Chapter 1

Sets and Basic Operations on Sets

1.1 INTRODUCTION

The concept of a *set* appears in all branches of mathematics. This concept formalizes the idea of grouping objects together and viewing them as a single entity. This chapter introduces this notion of a set and its members. We also investigate three basic operations on sets, that is, the operations union, intersection, and complement.

Although logic is formally treated in Chapter 10, we indicate here the close relationship between set theory and logic by showing how Venn diagrams, pictures of sets, can be used to determine the validity of certain arguments. The relation between set theory and logic will be further explored when we discuss Boolean algebra in Chapter 11.

1.2 SETS AND ELEMENTS

A *set* may be viewed as any well-defined collection of objects; the objects are called the *elements* or *members* of the set.

Although we shall study sets as abstract entities, we now list ten examples of sets:

(1) The numbers 1, 3, 7, and 10.
(2) The solutions of the equation $x^2 - 3x - 2 = 0$.
(3) The vowels of the English alphabet: a, e, i, o, u.
(4) The people living on the earth.
(5) The students Tom, Dick, and Harry.
(6) The students absent from school.
(7) The countries England, France, and Denmark.
(8) The capital cities of Europe.
(9) The even integers: 2, 4, 6, ...
(10) The rivers in the United States.

Observe that the sets in the odd-numbered examples are *defined*, that is, specified or presented, by actually listing its members; and the sets in the even-numbered examples are defined by stating properties or rules which decide whether or not a particular object is a member of the set.

Notation

A set will usually be denoted by a capital letter, such as,

$$A, B, X, Y, \ldots,$$

whereas lower-case letters, $a, b, c, x, y, z, \ldots$ will usually be used to denote elements of sets.

There are essentially two ways to specify a particular set, as indicated above. One way, if possible, is to list its elements. For example,

$$A = \{\text{a}, \text{e}, \text{i}, \text{o}, \text{u}\}$$

means that A is the set whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. This is sometimes called the *tabular form* of a set.

The second way is to state those properties which characterize the elements in the set, that is, properties held by the members of the set but not by nonmembers. Consider, for example, the expression

$$B = \{x : x \text{ is an even integer}, x > 0\}$$

which reads:

"B is the set of x such that x is an even integer and $x > 0$"

It denotes the set B whose elements are the positive even integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and". This is sometimes called the *set-builder form* or *property method* of specifying a set.

Two sets A and B are *equal*, written $A = B$, if they both have the same elements, that is, if every element which belongs to A also belongs to B, and vice versa. The negation of $A = B$ is written $A \neq B$.

The statement "p is an element of A" or, equivalently, the statement "p belongs to A" is written

$$p \in A$$

We also write

$$a, b \in A$$

to state that both a and b belong to A. The statement that p is not an element of A, that is, the negation of $p \in A$, is written

$$p \notin A$$

Remark: It is common practice in mathematics to put a vertical line "|" or slanted line "/" through a symbol to indicate the opposite or negative meaning of the symbol.

EXAMPLE 1.1

(*a*) The set A above can also be written as

$$A = \{x : x \text{ is a letter in the English alphabet}, x \text{ is a vowel}\}$$

Observe that $b \notin A$, $e \in A$, and $p \notin$ A.

(*b*) We cannot list all the elements of the above set B, although we frequently specify the set by writing

$$B = \{2, 4, 6, \ldots\}$$

where we assume everyone knows what we mean. Observe that $8 \in B$, but $9 \notin B$.

(*c*) Let $E = \{x : x^2 - 3x + 2 = 0\}$. In other words, E consists of those numbers which are solutions of the equation $x^2 - 3x + 2 = 0$, sometimes called the *solution set* of the given equation. Since the solutions are 1 and 2, we could also write $E = \{1, 2\}$.

(*d*) Let $E = \{x : x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$, and $G = \{1, 2, 2, 1, 6/3\}$. Then $E = F = G$ since each consists precisely of the elements 1 and 2. Observe that a set does not depend on the way in which its elements are displayed. A set remains the same even if its elements are repeated or rearranged.

Some sets of numbers will occur very often in the text, and so we use special symbols for them. Unless otherwise specified, we will let:

$\mathbf{N}$ = the set of nonnegative integers: $0, 1, 2, \ldots$
$\mathbf{P}$ = the set of positive integers: $1, 2, 3, \ldots$
$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbf{Q}$ = the set of rational numbers
$\mathbf{R}$ = the set of real numbers
$\mathbf{C}$ = the set of complex numbers

Even if we can list the elements of a set, it may not be practical to do so. For example, we would not list the members of the set of people born in the world during the year 1976 although theoretically it is possible to compile such a list. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

1.3 UNIVERSAL SET, EMPTY SET

All sets under investigation in any application of set theory are assumed to be contained in some large fixed set called the *universal set* or *universe*. For example, in plane geometry, the universal set consists of all the points in the plane, and in human population studies the universal set consists of all the people in the world. We will denote the universal set by

$$\mathbf{U}$$

unless otherwise specified.

Given a universal set $\mathbf{U}$ and a property P, there may be no element in $\mathbf{U}$ which has the property P. For example, the set

$$S = \{x : x \text{ is a positive integer, } x^2 = 3\}$$

has no elements since no positive integer has the required property. This set with no elements is called the *empty set* or *null set*, and is denoted by

$$\varnothing$$

(based on the Greek letter phi). There is only one empty set: If S and T are both empty, then $S = T$ since they have exactly the same elements, namely, none.

1.4 SUBSETS

Suppose every element in a set A is also an element of a set B; then A is called a *subset* of B. We also say that A is *contained* in B or B *contains* A. This relationship is written

$$A \subseteq B \qquad \text{or} \qquad B \supseteq A$$

If A is not a subset of B, that is, if at least one element of A does not belong to B, we write $A \not\subseteq B$ or $B \not\supseteq A$.

EXAMPLE 1.2

(*a*) Consider the sets

$$A = \{1, 3, 5, 8, 9\}, \qquad B = \{1, 2, 3, 5, 7\}, \qquad C = \{1, 5\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 5, the elements of C, are also elements of A and B. But $B \not\subseteq A$ since some of its elements, e.g., 2 and 7, do not belong to A. Furthermore, since the elements in the sets A, B, C must also belong to the universal set $\mathbf{U}$, it is clear that $\mathbf{U}$ must at least contain the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(*b*) Let $\mathbf{P}, \mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ be defined as in Section 1.2. Then:

$$\mathbf{P} \subseteq \mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

(*c*) The set $E = \{2, 4, 6\}$ is a subset of the set $F = \{6, 2, 4\}$, since each number 2, 4, and 6 belonging to E also belongs to F. In fact, $E = F$. Similarly, it can be shown that every set is a subset of itself.

The following properties of sets should be noted:

(i) Every set A is a subset of the universal set $\mathbf{U}$ since, by definition, all the elements of A belong to $\mathbf{U}$. Also the empty set $\varnothing$ is a subset of A.

(ii) Every set A is a subset of itself since, trivially, the elements of A belong to A.

(iii) If every element of A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(iv) If $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

We state these results formally.

Theorem 1.1: (i) For any set A, we have $\varnothing \subseteq A \subseteq \mathbf{U}$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proper Subset

If $A \subseteq B$, then it is still possible that $A = B$. When $A \subseteq B$ but $A \neq B$, we say that A is a *proper subset* of B. We will write $A \subset B$ when A is a proper subset of B. For example, suppose

$$A = \{1, 3\}, \qquad B = \{1, 2, 3\}, \qquad C = \{1, 3, 2\}$$

Then A and B are both subsets of C; but A is a proper subset of C, whereas B is not a proper subset of C.

Disjoint Sets

Two sets A and B are disjoint if they have no elements in common. For example, suppose

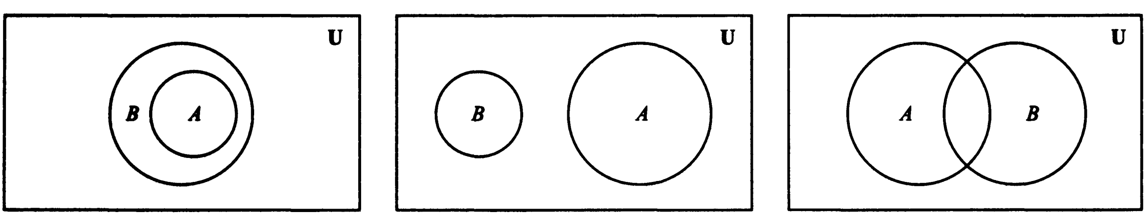
$$A = \{1, 2\}, \qquad B = \{2, 4, 6\}, \qquad C = \{4, 5, 6, 7\}$$

Note that A and B are not disjoint since they both contain the element 2. Similarly, B and C are not disjoint since they both contain the element 4, among others. On the other hand, A and C are disjoint since they have no element in common. We note that if two sets A and B are disjoint sets then neither is a subset of the other (unless one is the empty set).

1.5 VENN DIAGRAMS

A Venn diagram is a pictorial representation of sets where sets are represented by enclosed areas in the plane. The universal set $\mathbf{U}$ is represented by the points in a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B, as in Fig. 1-1(a). If A and B are disjoint, i.e., have no elements in common, then the disk representing A will be separated from the disk representing B, as in Fig. 1-1(b).

On the other hand, if A and B are two arbitrary sets, it is possible that some elements are in A but not B, some elements are in B but not A, some are in both A and B, and some are in neither A nor B; hence, in general, we represent A and B as in Fig. 1-1(c).



(a) $A \subseteq B$ (b) A and B are disjoint (c)

Fig. 1-1

1.6 SET OPERATIONS

The reader has learned to add, subtract, and multiply in the ordinary arithmetic of numbers; that is, to each pair of numbers a and b, we assign a number $a+b$ called the *sum* of a and b, a number $a-b$ called the *difference* of a and b, and a number ab called the *product* of a and b. These assignments are called the operations of addition, subtraction, and multiplication of numbers. This section defines a number of set operations, including the basic operations of union, intersection, and difference of sets, where new sets will be assigned to pairs of sets A and B. We will see that set operations have many properties similar to the above operations on numbers.

Union and Intersection

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or B; that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

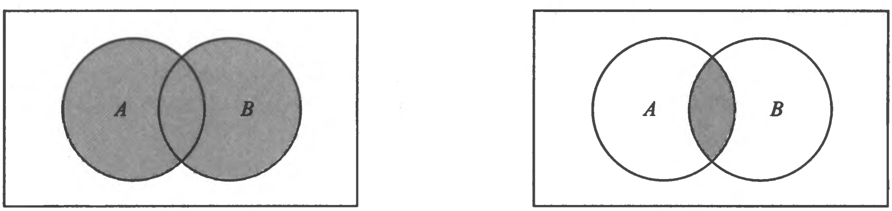
Here "or" is used in the sense of and/or. Figure 1-2(a) is a Venn diagram in which $A \cup B$ is shaded.

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of all elements which belong to both A and B; that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Figure 1-2(b) is a Venn diagram in which $A \cap B$ is shaded.

Recall that sets A and B are said to be disjoint if they have no elements in common. Accordingly, using the above notation, A and B are disjoint if $A \cap B = \varnothing$, the empty set.



(a) $A \cup B$ is shaded (b) $A \cap B$ is shaded

Fig. 1-2

EXAMPLE 1.3

(a) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}, \qquad A \cap B = \{3, 4\}$$
$$A \cup C = \{1, 2, 3, 4, 8, 9\}, \qquad A \cap C = \{2, 3\}$$
$$B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}, \qquad B \cap C = \{3\}$$

(b) Let $\mathbf{U}$ denote the set of students at a university, and let M and F denote, respectively, the set of male and female students at the university. Then

$$M \cup F = \mathbf{U}$$

since each student in $\mathbf{U}$ is either in M or in F. On the other hand,

$$M \cap F = \varnothing$$

since no student belongs to both M and F.

The following properties of the union and intersection of sets should be noted:

(i) Every element x in $A \cap B$ belongs to both A and B; hence x belongs to A and x belongs to B. Thus $A \cap B$ is a subset of A and of B, that is,

$$A \cap B \subseteq A \quad \text{and} \quad A \cap B \subseteq B$$

(ii) An element x belongs to the union $A \cup B$ if x belongs to A or x belongs to B; hence every element in A belongs to $A \cup B$, and also every element in B belongs to $A \cup B$. That is,

$$A \subseteq A \cup B \quad \text{and} \quad B \subseteq A \cup B$$

We state the above results formally.

Theorem 1.2: For any sets A and B, we have

$$A \cap B \subseteq A \subseteq A \cup B \quad \text{and} \quad A \cap B \subseteq B \subseteq A \cup B$$

The operation of set inclusion is also closely related to the operations of union and intersection, as shown by the following theorem, proved in Problem 1.13.

Theorem 1.3: The following are equivalent:

$$A \subseteq B, \qquad A \cap B = A, \qquad A \cup B = B$$

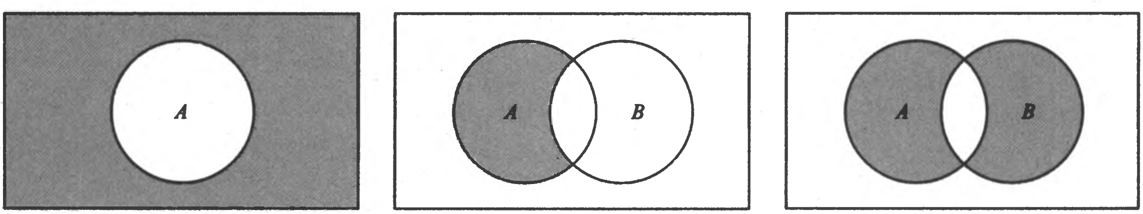
Other conditions equivalent to $A \subseteq B$ are given in Problem 1.51.

Complement

Recall that all sets under consideration at a particular time are subsets of a fixed universal set **U**. The *absolute complement*, or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to **U** but which do not belong to A; that is,

$$A^c = \{x : x \in \mathbf{U}, x \notin A\}$$

Some texts denote the complement of A by A' or $\bar{A}$. Figure 1-3(a) is a Venn diagram in which A^c is shaded.



(a) A^c is shaded (b) $A \backslash B$ is shaded (b) $A \oplus B$ is shaded

Fig. 1-3

EXAMPLE 1.4

(a) Let $\mathbf{U} = \{a, b, c, \ldots, y, z\}$, the English alphabet, be the universal set, and let

$$A = \{\mathrm{a, b, c, d, e}\}, \qquad B = \{\mathrm{e, f, g}\}, \qquad V = \{\mathrm{a, e, i, o, u}\}$$

Then

$$A^c = \{\mathrm{f, g, h, \ldots, y, z}\} \quad \text{and} \quad B^c = \{\mathrm{a, b, c, d, h, i, \ldots, y, z}\}$$

Since V consists of the vowels in **U**, V^c consists of the nonvowels, called consonants.

(b) Suppose the set **R** of real numbers is the universal set. Recall that **Q** denotes the set of rational numbers. Hence $\mathbf{Q}^c$ will denote the set of irrational numbers.

(*c*) Let U be the set of students at a university, and suppose M and F denote, respectively, the male and female students in U. Then

$$M^c = F \quad \text{and} \quad F^c = M$$

Difference and Symmetric Difference

Let A and B be sets. The *relative complement* of B with respect to A or, simply, the *difference* of A and B, denoted by $A\backslash B$, is the set of elements which belong to A but which do not belong to B; that is,

$$A\backslash B = \{x : x \in A, x \notin B\}$$

The set $A\backslash B$ is read "A minus B". Many texts denote $A\backslash B$ by $A - B$ or $A \sim B$. Figure 1-3(*b*) is a Venn diagram in which $A\backslash B$ is shaded.

The *symmetric difference* of the sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both A and B. That is,

$$A \oplus B = (A \cup B)\backslash(A \cap B) \quad \text{or} \quad A \oplus B = (A\backslash B) \cup (B\backslash A)$$

Figure 1-3(*c*) is a Venn diagram in which $A \oplus B$ is shaded. The fact that

$$(A \cup B)\backslash(A \cap B) = (A\backslash B) \cup (B\backslash A)$$

is proved in Problem 1.18.

EXAMPLE 1.5 Consider the sets

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{6, 7, 8, 9\}$$

Then

$$A\backslash B = \{1, 2\}, \quad B\backslash C = \{3, 4, 5\}, \quad B\backslash A = \{5, 6, 7\}, \quad C\backslash B = \{8, 9\}$$

Also,

$$A \oplus B = \{1, 2, 5, 6, 7\} \quad \text{and} \quad B \oplus C = \{3, 4, 5, 8, 9\}$$

Note that A and C are disjoint. This means

$$A\backslash C = A, \quad C\backslash A = C, \quad A \oplus C = A \cup C$$

1.7 ALGEBRA OF SETS, DUALITY

Sets under the above operations of union, intersection, and complement satisfy various laws (identities) which are listed in Table 1-1. In fact, we formally state:

Theorem 1.4: Sets satisfy the laws in Table 1-1.

Each of the laws in Table 1-1 follows from an equivalent logical law. Consider, for example, the proof of DeMorgan's law:

$$(A \cup B)^c = \{x : x \notin (A \text{ or } B)\} = \{x : x \notin A \text{ and } x \notin B\} = A^c \cap B^c$$

Here we use the equivalent (DeMorgan's) logical law:

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

Here $\neg$ means "not", $\vee$ means "or", and $\wedge$ means "and". Sometimes Venn diagrams are used to illustrate the laws in Table 1-1 (cf. Problem 1.16).

Table 1-1 Laws of the Algebra of Sets

Idempotent laws	
(1*a*) $A \cup A = A$	(1*b*) $A \cap A = A$
Associative laws	
(2*a*) $(A \cup B) \cup C = A \cup (B \cup C)$	(2*b*) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	
(3*a*) $A \cup B = B \cup A$	(3*b*) $A \cap B = B \cap A$
Distributive laws	
(4*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	
(5*a*) $A \cup \varnothing = A$	(5*b*) $A \cap \mathbf{U} = A$
(6*a*) $A \cup \mathbf{U} = \mathbf{U}$	(6*b*) $A \cap \varnothing = \varnothing$
Involution law	
(7) $(A^c)^c = A$	
Complement laws	
(8*a*) $A \cup A^c = \mathbf{U}$	(8*b*) $A \cap A^c = \varnothing$
(9*a*) $\mathbf{U}^c = \varnothing$	(9*b*) $\varnothing^c = \mathbf{U}$
DeMorgan's laws	
(10*a*) $(A \cup B)^c = A^c \cap B^c$	(10*b*) $(A \cap B)^c = A^c \cup B^c$

Duality

The identities in Table 1-1 are arranged in pairs, as, for example, (2*a*) and (2*b*). We now consider the principle behind this arrangement. Let E be an equation of set algebra. The *dual* E^* of E is the equation obtained by replacing each occurrence of $\cup, \cap, \mathbf{U}, \varnothing$ in E by $\cap, \cup, \varnothing, \mathbf{U}$, respectively. For example, the dual of

$$(\mathbf{U} \cap A) \cup (B \cap A) = A \quad \text{is} \quad (\varnothing \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Table 1-1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that, if any equation E is an identity, then its dual E^* is also an identity.

1.8 FINITE SETS, COUNTING PRINCIPLES

A set is said to be *finite* if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise a set is said to be infinite. For example, the empty set $\varnothing$ and the set of letters of the English alphabet are finite sets, whereas the set of even positive integers $\{2, 4, 6, \ldots\}$ is infinite. [Infinite sets will be studied in detail in Chapter 6.]

The notation $n(A)$ or $|A|$ will denote the number of elements in a finite set A.

First we begin with a special case.

Lemma 1.5: Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

Proof. In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. Since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$, as claimed.

Remark: A set C is called the *disjoint union* of A and B if

$$C = A \cup B \quad \text{and} \quad A \cap B = \varnothing$$

Lemma 1.5 tells us that, in such a case, $n(C) = n(A) + n(B)$.

Special Cases of Disjoint Unions

There are two special cases of disjoint unions which occur frequently.

(1) Given any set A, then the universal set U is the disjoint union of A and its complement A^c. Thus, by Lemma 1.5,

$$n(U) = n(A) + n(A^c)$$

Accordingly, bringing $n(A)$ to the other side, we obtain the following useful result.

Theorem 1.6: Let A be any set in a finite universal set U. Then

$$\boxed{n(A^c) = n(U) - n(A)}$$

For example, if there are 20 male students in a class of 35 students, then there are $35 - 20 = 15$ female students.

(2) Given any sets A and B, we show (Problem 1.37) that A is the disjoint union of $A \backslash B$ and $A \cap B$. This is pictured in Fig. 1-4. Thus Lemma 1.5 gives us the following useful result.

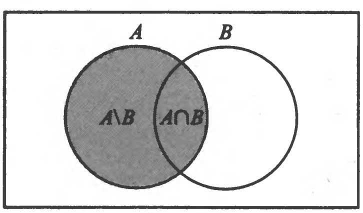
Theorem 1.7: Suppose A and B are finite sets. Then

$$\boxed{n(A \backslash B) = n(A) - n(A \cap B)}$$

For example, suppose an archery class A contains 35 students, and 15 of them are also in a bowling class B. Then

$$n(A \backslash B) = n(A) - n(A \cap B) = 35 - 15 = 20$$

That is, there are 20 students in the class A who are not in class B.



A is shaded

Fig. 1-4

Inclusion-Exclusion Principle

There is also a formula for $n(A \cup B)$ even when they are not disjoint, called the *inclusion-exclusion principle*. Namely:

Theorem 1.8: Suppose A and B are finite sets. Then $A \cap B$ and $A \cup B$ are finite, and

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since the elements in $A \cap B$ were counted twice.

We can apply this result to get a similar result for three sets.

Corollary 1.9: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.11) may be used to further generalize this result to any finite number of finite sets.

EXAMPLE 1.6 Consider the following data among 110 students in a college dormitory:

30 students are on a list A (taking Accounting),
35 students are on a list B (taking Biology),
20 students are on both lists.

Find the number of students: (*a*) on list or B, (*b*) on exactly one of the two lists, (*c*) on neither list.

(*a*) We seek $n(A \cup B)$. By Theorem 1.8,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45$$

In other words, we combine the two lists and then cross out the 20 student names which appear twice.

(*b*) List A contains 30 names and 20 of them are on list B; hence $30 - 20 = 10$ names are only on list A. That is,

$$n(A \backslash B) = n(A) - n(A \cup B) = 30 - 20 = 10$$

Similarly, list B contains 35 names and 20 of them are on list A; hence $35 - 20 = 15$ names are only on list B. That is,

$$n(B \backslash A) = n(B) - n(A \cup B) = 35 - 20 = 15$$

Thus there are $10 + 15 = 25$ students on exactly one of the two lists.

(*c*) The students on neither the A list nor the B list form the set $A^c \cap B^c$. By DeMorgan's law, $A^c \cap B^c = (A \cup B)^c$. Hence

$$n(A^c \cap B^c) = n((A \cup B)^c) = n(U) - n(A \cup B) = 110 - 45 = 65$$

EXAMPLE 1.7 Consider the following data for 120 mathematics students:

65 study French,	20 study French and German,
45 study German,	25 study French and Russian,
42 study Russian,	15 study German and Russian,

8 study all three languages

Let F, G, and R denote the sets of students studying French, German, and Russian, respectively.

(*a*) Find the number of students studying at least one of the three languages, i.e. find $n(F \cup G \cup R)$.

(*b*) Fill in the correct number of students in each of the eight regions of the Venn diagram of Fig. 1-5(*a*).

(*c*) Find the number k of students studying: (1) exactly one language, (2) exactly two languages.

(*a*) By Corollary 1.9,

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) - n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

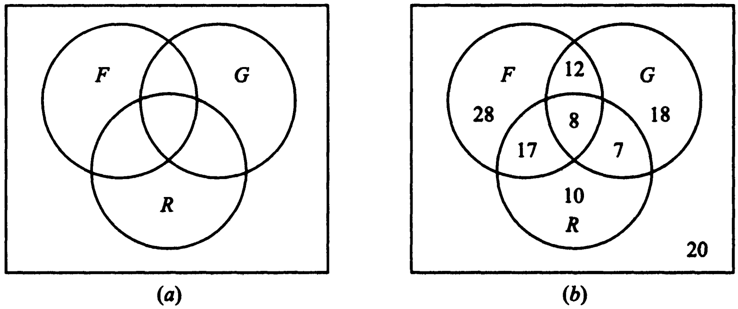


Fig. 1-5

(*b*) Using 8 study all three languages and 100 study at least one language, the remaining seven regions of the required Venn diagram Fig. 1-5(*b*) are obtained as follows:

$15 - 8 = 7$ study German and Russian but not French,
$25 - 8 = 17$ study French and Russian but not German,
$20 - 8 = 12$ study French and German but not Russian,
$42 - 17 - 8 - 7 = 10$ study only Russian,
$45 - 12 - 8 - 7 = 18$ study only German,
$65 - 12 - 8 - 17 = 28$ study only French,
$120 - 100 = 20$ do not study any of the languages.

(*c*) Use the Venn diagram of Fig. 1-5(*b*) to obtain:

(1) $k = 28 + 18 + 10 = 56$, (2) $k = 12 + 17 + 7 = 36$

1.9 CLASSES OF SETS, POWER SETS

Given a set S, we may wish to talk about some of its subsets. Thus we would be considering a "set of sets". Whenever such a situation arises, to avoid confusion, we will speak of a *class* of sets or a *collection* of sets. If we wish to consider some of the sets in a given class of sets, then we will use the term *subclass* or *subcollection*.

EXAMPLE 1.8 Suppose $S = \{1, 2, 3, 4\}$. Let $\mathscr{A}$ be the class of subsets of S which contain exactly three elements of S. Then

$$\mathscr{A} = [\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}]$$

The elements of $\mathscr{A}$ are the sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

Let $\mathscr{B}$ be the class of subsets of S which contain 2 and two other elements of S. Then

$$\mathscr{B} = [\{1,2,3\}, \{1,2,4\}, \{2,3,4\}]$$

The elements of $\mathscr{B}$ are $\{1,2,3\}$, $\{1,2,4\}$, and $\{2,3,4\}$. Thus $\mathscr{B}$ is a subclass of $\mathscr{A}$. (To avoid confusion, we will usually enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S, we may speak about the class of all subsets of S. This class is called the *power set* of S, and it will be denoted by $\mathscr{P}(S)$. If S is finite, then so is $\mathscr{P}(S)$. In fact, the number of elements in $\mathscr{P}(S)$ is 2 raised to the power of $n(S)$; that is,

$$n(\mathscr{P}(S)) = 2^{n(S)}$$

(This is the reason $\mathscr{P}(S)$ is called the power set of S; it is also sometimes denoted by 2^S.)

EXAMPLE 1.9 Suppose $S = \{1, 2, 3\}$. Then

$$\mathcal{P}(S) = [\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S]$$

Note that the empty set $\varnothing$ belongs to $\mathcal{P}(S)$ since $\varnothing$ is a subset of S. Similarly S belongs to $\mathcal{P}(S)$. As expected from the above remark, $\mathcal{P}(S)$ has $2^3 = 8$ elements.

1.10 ARGUMENTS AND VENN DIAGRAMS

Many verbal statements are essentially statements about sets and they can therefore be described by Venn diagrams. Hence Venn diagrams can sometimes be used to determine whether or not an argument is valid. This is illustrated in the following example.

EXAMPLE 1.10 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1-6.

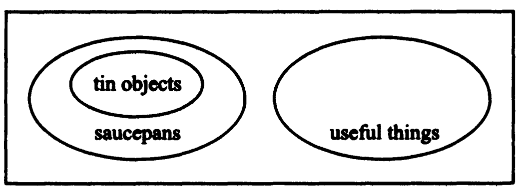


Fig. 1-6

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1-7.

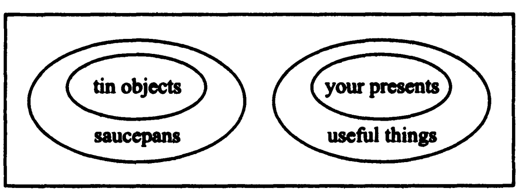


Fig. 1-7

The conclusion is clearly valid by the Venn diagram in Fig. 1-7 because the set of "your presents" is disjoint from the set of tin objects.

1.11 MATHEMATICAL INDUCTION

Consider the set $\mathbf{P} = \{1, 2, \ldots\}$ of positive integers (or counting numbers). We say that an assertion $A(n)$ is defined on $\mathbf{P}$ if $A(n)$ is true or false for each $n \in \mathbf{P}$. An essential property of $\mathbf{P}$, which is used in many proofs, follows.

Principle of Mathematical Induction I: Let $A(n)$ be an assertion defined on $\mathbf{P}$, that is, $A(n)$ is true or false for each integer $n \geq 1$. Suppose $A(n)$ has the following two properties:

(1) $A(1)$ is true.
(2) $A(n+1)$ is true whenever $A(n)$ is true.

Then $A(n)$ is true for every $n \geq 1$.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when $\mathbf{P}$ is developed axiomatically.

EXAMPLE 1.11 Let $A(n)$ be the assertion that the sum of the first n odd integers is n^2; that is,

$$A(n): \quad 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

[The nth odd integer is $2n-1$ and the next odd integer is $2n+1$.] Observe that $A(n)$ is true for $n = 1$, that is,

$$A(1): \quad 1 = 1^2$$

Assuming $A(n)$ is true, we add $2n+1$ to both sides of $A(n)$, obtaining:

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

However, this is $A(n+1)$. That is, $A(n+1)$ is true whenever $A(n)$ is true. By the principle of mathematical induction, $A(n)$ is true for all $n \geq 1$.

There is another form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the above principle of induction.

Principle of Mathematical Induction II: Let $A(n)$ be an assertion defined on the set $\mathbf{P}$ of positive integers which satisfies the following two conditions:

(1) $A(1)$ is true.
(2) $A(n)$ is true whenever $A(k)$ is true for $1 \leq k < n$.

Then $A(n)$ is true for every $n \geq 1$.

The above two principles may also be stated in terms of subsets of $\mathbf{P}$ rather than in terms of assertions defined on $\mathbf{P}$. (See Problem 1.40.) Although the languages are different, they are logically equivalent.

Remark: Sometimes one wants to prove that an assertion A is true for a set of integers of the form

$$\{a, a+1, a+2, \ldots\}$$

where a is any integer, possibly 0. This can be done by simply replacing 1 by the integer a in either of the above principles of mathematical induction.

1.12 AXIOMATIC DEVELOPMENT OF SET THEORY

Any axiomatic development of a branch of mathematics begins with the following:

(1) undefined terms,
(2) undefined relations,
(3) axioms relating the undefined terms and undefined relations.

Then, one develops theorems based upon the axioms and definitions.

Consider, for example, the axiomatic development of plane Euclidean geometry. It begins with the following:

(1) "points" and "lines" are undefined terms;
(2) "point on a line" or, equivalently, "line contains a point" is an undefined relation.

Two of the many axioms of Euclidean geometry follow:

Axiom 1: Two distinct points are on one and only one line.

Axiom 2: Two distinct lines cannot contain more than one point in common.

The axiomatic development of set theory begins with the following:

(1) "element" and "set" are undefined terms;
(2) "element belongs to a set" is the undefined relation.

Two of the axioms (called principles) of set theory follow:

Principle of Extension: Two sets A and B are equal if and only if they have the same elements, that is, if every element in A belongs to B and every element in B belongs to A.

Principle of Abstraction: Given any set $\mathbf{U}$ and any property P, there is a set A such that the elements of A are exactly those elements in $\mathbf{U}$ which have the property P; that is,

$$A = \{x : x \in \mathbf{U}, P(x) \text{ is true}\}$$

There are other axioms which are not listed. As our treatment of set theory is mainly intuitive, especially Part I, we will refrain from any further discussion of the axiomatic development of set theory.

Solved Problems

SETS AND SUBSETS

1.1. Which of these sets are equal: $\{r, t, s\}$, $\{s, t, r, s\}$, $\{t, s, t, r\}$, $\{s, r, s, t\}$?

They are all equal. Order and repetition do not change a set.

1.2. List the elements of the following sets where $\mathbf{P} = \{1, 2, 3, \ldots\}$.

(*a*) $A = \{x : x \in \mathbf{P},\ 3 < x < 12\}$
(*b*) $B = \{x : x \in \mathbf{P},\ x \text{ is even},\ x < 15\}$
(*c*) $C = \{x : x \in \mathbf{P},\ 4 + x = 3\}$
(*d*) $D = \{x : x \in \mathbf{P},\ x \text{ is a multiple of } 5\}$.

(*a*) A consists of the positive integers between 3 and 12; hence

$$A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

(*b*) B consists of the even positive integers less than 15; hence

$$B = \{2, 4, 6, 8, 10, 12, 14\}$$

(*c*) There are no positive integers which satisfy the condition $4 + x = 3$; hence C contains no elements. In other words, $C = \varnothing$, the empty set.