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix} \qquad \text{and} \qquad M_S = \begin{matrix} \\ a \\ b \\ c \\ d \end{matrix} \begin{matrix} \begin{matrix} x & y & z \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Multiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

The nonzero entries in this matrix tell us which elements are related by $R \circ S$. Thus $M = M_R M_S$ and $M_{R \circ S}$ have the same nonzero entries.

3.6 TYPES OF RELATIONS

Consider a given set A. This section discusses a number of important types of relations which are defined on A.

(1) **Reflexive Relations**: A relation R on a set A is *reflexive* if $a\,R\,a$ for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.

(2) **Symmetric Relations**: A relation R on a set A is *symmetric* if whenever $a\,R\,b$ then $b\,R\,a$, that is, if whenever $(a, b) \in R$, then $(b, a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

(3) **Antisymmetric Relations**: A relation R on a set A is *antisymmetric* if whenever $a\,R\,b$ and $b\,R\,a$ then $a = b$, that is, if whenever (a, b) and (b, a) belong to R then $a = b$. Thus R is not antisymmetric if there exist $a, b \in A$ such that (a, b) and (b, a) belong to R, but $a \neq b$.

(4) **Transitive Relations**: A relation R on a set A is *transitive* if whenever $a\,R\,b$ and $b\,R\,c$ then $a\,R\,c$, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. Thus R is not transitive if there exist $a, b, c \in A$ such that $(a, b), (b, c) \in R$, but $(a, c) \notin R$.

EXAMPLE 3.6 Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$$R_1 = \{(1,1),\ (1,2),\ (2,3),\ (1,3),\ (4,4)\}$$
$$R_2 = \{(1,1),\ (1,2),\ (2,1)\,(2,2),\ (3,3), (4,4)\}$$
$$R_3 = \{(1,3),\ (2,1)\}$$
$$R_4 = \varnothing, \text{ the empty relation}$$
$$R_5 = A \times A, \text{ the universal relation}$$

Determine which of the relations are: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

(*a*) Since A contains the four elements 1, 2, 3, 4, a relation R on A is reflexive if it contains the four pairs $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, $(2, 2)$ does not belong to any of them.

(*b*) R_1 is not symmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$. R_3 is not symmetric since $(1, 3) \in R_3$ but $(3, 1) \notin R_3$. The other relations are symmetric.

(*c*) R_2 is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_5 is not antisymmetric. All the other relations are antisymmetric.

(*d*) The relation R_3 is not transitive since $(2, 1), (1, 3) \in R_3$ but $(2, 3) \notin R_3$. All the other relations are transitive.

EXAMPLE 3.7 Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers.
(2) Set inclusion $\subseteq$ on a collection $\mathscr{C}$ of sets.
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
(4) Relation $\|$ (parallel) on the set L of lines in the plane.
(5) Relation $|$ of divisibility on the set $\mathbf{P}$ of positive integers. (Recall that $x|y$ if there exists z such that $xz = y$.)

Determine which of the relations are: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

(*a*) The relation (3) is not reflexive since no line is perpendicular to itself. Also, (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every integer x in $\mathbf{Z}$, $A \subseteq A$ for any set A in $\mathscr{C}$, and $n|n$ for every positive integer n in $\mathbf{P}$.

(*b*) The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\|$ is symmetric since if line a is parallel to line b then b is parallel to a. The other relations are not symmetric. For example, $3 \leq 4$ but $4 \leq 3$; $\{1,2\} \subseteq \{1,2,3\}$ but $\{1,2,3\} \not\subseteq \{1,2\}$; and $2|6$ but $6|2$.

(*c*) The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on $\mathbf{P}$ is antisymmetric since whenever $m|n$ and $n|m$ then $m = n$. (Note that divisibility on $\mathbf{Z}$ is not antisymmetric since $3|-3$ and $-3|3$ but $3 \neq -3$.) The relation $\perp$ is not antisymmetric since we can have distinct lines a and b such that $a \perp b$ and $b \perp a$. Similarly, $\|$ is not antisymmetric.

(*d*) The relations $\leq$, $\subseteq$ and $|$ are transitive. That is:

(i) If $a \leq b$ and $b \leq c$, then $a \leq c$.
(ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iii) If $a|b$ and $b|c$, then $a|c$.

On the other hand, the relation $\perp$ is not transitive. If $a \perp b$ and $b \perp c$, then it is not true that $a \perp c$. Since no line is parallel to itself, we can have $a \| b$ and $b \| a$, but $a \nparallel a$. Thus $\|$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane.)

Remark 1: The properties of being symmetric and antisymmetric are not negatives of each other. For example, the relation $R = \{(1,3), (3,1), (2,3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1,1), (2,2)\}$ is both symmetric and antisymmetric.

Remark 2: The property of transitivity can also be expressed in terms of the composition of relations. Recall that, for a relation R on a set A, we defined

$$R^2 = R \circ R \qquad \text{and, more generally,} \qquad R^n = R^{n-1} \circ R$$

Then one can show (Problem 3.66) that a relation R is transitive if and only if $R^n \subseteq R$ for every $n \geq 1$.

3.7 CLOSURE PROPERTIES

Let $\mathscr{P}$ denote a property of relations on a set A such as being symmetric or transitive. A relation on A with property $\mathscr{P}$ will be called a *$\mathscr{P}$-relation*.

Now let R be a given relation on A with or without property $\mathscr{P}$. The *$\mathscr{P}$-closure* of R, written $\mathscr{P}(R)$, is a relation on A containing R such that

$$R \subseteq \mathscr{P}(R) \subseteq S$$

for any other $\mathscr{P}$-relation S containing R. Clearly $R = \mathscr{P}(R)$ if R itself has property $\mathscr{P}$.

The reflexive, symmetric, and transitive closures of a relation R will be denoted respectively by:

$$\text{reflexive}(R), \qquad \text{symmetric}(R), \qquad \text{transitive}(R)$$

Reflexive and Symmetric Closures

The next theorem tells us how to easily obtain the reflexive and symmetric closures of a relation. Here $\Delta_A = \{(a, a): a \in A\}$ is the *diagonal* or *equality* relation on A.

Theorem 3.2: Let R be a relation on a set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R.

In other words, reflexive(R) is obtained by simply adding to R those elements (a, a) in the diagonal which do not already belong to R, and symmetric(R) is obtained by adding to R all pairs (b, a) whenever (a, b) belongs to R.

EXAMPLE 3.8 Consider the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1,1),\ (1,3),\ (2,4),\ (3,1),\ (3,3),\ (4,3)\}$$

Then

$$\begin{aligned}\text{reflexive}(R) &= R \cup \{(2,2),\ (4,4)\}\\ &= \{(1,1),\ (1,3),\ (2,4),\ (3,1),\ (3,3),\ (4,3),\ (2,2),\ (4,4)\}\end{aligned}$$

and

$$\begin{aligned}\text{symmetric}(R) &= R \cup \{(4,2),\ (3,4)\}\\ &= \{(1,1),\ (1,3),\ (2,4),\ (3,1),\ (3,3),\ (4,3),\ (4,2),\ (3,4)\}\end{aligned}$$

Transitive Closure

Let R be a relation on a set A. Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$. We define

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

The following theorem applies.

Theorem 3.3: R^* is the transitive closure of a relation R.

Suppose A is a finite set with n elements. Using graph theory, one can easily show that

$$R^* = R \cup R^2 \cup \cdots \cup R^n$$

This gives us the following result.

Theorem 3.4: Let R be a relation on a set A with n elements. Then

$$\text{transitive}(R) = R \cup R^2 \cup \cdots \cup R^n$$

Finding transitive(R) can take a lot of time when A has a large number of elements. Here we give a simple example where A has only three elements.

EXAMPLE 3.9 Consider the following relation R on $A = \{1, 2, 3\}$:

$$R = \{(1,2),\ (2,3),\ (3,3)\}$$

Then

$$R^2 = R \circ R = \{(1,3),\ (2,3),\ (3,3)\} \qquad \text{and} \qquad R^3 = R^2 \circ R = \{(1,3),\ (2,3),\ (3,3)\}$$

Accordingly,

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(1,2),\ (2,3),\ (3,3),\ (1,3)\}$$

3.8 PARTITIONS

Let S be a nonempty set. A *partition* of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a partition of S is a collection $P = \{A_i\}$ of nonempty subsets of S such that

(i) Each $a \in S$ belongs to one of the A_i.

(ii) The sets $\{A_i\}$ are mutually disjoint; that is,

$$\text{If } A_i \neq A_j, \text{ then } A_i \cap A_j = \emptyset$$

The subsets in a partition are called *cells*. Thus each $a \in S$ belongs to exactly one of the cells. Figure 3-6 is a Venn diagram of a partition of the rectangular set S of points into five cells: A_1, A_2, A_3, A_4, A_5.

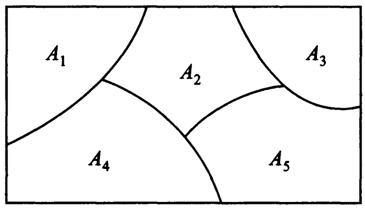


Fig. 3-6

EXAMPLE 3.10 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $P_1 = [\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$

(ii) $P_2 = [\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$

(iii) $P_3 = [\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

Then P_1 is not a partition of S since $7 \in S$ does not belong to any of the subsets. P_2 is not a partition of S since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. On the other hand, P_3 is a partition of S.

Remark: Given a partition $P = \{A_i\}$ of a set S, any element $b \in A_i$ is called a *representative* of the cell A_i, and a subset B of S is called a *system* of representatives if B contains exactly one element of each of the cells of P. Note $B = \{1, 2, 7\}$ is a system of representatives of the partition P_3 in Example 3.10.

3.9 EQUIVALENCE RELATIONS

Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, $a R a$.

(2) If $a R b$, then $b R a$.

(3) If $a R b$ and $b R c$, then $a R c$.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike". In fact, the relation $=$ of equality on any set S is an equivalence relation; that is,

(1) $a = a$ for every $a \in S$.

(2) If $a = b$, then $b = a$.

(3) If $a = b$ and $b = c$, then $a = c$.

For this reason, one frequently uses $\sim$ or $\equiv$ to denote an equivalence relation.

Examples of equivalence relations other than equality follow.

EXAMPLE 3.11

(*a*) Consider the set L of lines and the set T of triangles in the Euclidean plane. The relation "is parallel to or identical to" is an equivalence relation on L, and congruence and similarity are equivalence relations on T.

(*b*) The classification of animals by species, that is, the relation "is of the same species as," is an equivalence relation on the set of animals.

(*c*) The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

(*d*) Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, written

$$a \equiv b \pmod{m}$$

if m divides $a - b$. For example, for $m = 4$ we have $11 \equiv 3 \pmod 4$ since 4 divides $11 - 3$, and $22 \equiv 6 \pmod 4$ since 4 divides $22 - 6$. This relation of congruence modulo m is an equivalence relation.

Equivalence Relations and Partitions

Suppose R is an equivalence relation on a set S. For each a in S, let $[a]$ denote the set of elements of S to which a is related under R; that is,

$$[a] = \{x : (a, x) \in R\}$$

We call $[a]$ the *equivalence class* of a in S under R. The collection of all such equivalence classes is denoted by S/R, that is,

$$S/R = \{[a] : a \in S\}$$

It is called the *quotient* set of S by R.

The fundamental property of an equivalence relation and its quotient set is contained in the following theorem (which is proved in Problem 3.28).

Theorem 3.5: Let R be an equivalence relation on a set S. Then the quotient set S/R is a partition of S. Specifically:

(i) For each a in S, we have $a \in [a]$.

(ii) $[a] = [b]$ if and only if $(a, b) \in R$.

(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

The converse of the above theorem (proved in Problem 3.29) is also true. That is,

Theorem 3.6: Suppose $P = \{A_i\}$ is a partition of a set S. Then there is an equivalence relation $\sim$ on S such that the set $S/\sim$ of equivalence classes is the same as the partition $P = \{A_i\}$.

Specifically, for $a, b \in S$, the equivalence $\sim$ in Theorem 3.6 is defined by $a \sim b$ if a and b belong to the same cell in P.

Thus we see there is a one-to-one correspondence between the equivalence relations on a set S and the partitions of S. Accordingly, for a given equivalence relation R on a set S, we can talk about a system B of representatives of the quotient set S/R which would contain exactly one representative from each equivalence class.

EXAMPLE 3.12

(*a*) Consider the following relation R on $S = \{1, 2, 3, 4\}$:

$$R = \{(1,1),\ (2,2),\ (1,3),\ (3,1),\ (3,3),\ (4,4)\}$$

One can show that R is reflexive, symmetric and transitive, that is, that R is an equivalence relation. Under the relation R,

$$[1] = \{1,3\}, \qquad [2] = \{2\}, \qquad [3] = \{1,3\}, \qquad [4] = \{4\}$$

Observe that $[1] = [3]$ and that $S/R = \{[1],\ [2],\ [4]\}$ is a partition of S. One can choose either $\{1, 2, 4\}$ or $\{2, 3, 4\}$ as a system of representatives of the equivalence classes.

(*b*) Let R_5 be the relation on the set $\mathbf{Z}$ of integers defined by

$$x \equiv y \pmod 5$$

which reads "x is congruent to y modulo 5" and which means that the difference $x - y$ is divisible by 5. Then R_5 is an equivalence relation on $\mathbf{Z}$. There are exactly five equivalence classes in the quotient set $\mathbf{Z}/R_5$ as follows:

$$\begin{aligned} A_0 &= \{\ldots, -10, -5, 0, 5, 10, \ldots\} \\ A_1 &= \{\ldots, -9, -4, 1, 6, 11, \ldots\} \\ A_2 &= \{\ldots, -8, -3, 2, 7, 12, \ldots\} \\ A_3 &= \{\ldots, -7, -2, 3, 8, 13, \ldots\} \\ A_4 &= \{\ldots, -6, -1, 4, 9, 14, \ldots\} \end{aligned}$$

Observe that any integer x, which can be uniquely expressed in the form $x = 5q + r$ where $0 \le r < 5$, is a member of the equivalence class A_r where r is the remainder. As expected, the equivalence classes are disjoint and

$$\mathbf{Z} = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$$

This quotient set $\mathbf{Z}/R_5$ is usually denoted by

$$\mathbf{Z}/5\mathbf{Z} \text{ or simply } \mathbf{Z}_5$$

Usually one chooses $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$ as a system of representatives of the equivalence classes.

3.10 PARTIAL ORDERING RELATIONS

This section defines another important class of relations. A relation R on a set S is called a *partial ordering* of S or a *partial order* on S if it has the following three properties:

(1) For every $a \in S$, we have $a\,R\,a$.
(2) If $a\,R\,b$ and $b\,R\,a$, then $a = b$.
(3) If $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$.

That is, R is a partial ordering of S if R is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a *partially ordered set* or *poset*. Partially ordered sets will be studied in more detail in Chapter 7, so here we simply give some examples.

EXAMPLE 3.13

(*a*) The relation $\subseteq$ of set inclusion is a partial ordering of any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set A.
(2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
(3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(*b*) The relation $\leq$ on the set R of real numbers is reflexive, antisymmetric, and transitive. Thus $\leq$ is a partial ordering.

(*c*) The relation "a divides b" is a partial ordering of the set p of positive integers. However, "a divides b" is not a partial ordering of the set $\mathbf{Z}$ of integers since $a|b$ and $b|a$ does not imply $a = b$. For example, $3|-3$ and $-3|3$ but $3 \neq -3$.

3.11 *n*-ARY RELATIONS

All the relations discussed above were binary relations. By an *n-ary relation*, we mean a set of ordered n-tuples. For any set S, a subset of the product set S^n is called an n-ary relation on S. In particular, a subset of S^3 is called a *ternary relation* on S.

EXAMPLE 3.14

(*a*) Let L be a line in the plane. Then "betweenness" is a ternary relation R on the points of L; that is, $(a, b, c) \in R$ if b lies between a and c on L.

(*b*) The equation $x^2 + y^2 + z^2 = 1$ determines a ternary relation T on the set $\mathbf{R}$ of real numbers. That is, a triple (x, y, z) belongs to T if (x, y, z) satisfies the equation which means that (x, y, z) is the coordinates of a point in $\mathbf{R}^3$ on the sphere S with radius 1 and center at the origin $0 = (0, 0, 0)$.

Solved Problems

ORDERED PAIRS AND PRODUCT SETS

3.1. Let $A = \{1, 2\}$, $B = \{x, y, z\}$, $C = \{3, 4\}$. Find $A \times B \times C$.

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram (Fig. 3-7). The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$ and, as expected,

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

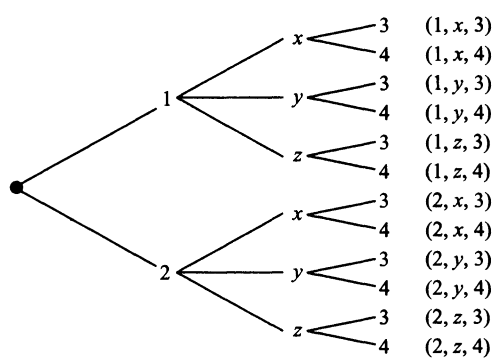


Fig. 3-7