* ORDINARY DIFFERENTIAL EQUATION:- OF HIGHER ORDER HYPER GEOMETRIC D.Eq

In mathematics the hypergeometric D.E is a 2nd order linear ordinary D.E whose solutions are given by the hypergeometric series.

* Hypergeometric Series:-

A hypergeometric series is a power series in which the ratios of successive coefficient is a rational function the series of convergent will define a hypergeometric function.

Hypergeometric function generalize many special functions including the Bessel function, The incomplet GAMMA function, The ERROR function.

Symbolically hypergeometric function can be written as

$$F(a, b, c; z) = F\left[\begin{matrix} a, & b \\ & c, \; z \end{matrix}\right]$$

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n$$

where $(a)_n$ read as a factorial function of order n or generlize factorial

Now we take $a=c$ and $b=1$ we get

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(c)_n (1)_n}{(c)_n \, n!} z^n$$

$$F(a,b,c;z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

* GAMMA FUNCTION:: The GAMMA Function represented by the capitalized Greek letter "Γ" which is an extension of factorial function to the real and complex numbers z. with positive (+ve) real part. ~~Γ~~

NOTATION $\Gamma(z)$ = Gamma z It is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

Properties:

$\Gamma(1) = 1$

using

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

Put $z=1$

$$\Gamma(1) = \lim_{h\to\infty}\int_0^h \bar{e}^t\,dt.$$

$$= \lim_{h\to\infty}\left|-\bar{e}^t\right|_0^h$$

$$= \lim_{h\to\infty}\left|-\frac{1}{e^h}+\frac{1}{e^0}\right|$$

$$= \lim_{h\to\infty}\left|-\frac{1}{e^\infty}+1\right|$$

$$= \cancel{\lim_{h\to\infty}}\,|-0+1|$$

$$= \cancel{\lim_{h\to\infty}}\,(1)$$

Hence $\Gamma(1) = 1$ Ans.

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ii)

$$\Gamma(z+1) = z\,\Gamma(z).$$

$$\Gamma(z+1) = \int_0^\infty \bar{e}^t.\,t^{z+1-1}\,dt.$$

$$= \int_0^\infty \bar{e}^t.\,t^z\,dt.$$

$$= \lim_{h\to\infty}\int_0^\infty \bar{e}^t t^z\,dt.$$

$$= \lim \left[t^z\bar{e}^t\Big|^h - \int^h \bar{e}^t\,z.\,t^{z-1}\,dt\right]$$

$$= 0 + \lim_{h \to \infty} z \int_0^h e^{-t} t^{z-1}\, dt.$$

$$= z \lim_{h \to \infty} \int_0^h e^{-t} t^{z-1}\, dt.$$

$$= z \int_0^\infty e^{-t} t^{z-1}\, dt.$$

Hence

$$\Gamma(z+1) = z\,\Gamma z$$

prove $\Gamma(z+1) = z!$

Since

$$\Gamma(z+1) = z \cdot \Gamma(z)$$

$$= z \cdot (z-1)\Gamma(z-1).$$

$$= z \cdot (z-1)(z-2)\Gamma(z-2)$$

$$= z \cdot (z-1)(z-2)(z-3)\Gamma(z-3)$$

$$\vdots$$

$$z \cdot (z-1)(z-2)(z-3) \cdots \cdot \Gamma 1.$$

$$= z \cdot (z-1)(z-2)(z-3) \cdots 1.$$

$$\Gamma(z+1) = z!$$

✓ Show that $0! = 1$.

we know that

$$n! = n(n-1)!$$

$$\Rightarrow (n-1)! = \frac{n}{n!}$$

Beta function:-

Notation: $B(x,y)$

It is defined as.

$$B(x,y) = \int_0^1 (t)^{x-1}(1-t)^{y-1}\,dt \text{ where } x,y \in Z.$$

* Relation b/w Gamma & Beta function:-

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)}.$$

✓ $B(x,y) = B(y,x)$

i.e Beta function is Symmetric.

Some useful Result:

i) Binomial Theorem:-

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{n!}$$

ii) $\Gamma(\alpha+n) = (\alpha+n-1)\,\Gamma(\alpha+n-1)$

$$= (\alpha+n-1)(\alpha+n-2)\,\Gamma(\alpha+n-2).$$

(10.b)

Question

If $|z| < 1$ & $Re(\gamma) > Re(\beta) > 0$

Then
$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{(t)^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-tz)^{\alpha}} dt$$

Sol:- Since
$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} z^n \quad \text{--- (1)}$$

take
$$\frac{(\beta)_n}{(\gamma)_n} = \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} \quad \because (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)} \quad \left| \begin{array}{l} x = \beta+n \\ y = \gamma-\beta \\ x+y = \gamma+n \end{array} \right.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot B(\beta+n, \gamma-\beta)$$ By defn of Beta function

$$\frac{(\beta)_n}{(\gamma)_n} = \frac{\Gamma(\gamma)}{\ldots} \cdot \int_0^1 (t)^{\beta+n-1} (1-t)^{\gamma-\beta-1}$$

$$F(\alpha, \beta, \gamma ; z) = \sum_{n=0}^{\infty} \frac{\Gamma\gamma}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 (t)^{\beta+n-1} (1-t)^{\gamma-\beta-1} dt \, \frac{(\alpha)_n z^n}{n!},$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \sum_{n=0}^{\infty} (t)^{\beta-1} \cdot t^n (1-t)^{\gamma-\beta-1} dt \, \frac{(\alpha)_n}{n!} z^n.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (tz)^n dt.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \times \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt.$$

$$\because (1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n.$$

Hence.

$$F(\alpha, \beta, \gamma ; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt.$$ //

Qu $\operatorname{Re}(c-a-b)>0$ & $\operatorname{Re}(c)>\operatorname{Re}(b)>0$

Then

$$F(a,b,c;1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\cdot\Gamma(c-b)}.$$

Solution Since

$$F(a,b,c;1)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}(1)^n.$$

$$F(a,b,c;1)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}\quad\text{———}\ (1)$$

take

$$\frac{(b)_n}{(c)_n}=\frac{\Gamma(b+n)}{\Gamma(b)}\cdot\frac{\Gamma(c)}{\Gamma(c+n)}.$$

$$=\frac{\Gamma(c)\,\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(b)\,\Gamma(c-b)\,\Gamma(c+n)}$$

$$=\frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}B(b+n,\,c-b)\qquad\because B(x,y)=\frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$$

$$\frac{(b)_n}{(c)_n}=\frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\cdot\int_0^1(t)^{b+n-1}\cdot(1-t)^{c-b-1}\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \sum_{n=0}^{\infty} (t)^{b-1} \cdot t^n (1-t)^{c-b-1} \cdot \frac{(a)_n}{n!} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-a} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-a-b-1} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot B(b, c-a-b)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b) \cdot \Gamma(c-a-b)}{\Gamma(\cancel{b}+c-a-\cancel{b})}$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)}\Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)} \cdot \Gamma(c-b-a)}{\Gamma(c-a)}.$$

$$= \frac{\Gamma(c)\,\Gamma(c-b-a)}{\Gamma(c-b) \cdot \Gamma(c-a)}$$ is the required

Ass

Q. $F(-n, 1-b-n, a; 1) = \dfrac{(a+b-1)_{2n}}{(a)_n\,(a+b-1)_n}$.

Sol:-

Since
$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

$$F(-n, 1-b-n, a; 1) = \frac{\Gamma(a)\Gamma(a-(-n)-(1-b-n))}{\Gamma(a-(-n))\Gamma(a-(1-b-n))}$$

$$= \frac{\Gamma(a)\Gamma(a+n-1+b+n)}{\Gamma(a+n)\Gamma(a-1+b+n)}.$$

$$= \frac{\Gamma(a)\Gamma(a+b+2n-1)}{\Gamma(a+n)\Gamma(a+b-1+n)}$$

$$= \frac{\Gamma(a)\Gamma(a+b-1+2n)}{\Gamma(a+n)\Gamma(a+b-1+n)}$$

$$= \frac{1}{(a)_n}\cdot\frac{\Gamma(a+b-1+2n)}{\Gamma(a+b-1+n)}$$

$\because (a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$.

$$= \frac{1}{(a)_n}\cdot\frac{\Gamma(a+b-1+2n)}{\Gamma(a+b-1)}\cdot\frac{\Gamma(a+b-1)}{\Gamma(a+b-1+n)}.$$

Q11 $F(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n}$.

Sol:-

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n (b)_n}{(c)_n\, n!} (1)^n$$

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n (b)_n}{(c)_n\, n!} \quad \text{--- (1)}$$

take

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}$$

$$= \frac{\Gamma(c) \cdot \Gamma(b+n) \cdot \Gamma(c-b)}{\Gamma(b) \cdot \Gamma(c-b) \cdot \Gamma(c+n)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b+n, c-b) \quad \because B(x,y) = \frac{\Gamma x \cdot \Gamma y}{\Gamma(x+y)}$$

$$= \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c-b)} \int_0^1 t^{(b+n-1)} \cdot (1-t)^{(c-b-1)} \, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \cdot t^n \, dt$$

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n}{n!} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} t^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(-n)_n t^n}{n!} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-(-n)} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b+n-1} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot B(b, c-b+n) \quad \text{By Def of Beta ftn}$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)} \cdot \Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)} \cdot \Gamma(c-b+n)}{\Gamma(\cancel{b}+c-\cancel{b}+n)} \quad \because B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(c)}{\Gamma(c-b)} \cdot \frac{\Gamma(c-b+n)}{\Gamma(c+n)}.$$

$$= \frac{(c-b)_n}{(c)_n}$$

Hence

$$(a+1-c)_n = (a+1-c)(a+1-c+1)(a+1-c+2) + \cdots + (a+1-c+(n-1))$$

$$= (a+1-c)(a+2-c)(a+3-c)\cdots(a+n-c). \quad (113)$$

Qn.

Let $F(-n, a+n, c; 1) = \dfrac{(-1)^n (a+1-c)_n}{(c)_n}$.

Sol. As

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Then

$$F(-n, a+n, c; 1) = \frac{\Gamma(c)\Gamma(c+\cancel{n}-a-\cancel{n})}{\Gamma(c+n)\cdot\Gamma(c-a-n)}.$$

$$= \frac{\Gamma(c)\cdot\Gamma(c-a)}{\Gamma(c+n)\cdot\Gamma(c-a-n)}.$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)}\cdot\frac{(c-a-1)\Gamma(c-a-1)}{\Gamma(c-a-n)}.$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)}\cdot\frac{(c-a-1)(c-a-2)\cdots(c-a-n)\cancel{\Gamma(c-a-n)}}{\cancel{\Gamma(c-a-n)}}$$

$$= \frac{1}{(c)_n}\cdot(-1)^n(a+1-c)(a+2-c)\cdots(a+n-c)$$

(114) $(a)_1 = a$, $(a)_2 = a(a+1)$.

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Q. $\frac{d}{dx} F(a,b,c;x) = \frac{ab}{c} F(a+1, b+1, c+1; x)$

Sol:
we know that,

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n$$

$$= 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{c(c+1)2!} x^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\,3!} x^3 + \cdots$$

Differentiating both sides

$$\frac{d}{dx} F(a,b,c;x) = \frac{ab}{c} + \frac{2ab(a+1)(b+1)}{c(c+1)2!} x$$

$$+ \frac{3ab(a+1)(a+2)(b+1)(b+2)}{c(c+1)(c+2)\,3!} x^2 + \cdots$$

$$= \cancel{\frac{abc}{}}\ \frac{ab}{c}\left[1 + \frac{(a+1)(b+1)}{(c+1)} x + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)2!} x^2 + \cdots\right]$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n\, n!} x^n$$

$$= \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Hence

Q11 Show That

$$\frac{d^n}{dx^n}\left[x^{a+n-1} F(a,b,c;x)\right] = (a)_n x^{a-1} F(a+n,b,c;x)$$

let

$$g(x) = x^{a+n-1} F(a,b,c;x)$$

$$= x^{a+n-1}\left[\sum_{n=0}^{n} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n\right]$$

$$= x^{a+n-1}\left[1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!}x^3 + \cdots\right]$$

$$g(x) = x^{a+n-1} + \frac{ab}{c}x^{a+n} + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^{a+n+1} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!}x^{a+n+2} + \cdots$$

$$\frac{d}{dx}g(x) = (a+n-1)x^{a+n-2} + \frac{ab}{c}(a+n)x^{a+n-1} + \frac{a(a+1)b(b+1)(a+n+1)}{c(c+1)2!}x^{a+n}$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)(a+n+2)}{\quad} x^{a+n+1} + \cdots$$

$$\frac{d^2}{dx^2}\left[g(x)\right] = \Big[(a+n-1)(a+n-2)x^{a+n-3} + \frac{ab}{c}(a+n)(a+n-1)x^{a+n-2}$$

$$+ \frac{a(a+1)b(b+1)}{c(c+1)2!}(a+n+1)(a+n)x^{a+n-1}$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}(a+n+2)(a+n+1)x^{a+n} + \cdots\Big]$$

Generally:

$$\frac{d^n}{dx^n}\left[g(x)\right] = (a+n-1)(a+n-2)(a+n-3)\cdots(a+n-(n-1))a\,x^{a-1}$$

$$+ \frac{ab}{c}(a+n)(a+n-1)(a+n-2)\cdots(a+1)\,x^{a}$$

$$+ \frac{a(a+1)b(b+1)}{c(c+1)2!}(a+n+1)(a+n)(a+n-1)(a+n-2)\cdots(a+2)\,x^{a+1} + \cdots\Big]$$

$$\frac{d^n}{dx^n}\left[g(x)\right] = a(a+1)(a+2)\cdots(a+(n-1)\,x^{a-1}$$

$$+ a(a+1)(a+2)\cdots(a+n-1)(a+n)\frac{b}{c}x^{a}$$

$$+ a(a+1)(a+2)(a+3)\cdots(a+n-1)(a+n)(a+n+1)\frac{b(b+1)}{c(c+1)2!}x^{a+1} + \cdots$$

$$\frac{d^n[g(x)]}{dx^n} = (a)_n x^{a-1} + (a)_n \frac{(a+n)}{c} x^{a} + (a)_n \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)\,2!} x^{a+1} \cdots$$

$$= (a)_n x^{a-1}\left[1 + \frac{(a+n)}{c} b x^{1} + \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)\,2!} x^{2} + \cdots\right]$$

$$= (a)_n x^{a-1}\left[\sum_{n=0}^{\infty} \frac{(a+n)_n (b)_n}{(c)_n \, n!} x^n\right]$$

$$= (a)_n x^{a-1} F(a, b, c; x).$$

Hence

$$\frac{d^n}{dx^n}\left[x^{a+n-1} F(a, b, c; x)\right] = (a)_n x^{a-1} F(a+n, b, c; x)$$

(118) Results Prove That

i) $B(x+1, y) = \frac{x}{x+y} B(x, y)$.

ii) $B(x, y+1) = \frac{y}{x+y} B(x, y)$.

Proof:

$$B(x+1, y) = \frac{\Gamma(x+1)\Gamma y}{\Gamma(x+y+1)}.$$

$$= \frac{x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)} \qquad \because \Gamma(z+1) = z\Gamma(z)$$

$$= \frac{x}{x+y} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$$B(x+1, y) = \frac{x}{x+y} B(x, y)$$

ii) $B(x, y+1) = \frac{y}{x+y} B(x, y)$.

L.H.S

$$B(x, y+1) = \frac{\Gamma(x)\,\Gamma(y+1)}{\Gamma(x+y+1)}.$$

$$B(x, y+1) = \frac{y}{(x+y)} \cdot \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$B(x, y+1) = \frac{y}{x+y} B(x, y).$$

Imp Q. Prove That.

$$\Gamma(z) = 2 \int_0^\infty e^{-t^2} \cdot t^{2z-1} \, dt.$$

Proof

$$\text{Since } \Gamma(z) = \int_0^\infty e^{-t} \cdot t^{z-1} \, dt. \quad \text{---} (1)$$

Put $t = u^2$ $\Rightarrow$ $dt = 2u\,du$.

when $t \to 0$ Then $u \to 0$
when $t \to \infty$ Then $u \to \infty$

(1) $\Rightarrow$

$$\Gamma(z) = \int_0^\infty e^{-u^2} (u^2)^{z-1} (2u\,du).$$

$$= 2 \int_0^\infty e^{-u^2} u^{2z-2} \cdot u\,du.$$

$$= 2 \int_0^\infty e^{-u^2} u^{2z-1} \, du$$

Theorem:-

$$\int_0^{\pi/2} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, d\theta = \frac{\Gamma(x)\,\Gamma(y)}{2\,\Gamma(x+y)}.$$

Proof:- Consider

$$I = \iint_R e^{-t^2} \cdot e^{-u^2} \cdot t^{2x-1}\, u^{2y-1}\, dt\, du$$

where R is the region of the tu-plane consisting of first quardrant only.

R: $0 \le t < \infty$, $0 \le u < \infty$

$$I = \int_0^{\infty}\int_0^{\infty} e^{-t^2} \cdot e^{-u^2} \cdot t^{2x-1}\, u^{2y-1}\, dt\, du.$$

$$I = \int_0^{\infty} \left(\int_0^{\infty} e^{-t^2} \cdot t^{2x-1}\, dt \right) e^{-u^2} u^{2y-1}\, du.$$

$$I = \int_0^{\infty} \frac{1}{2}\Gamma(x)\, e^{-u^2} u^{2y-1}\, du \quad \because \; \Gamma(x) = 2\int_0^{\infty} e^{-t^2} \cdot t^{2x-1}\, dt$$

$$= \frac{1}{2}\Gamma(x) \cdot \frac{1}{2}\Gamma(y) \quad \text{--- (1)}$$

Now we solve I in polar coordinates

put
$$t = r\cos\theta, \quad u = r\sin\theta.$$

$$dA = r\,dr\,d\theta.$$

where
$$R: 0 \le r < \infty \text{ and } 0 \le \theta < \pi/2.$$

$$I = \int_0^\infty \int_0^{\pi/2} e^{-r^2\cos^2\theta} \cdot e^{-r^2\sin^2\theta} \cdot (r\cos\theta)^{2x-1} (r\sin\theta)^{2y-1} r\,dr\,d\theta$$

$$I = \int_0^\infty \int_0^{\pi/2} e^{-r^2(\cos^2\theta+\sin^2\theta)} \cdot r^{2(x+y)-1} \cos^{2x-1}\theta \sin^{2y-1}\theta\, dr\, d\theta$$

$$I = \int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(x+y)-1} \cos^{2x-1}\theta \sin^{2y-1}\theta\, dr\, d\theta$$

$$I = \int_0^{\pi/2} \frac{1}{2}\Gamma(x+y) \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta. \quad \because 2\int_0^\infty r^{2(x+y)-1} e^{-r^2} dr = \Gamma(x+y).$$

$$I = \frac{1}{2}\Gamma(x+y) \int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \text{ --- } (2)$$

Comparing eqn (1) & (2) we get

$$\frac{1}{2}\Gamma(x+y) \int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta = \frac{1}{2}\cdot\frac{1}{2}\Gamma(x)\Gamma(y)$$

Theorem.

Imp Q. $\Gamma(1/2) = \sqrt{\pi}$

Proof. Since $\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta$

$$= \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$$

put $x = y = 1/2$.

Then $\int_0^{\pi/2} \cos^0\theta \cdot \sin^0\theta \, d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

$\Rightarrow \int_0^{\pi/2} (1)(1)\, d\theta = \frac{\Gamma(1/2)\Gamma(1/2)}{2\Gamma(1)}$ $\because x^0 = 1$

$\Rightarrow \int_0^{\pi/2} d\theta = \frac{(\Gamma(1/2))^2}{2}$

$\Rightarrow \theta \Big|_0^{\pi/2} = \frac{(\Gamma(1/2))^2}{2}$

$\Rightarrow \frac{\pi}{\cancel{2}} = \frac{(\Gamma(1/2))^2}{\cancel{2}}$

$\pi = (\Gamma(1/2))^2$

I. from Stat Notes

Assignment

Prove That

$$\Gamma(x)\cdot\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Proof: Since we know That

$$\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta = \frac{\Gamma(x)\,\Gamma(y)}{2\,\Gamma(x+y)}$$

put $y = 1-x$.

$$\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{2(1-x)-1}\theta \, d\theta = \frac{\Gamma(x)\cdot\Gamma(1-x)}{2\,\Gamma(x+(1-x))}.$$

$$\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{1-2x}\theta \, d\theta = \frac{\Gamma(x)\cdot\Gamma(1-x)}{2\,\Gamma(1)}$$

$$\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{1-2x}\theta \, d\theta = \frac{\Gamma(x)\cdot\Gamma(1-x)}{2}$$

$$\Rightarrow 2\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{1-2x}\theta \, d\theta = \Gamma(x)\,\Gamma(1-x)$$

Prove That $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$

Proof

$$\Gamma(x) = 2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt.$$

Put $x = 1/2$

$$\Gamma(1/2) = 2\int_0^\infty e^{-t^2} \cdot t^{2(1/2)-1}\, dt.$$

$$\Gamma(1/2) = 2\int_0^\infty e^{-t^2} dt.$$

$$\sqrt{\pi} = 2\int_0^\infty e^{-t^2} dt.$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-t^2} dt$$

Hence $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2.$

Prove That.

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Proof Since

$$B(x, y) = \int_0^1 t^{x-1} \cdot (1-t)^{y-1}\, dt.$$

$$B(x,y) = \int_{\pi/2}^{0} (\cos^2\theta)^{x-1} (1-\cos^2 x)^{y-1} (-2\cos\theta\sin\theta)\,d\theta.$$

$$= 2\int_0^{\pi/2} \cos^{2x-2}\theta \cdot \sin^{2y-2}\theta \cdot \cos\theta \sin\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, d\theta.$$

$$= \cancel{2} \cdot \frac{\Gamma(x)\cdot\Gamma(y)}{\cancel{2}\,\Gamma(x+y)}$$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

— x — x — x — x —

imp 4. Express $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$ in term of Gamma ftn or Beta functi and simplify if possible.

Proof. $\int_0^{\pi/2} \sqrt{\quad}\ d\theta = \int_0^{\pi/2} (\sin\theta)^{1/2} d\theta$

$$= \int_0^{\pi/2} \cos^{-1/2}\theta \sin^{1/2}\theta \, d\theta \quad \text{—} \quad (1)$$

Since $\displaystyle\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$ — (2)

put $2x-1 = -1/2 \Rightarrow 2x = 1/2 \Rightarrow x = 1/4$
and
$2y-1 = 1/2 \Rightarrow 2y = 3/2 \Rightarrow y = 3/4$

putting value of x & y in eqn (2)

$$\int_0^{\pi/2} \cos^{-1/2}\theta \sin^{1/2}\theta \, d\theta = \frac{\Gamma(1/4)\cdot\Gamma(3/4)}{2\Gamma(1/4+3/4)}$$

$$= \frac{\Gamma(1/4)\cdot\Gamma(1-1/4)}{2\Gamma(1)}$$

$$= \frac{\Gamma(1/4)\cdot\Gamma(1-1/4)}{2}$$

$$= \frac{\dfrac{\pi}{\sin\pi(1/4)}}{2} \quad \because \Gamma x\cdot\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

Bessal Function:-

Notation: $J_n(x)$

It is defined as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n}\, m!\, \Gamma(m+n+1)}$$

is called Bessal function of index 'n' order 'n' and we obtain the Bessal function by solving Bessal differential equation of order '2',

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

where 'n' is non-negative integer.

* Show That

$$J_{-n}(x) = (-1)^n J_n(x).$$

Proof:- we know

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n}\, m!\, \Gamma(m+n+1)}$$

Put $n = -n$.

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-n}}{2^{2m-n}\, m!\, \Gamma(m-n+1)}$$

$$\boxed{\begin{aligned}\Gamma(z+1) &= (z)\Gamma(z)\\ \Gamma(z+2) &= (z+1)z\Gamma z.\end{aligned}}$$

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for This value

$$J_{-n}(x) = 0 \qquad \text{so } m \geq n.$$

~~Since~~

$$\Rightarrow J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m (x)^{2m-n}}{2^{2m-n}(m!)\Gamma(m-n+1)}$$

Let

$$k = m-n \quad \Rightarrow m = k+n.$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(x)^{2(k+n)-n}}{2^{2(k+n)-n}\cdot(k+n)!\,\Gamma(k+\cancel{n}-\cancel{n}+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(x)^{2k+2n-n}}{2^{2k+2n-n}\cdot(k+n)!\,\Gamma(k+1)}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^n\cdot(x)^{2k+n}}{2^{2k+n}(k+n)(k+(n-1))\cdots(k+1)k!\,\Gamma(k+1)}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+n}}{2^{2k+n}k!\,\Gamma(k+n+1)}$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sqrt{z} = (z-1)(z-1)$
$= (z-1)(z-)$

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Show That $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$

Sol: As
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+n}}{2^{2k+n} . k! \, \Gamma(k+n+1)}$$

Put $n = 1/2$.

$$J_{1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1/2}}{2^{2k+1/2} . k! \, \Gamma(k+1/2+1)}$$

$$J_{1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1/2}}{2^{2k+1/2} . k! \, (\Gamma(k+3/2)}.$$

$$= \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)} + \frac{(-1)^1 (x)^{5/2}}{2^{5/2} 1! \, \Gamma(5/2)} + \frac{(-1)^2 (x)^{9/2}}{2^{9/2} . 2! \, \Gamma(9/2)} + \cdots.$$

$$J_{1/2}(x) = \frac{\left(\frac{x}{2}\right)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2! \, \Gamma(9/2)}$$

$$- \frac{(x/2)^{11/2}}{3! \, \Gamma(9/2)} + \cdots.$$

$$J(x) = \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\frac{1}{2}\Gamma(1/2)} - \frac{(x/2)^2}{\frac{3}{2}.\frac{1}{2}\Gamma(1/2)} + \frac{(x/2)^4}{2! \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}} \right.$$

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$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})}\left[1 - \frac{x^2}{2^2 \cdot 3/2} + \frac{x^4}{2^4 \cdot 2! \cdot \frac{5}{2} \cdot \frac{3}{2}} - \frac{x^6}{2^6 \cdot 3! \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} + \cdots\right]$$

$$= \frac{x^{1/2}}{\frac{1}{2} \cdot 2^{1/2}\sqrt{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right]$$

$$= \frac{2^{1/2} \cdot x^{1/2}}{x\sqrt{\pi}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sqrt{\frac{2}{x\pi}} \sin x.$$

Hence

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

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Show That
$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x.$$

Sol:- As by the definition Bessal function
$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (x)^{2m+n}}{2^{2m+n} \cdot m!\,\Gamma(m+n+1)}$$

Put
$n = -1/2$.
$$J_{(-1/2)}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-1/2}}{2^{2m-1/2} \cdot m!\,\Gamma(m-1/2+1)}.$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-1/2}}{2^{2m-1/2}\, m!\,\Gamma(m+1/2)}.$$

$$= \frac{x^{-1/2}}{2^{-1/2} \cdot \Gamma(1/2)} + \frac{(-1)^1 (x)^{3/2}}{2^{3/2} (1!)\,\Gamma(3/2)} + \frac{(-1)^2 (x)^{7/2}}{2^{7/2} (2!)\,\Gamma(5/2)}$$
$$+ \frac{(-1)^3 \cdot (x)^{11/2}}{2^{11/2} \cdot (3!)\,\Gamma(7/2)} + \cdots$$

$$= \frac{x^{-1/2}}{2^{-1/2}} \left[\frac{1}{\Gamma(1/2)} - \frac{x^2}{2^2 \cdot \frac{1}{2}\Gamma(1/2)} + \frac{x^4}{2^4 \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \Gamma(1/2)} - \cdots \right]$$

$$= \sqrt{\frac{2}{x}} \cdot \frac{1}{\sqrt{\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x.$$

Hence $J_{(-1/2)}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$

— x — x — x — x — x —

V-Imp

* Differential Recurence Relation of Bessal function:-

As $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)}$ — (1)

Multiplying (x^n) on both sides

$$x^n J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n} m! (\Gamma(m+n+1))}$$

$$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+2n}}{2^{2m+n} m! \cdot \Gamma(m+n+1)}$$

On Differentiating

Show $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ / $\frac{d}{dx} x^5 J_5(x) = x^5 J_4(x)$

$= x^n J_{n-1}(x)$

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$$x^n J_n'(x) + n x^{n-1} J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+2n-1}}{2^{2m+n-1} \cdot m! \, \Gamma(m+n)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n-1} \cdot (x)^n}{2^{2m+n-1} \cdot m! \, \Gamma(m+n)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+\underline{n-1}}}{2^{2m+n-1} \cdot m! \, \Gamma(m+n-1+1)}$$

$$= x^n J_{n-1}(x)$$

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x).$$

$$J_n'(x) = J_{n-1}(x) - n x^{-1} J_n(x) \text{ --- } (2)$$

Now Multiplying x^{-n} on both sides of (1)

$$x^{-n} J_n(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n}}{2^{2m+n} \cdot m! \, \Gamma(m+n+1)}$$

on Differentiating

$$\therefore \; x^{-n} J_n'(x) - n x^{-n-1} J_n(x)$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m \cdot 2m \cdot x^{2m-1}}{2^{2m+n} \cdot m! \, \Gamma(m+n+1)}$$

put

$$m-1 = k \quad \Rightarrow m = k+1$$

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot 2(k+1) \cdot x^{2k+1}}{2^{2k+n+2} \cdot (k+1)! \, \Gamma(k+n+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (-1)^1 \cdot \cancel{(k+1)} \, x^{2k+1}}{2^{2k+n+1} \cdot \cancel{(k+1)} \, k! \, \Gamma(k+n+2)}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+n+1} \cdot x^{-n}}{2^{2k+n+1} \cdot k! \, \Gamma(k+n+1+1)} \checkmark$$

$$= -x^{-n} \sum \frac{(-1)^k \cdot x^{2k+n+1}}{\ldots}$$

i) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$ when multiplying by x^n & differentiate.

ii) $x J_n'(x) = n J_n(x) - x J_n(x)$ when multiply by x^{-n} & diff

iii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

iv) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$$\Rightarrow + J_n'(x) - n x^{-1} J_n(x) = - J_{n+1}(x)$$

$$J_n'(x) = n x^{-1} J_n(x) - J_{n+1}(x) \quad (3)$$

adding (2) & (3) we have.

$$\boxed{2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)}$$

This relation is called differential recurence relation.

now subtracting (3) from (2)

$$0 = J_{n-1}(x) - 2n x^{-1} J_n(x) + J_{n+1}(x)$$

$$2n x^{-1} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\boxed{J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]}$$

This relation is called pure recurrence or non-differential

V.Imp Spherical Bessal Function:

General Representation of Spherical Bessal function is

$$J_{n+1/2}(z) = A(z)\sin z + B(z)\cos z$$

where $A(z)$ and $B(z)$ are polynomial in z.

* Properties of Bessal function:-

1) $$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{\sin z - z\cos z}{z}\right]$$

Sol:- we know

$$J_n(z) = \frac{z}{2n}\left[J_{n-1}(z) + J_{n+1}(z)\right]$$

Put $n = 1/2$.

$$J_{1/2}(z) = \frac{z}{2(1/2)}\left[J_{1/2-1}(z) + J_{1/2+1}(z)\right]$$

$$J_{1/2}(z) = z\left[J_{-1/2}(z) + J_{3/2}(z)\right]$$

$$= z\,J_{-1/2}(z) + z\,J_{3/2}(z)$$

$$J_{3/2}(z) = z^{-1}\left[J_{1/2}(z) - z\,J_{-1/2}(z)\right]$$

$$= z^{-1}\left[\sqrt{\frac{2}{\pi z}}\sin z - z\sqrt{\frac{2}{\pi z}}\cos z\right]$$

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{\sin z - z\cos z}{z}\right]$$

<u>**Assignment**</u>

$$J_{-3/2}\,,\quad J_{5/2}\,,\quad J_{-5/2}$$

(9). Show that $\int x^n J_{n-1}(x)\,dx$

Sol. $\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + C$

$$= \int \frac{d}{dx}\left(x^n J_n(x)\right)dx$$

$$= x^n J_n(x) + C$$

$\frac{d}{dx}(x^{-n} J_n(x))$

b) $\int x^{-n} J_{n+1}(x)\, dx.$

$= -\int \frac{d}{dx}\{x^{-n} J_n(x)\}\, dx.$

$= -x^{-n} J_n + C.$

e.g. $\int x^{-3} J_4(x)\, dx.$

$= -x^{-3} J_3(x) + C.$

---

* Find $J_{-3/2}(x)$.

we know That

$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x) \right]$

put $n = -1/2$.

$J_{-1/2}(x) = \frac{x}{2(-1/2)}\left[J_{-1/2-1}(x) + J_{-1/2+1}(x) \right]$?

$J_{-1/2}(x) = -x\left[J_{-3/2}(x) + J_{1/2}(x) \right]$

$J_{-1/2}(x) = -x J_{-3/2}(x) - x J_{1/2}(x).$

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$$J_{-3/2}(x) = \frac{1}{x}\left[J_{-\frac{1}{2}}(x) - x\, J_{1/2}(x)\right]$$

$$J_{-3/2}(x) = \frac{1}{x}\left[\sqrt{\frac{2}{\pi x}}\cos x - x\left[\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x - x\cos x}{x}\right)\right]\right]$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cdot\cos x - \sqrt{\frac{2}{\pi x}}\sin x + x\cos x\cdot\sqrt{\frac{2}{\pi x}}$$

$$= \left(\frac{1}{x} + x\right)\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x.$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(x\cos x + \frac{1}{x}\cos x - \sin x\right)$$

Find $J_{5/2}(x)$. ?

we know That

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Put $n = 3/2$.

$$J_{3/2}(x) = \frac{x}{\cancel{2}(3/\cancel{2})}\left[J_{1/2}(x) + J_{5/2}(x)\right]$$

$$J_{3/2}(x) = \frac{x}{3}J_{1/2}(x) + \frac{x}{3}J_{5/2}(x).$$

$$J_{5/2}(x) = \frac{3}{x}\cdot\left[\sqrt{2/\pi x}\left(\frac{\sin x - x\cos x}{x}\right)\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3(\sin x - x\cos x)}{x^2} - \sin x\right]$$

$$J_{5/2}(x) = \sqrt{2/\pi x}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$$

---

* ~~Find~~ $J_{-5/2}(x)$. ?

we know that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

put $n = -3/2$.

$$J_{-3/2}(x) = \frac{x}{\cancel{2}(-3/\cancel{2})}\left[J_{-5/2}(x) + J_{-1/2}(x)\right]$$

$$J_{-3/2}(x) = -\frac{x}{3}\left[J_{-5/2}(x) + J_{-1/2}(x)\right]$$

$$J_{-5/2}(x) = \frac{3}{x}\left[-\frac{x}{3}J_{-1/2} - J_{-3/2}(x)\right]$$

$$J_{-5/2}(x) = \left[-J_{-1/2} - \frac{3}{x}J_{-3/2}(x)\right]$$

* i) $\int z^n J_{n-1}(z)\,dz = z^n J_n(z)$ — If power is greater then order of bessel function

ii) $\int z^{-n} J_{n+1}(z)\,dz = -z^{-n} J_n(z)$

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Qu Evaluate $\int x^3 J_3(x)\,dx$.

$$= \int x^5 \cdot x^{-2} J_3(x)\,dx$$

$$= x^5 \cdot \int x^{-2} J_3(x)\,dx - \int (5x^4)\left(-x^{-2} J_2(x)\right)dx$$

$$= x^5 \cdot \left(-x^{-2} J_2(x)\right) + 5\int x^2 J_2(x)\,dx$$

$$= -x^3 J_2(x) + 5\int x^{-1} \cdot x^3 J_2(x)\,dx$$

$$= -x^3 J_2(x) + 5\left[x^3 \int x^{-1} J_2(x)\,dx - \int 3x^2 \cdot \left(-x^{-1} J_1(x)\right)dx\right]$$

$$= -x^3 J_2(x) + 5\left[-x^3 \cdot x^{-1} J_1(x) + 3\int x^2 x^{-1} J_1(x)\,dx\right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15\int x J_1(x)\,dx$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15\left[\int x^1 \cdot x^{-0} J_1(x)\,dx\right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15\left[x \int x^{-0} J_1(x) - \int \left(-x^{-0} J_0(x)\right)\right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15x\left(-x^0 J_0(x)\right) + 15\left(x^{-0} J_0(x)\right)$$

Q11. a) $\int x^4 J_1(x)\,dx.$

b) $\int x^4 J_4(x)\,dx.$

c) $\int x^3 J_2(x)\,dx.$

i) $\frac{d}{dx} x^n J_n(x) = x^n J_{n-1}(x).$

ii) $\int x^n J_{n-1}(x) = x^n J_n(x)$

iii) $\int x^{-n} J_{n+1}\,dx = -x^{-n} J_n(x).$

Sol:

a) $\int x^4 J_1(x)\,dx$

$= \int x^2 x^2 J_1(x)\,dx$

$= x^2 \int x^2 J_1(x)\,dx - \int 2x \cdot x^2 J_2(x)\,dx$

$= x^2 \cdot x^2 J_2(x) - 2\int x^3 J_2(x)\,dx.$

$x^4 J_2(x) - 2x^3 J_3(x)$

$\cancel{= x^4 J_2(x) - 2\left[\int x \cdot x^2 J_2(x)\,dx\right]}$

$\cancel{= x^4 J_2(x) - 2\left[x \cdot x^2 J_2(x) - \int x^2 J_2(x)\,dx\right]}$

$\cancel{= x^4 J_2(x) - 2\left[x^3 J_2(x) - \int x^3 x^{-1} J_2(x)\,dx\right]}$

$\cancel{= x^4 J_2(x) - 2(x^3 J_2(x)) + 2\left[x^3(-x^{-1} J_1(x)) + \int 3x^2 x^{-1} J_1(x)\,dx\right]}$

$= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6\int x J_1(x)\,dx.$

$= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6\left[\int x \cdot x^0 J_1(x)\,dx\right]$

(244) Solve

b) $\int x^4 J_4(x)\,dx.$

$= \int x^7 \cdot x^{-3} J_4(x)\,dx.$

$= x^7 \left(-x^{-3} J_3(x)\right) + 7\int x^6 \cdot x^{-3} J_3(x)\,dx.$

$= -x^4 J_3(x) + 7\int x^3 J_3(x)\,dx.$

$= -x^4 J_3(x) + 7\int x^5 x^{-2} J_3(x)\,dx.$

$= -x^4 J_3(x) + 7\left[x^5\left(-x^{-2} J_2(x)\right) + 5\int x^4 \cdot x^{-2} J_2(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) + 35\int x^2 J_2(x)\,dx$

$= -x^4 J_3(x) - 7x^3 J_2(x) + 35\left[\int x^3 x^{-1} J_2(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) + 35\left[-x^3 \cdot x^{-1} J_1(x) + 3\int x^2 x^{-1} J_1(x)\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105\int x J_1(x)\,dx$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105\left[\int x \cdot x^{0} J_1(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105\left[-x \cdot x^{0} J_0(x) + \int J_0(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) - 105 x J_0(x) + \int J_0(x)\,dx$

i) $\int x^n J_{n-1}(x)\,dx = x^n J_n(x)$ (145)

$\int$ ve ii) $\int x^{-n} J_{n+1}\,dx = -x^{-n} J_n$

c) $\int x^3 J_2(x)\,dx$.

$= x^3 J_3(x)$.

a) $\int x^4 J_1(x)\,dx$.

$= \int x^2 \cdot x^2 J_1(x)\,dx$.

$= x^2 \int x^2 J_1(x) - \int 2x \cdot \int x^2 J_1(x)\,dx$.

$= x^2 \int x^2 J_1(x) - 2\int x \cdot x^2 J_2(x)\,dx$.

$= x^4 J_2(x) - 2\int x^3 J_2(x)$

b) x)

$= x^4 J_2(x) - 2x^3 J_3(x)$

(146)

These notes are available on at

http://www.mathcity.org/msc/notes

$\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1\cdot \Gamma(1) = 2\cdot 1\cdot$

imp (147)

Q11 Show That

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

Sol:

we know

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+n}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}$$

Put $n = 1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\cdot x^{2k+1}}{2^{2k+1}\cdot k!\cdot \Gamma(k+2)}$$

$$= \frac{x}{2\cdot 0!\,\Gamma(2)} + \frac{(-1)^1 x^3}{2^3\cdot 1!\,\Gamma(3)} + \frac{(-1)^2 x^5}{2^5\cdot 2!\,\Gamma(4)}$$

$$+ \frac{(-1)^3 x^7}{2^7\cdot 3!\cdot \Gamma(5)} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\cdot 2} + \frac{x^5}{2^6\cdot 3\cdot 2} - \frac{x^7}{2^7\, 3!\cdot 4\cdot 3\cdot 2\cdot 1} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

which is required

$F_n \longrightarrow$ [illegible]

★ Hypergeometric function:-

we define the general hypergeometric function with 'm' upper perameter and 'n' Lower parameter written in

$$F_n (a_1, a_2, a_3, ---, a_m, b_1, b_2, b_3, --, b_n ; z)$$

by

$$_mF_n (a_1, a_2, a_3, --, a_m ; b_1, b_2, b_3, --, b_n ; z)$$

$$_mF_n = \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r (a_3)_r --- (a_m)_r}{(b_1)_r (b_2)_r (b_3)_r --- (b_n)_r} \frac{z^n}{n!}$$

The notation $_mF_n \left[\begin{matrix} a_1, a_2, a_3, ----, a_m ; x \\ b_1, b_2, b_3, ----, b_n ; \end{matrix} \right]$

is also often used.

Now we discussed Two cases

If

$$m = n = 1$$

Then The general hypergeometric fn will be called the "confluent" hypergeometric function. Thes Confluent Hyper geometric function.

$_1F_1 (a ; b ; x)$ is often denoted by

$M (a ; b ; x)$.

Some special cases of $_mF_n$.

i) when there are no upper and low upper parameters.

$${}_0F_0(\,;\,;\pm x) = e^{\pm x}$$

As

$${}_0F_0(\,;\,;x) = \sum_{r=0}^{\infty} \frac{(x)^r}{r!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

$$= e^{x} \quad \text{—— (1)}$$

Also

$${}_0F_0(\,;\,;-x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!}$$

$$= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!}$$

$$= e^{-x} \quad \text{—— (2)}$$

Combining (1) & (2)

$${}_0F_0(\,;\,;\pm x) = e^{\pm x}$$

$(1/2)_2 = 1/2(1/2+1)$ $\quad (a)_n = a(a+1)(a+2)\cdots(a+(n-1))$

$= 3/6$

$= 3/2(3/2+1)$ $\quad (1/2)_1 = 1/2+0 = 1/2$

(150)

2) $_1F_0(a;\ ;x) = (1-x)^{-a}$

Proof

$$_1F_0(a;\ ;x) = \sum_{r=0}^{\infty} \frac{(a)_r}{r!} (x)^r$$

$$= (1-x)^{-a}$$

3) $x \cdot {}_2F_1(1/2, 1/2; 3/2; x)$

$$= x + \frac{x^2}{2 \cdot 3} + \frac{3x^3}{3 \cdot 5}$$

4) $x \cdot {}_2F_1(1,1;2;-x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

$= \log(1+x)$.

3) L.H.S.

Sol:

$x \cdot {}_2F_1(1/2, 1/2; 3/2; x)$.

$$= x \cdot \left[\sum_{r=0}^{\infty} \frac{(1/2)_r (1/2)_r}{(3/2)_r} \frac{x^r}{r!} \right]$$

$$= x \left[\frac{(1/2)_0 (1/2)_0}{(3/2)_0\, 0!} x^0 + \frac{(1/2)_1 (1/2)_1}{(3/2)_1\, 1!} x + \frac{(1/2)_2 (1/2)_2}{(3/2)_2\, 2!} x^2 + \cdots \right]$$

$$= x\left[1 + \frac{1}{6} x + \cancel{\frac{3/4 \cdot 3/4\, x^2}{9/10 \cdot 2}} \right.$$

$$(1)_2 = (1+0)(1+1) = 2$$
$$(2)_2 = (2)(2+1) = 6$$

(151)

Q11 $\quad x\,{}_2F_1(1,1;2;-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

L.H.S

$$x\,{}_2F_1(1,1;2;-x)$$
$$= x\sum_{r=0}^{\infty} \frac{(1)_r(1)_r}{(2)_r\, r!}(-x)^r$$

$$= x\left[1 + \frac{(1)_1(1)_1}{(2)_1 1!}(-x)^1 + \frac{(1)_2(1)_2}{(2)_2 2!}(-x)^2 + \dots\right]$$

$$= x\left[1 + \frac{1}{2}(-x)^1 + \frac{\cancel{2}\cdot\cancel{2}}{\cancel{6}\cdot\cancel{2!}}(-x)^2 + \dots\right]$$

$$= x\left[1 - \frac{1}{2}x + \frac{\cancel{4}\,1}{\cancel{12}\,3}x^2 + \dots\right]$$

$$= x\left[1 - \frac{1}{2!}x + \frac{x^2}{3} - \dots\right]$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$(1/2)_2 = \tfrac{1}{2}(1/2+1) = \tfrac{1}{2}\cdot\tfrac{3}{2} = 3/4$$
$$(3/2)_2 = 3/2(3/2+1) = 3/2\cdot 5/2 = 15/4$$

(152).

(5) $x\, {}_2F_1\left(1/2, 1; 3/2; x^2\right)$.

$$= x \sum_{r=0}^{\infty} \frac{(1/2)_r (1)_r}{(3/2)_r\, r!} (x^2)^r.$$

$$= x\left[1 + \frac{(1/2)_1 (1)_1}{(3/2)_1\, 1!}(x^2)^1 + \frac{(1/2)_2 (1)_2}{(3/2)_2\, 2!}(x^2)^2 + \cdots\right]$$

$(1+0)(1+1)$

$$= x\left[1 + \frac{1/2}{3/2} x^2 + \frac{(3/4)(2)}{15/4\cdot 2!} x^4 + \cdots\right]$$

$$= x\left(1 + \frac{1}{3}x^2 + \frac{1}{5} x^4 + \cdots\right)$$

$$= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

5) $x\,{}_2F_1(1/2, 1; 3/2; x^2)$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$= \tanh^{-1} x.$$

6) $x\,{}_2F_1(1/2, 1; 3/2; -x^2) = \tan^{-1} x.$

7) $x\,{}_2F_1(1/2, 1/2; 3/2; x^2) = \sin^{-1} x$

8) $x\,{}_0F_1(\;; 3/2; +x^2/4) = \sin x.$

9) $x\,{}_0F_1(\;; 3/2; -x^2/4) = \cos x.$

Objective
Problem [redacted]

$$J_0'(x) = -J_1(x).$$

Sol: we know that

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x).$$

Put $n = 0$

$$J_0'(x) = -J_1(x) + \frac{0}{x} J_n(x)$$

(154)

Show that

$$\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x).$$

Sol: we know that

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} \cdot k! \, \Gamma(k+n+1)}.$$

Put $n=1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{2^{2k+1} \cdot k! \, \Gamma(k+2)}.$$

Multiplying by x.

$$x J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+2}}{2^{2k+1} \cdot k! \, \Gamma(k+2)}.$$

Differentiating on both sides

$$\frac{d}{dx} x J_1(x) = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+2}}{2^{2k+1} \cdot k! \, \Gamma(k+2)}.$$

$$= \sum^{\infty} (-1)^k \cdot 2(k+1) x^{2k+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2^{2n} \cdot n! \, \Gamma(n+1)}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{2^{2n} \cdot n! \, \Gamma(n+1)}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+0}}{2^{2n+0} \, n! \, \Gamma(n+1+0)}$$

$$= x \, J_0(x)$$

Hence

$$\frac{d}{dx} x J_1(x) = x J_0(x)$$

Q: Find $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Sol: we know that

$$2n \, J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right) \quad (A)$$

put $n = 2$

$$2(2) J_2(x) = x\left(J_1(x) + J_3(x)\right)$$

Again putting $n=1$ in (A)

$$2(1)\,J_1(x) = x\left(J_0(x) + J_2(x)\right)$$

$$x J_2(x) = 2J_1(x) - x J_0(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - \frac{x}{x} J_0(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

using this value in equation (1)

$$x\, J_3(x) = 4\left(\frac{2}{x} J_1(x) - J_0(x)\right) - x J_1(x)$$

$$J_3(x) = \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x).$$

Q11 Show that

$$J_n''(x) = \frac{1}{2}\left(J_{n-1}(x) - 2J_n(x) + J_{n+2}(x)\right)$$

we know that

$$J_n'(x) = n x^{-1} J_n(x) - J_{n+1}(x)$$

differentiating we have.

Q11

Show $J_n''(x) = \frac{1}{2}\left[J_{n-1}(x) - 2J_n(x) + J_{n+2}(x)\right]$

As we know that

$$2J_n'(x) = J_{n-1}(x) + J_{n+1}(x).$$

Differentiating we have

$$2J_n''(x) = J_{n-1}'(x) + J_{n+1}'(x).$$

$$2J_n''(x) = \frac{1}{2}\left[J_{n-2}(x) + J_n(x)\right] + \frac{1}{2}\left[J_n(x) - J_{n+2}\right]$$

$$= \frac{1}{2}\left[J_{n-2} + 2J_n(x) - J_{n+2}\right]$$

$$J_n''(x) = \frac{1}{4}\left[J_{n-2} + 2J_n(x) - J_{n+2}\right] \quad (A)$$

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{3!}x^2\right.$$

$$\left. - \frac{(n-2)(n-4)(n+1)(n+3)(n+5)}{4!}x^4 + \cdots\right)$$

$$y_2(x) = c_1\left(x - n - \right.$$

* Legender Polynomial:: (159)

The equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

or

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$ is called

The Legender equation.

By the method of series Solution (Forbenion Solution).

we get two independ solution of above equation of the form

$$y_1(x) = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \frac{(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

&

$$y_2(x) = C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots\right]$$

If n is even Then $y_1(x)$ (1^{st} eqn) get terminate and we get $y(n)$ as a polynomial of degree n. while 2^{nd} series is an infinit series.

Now If n is odd Then 2^{nd} eqn get terminate and we get $y(n)$ a polynomial of degree n and 1^{st} series an infinite series. These polynomial are known as Legender Polynomial.

Initial Condition for Legendre polynomial

$$P_n(1) = 1. \quad \forall\, n \in N.$$

If $n = 2$

$$P_2(x) = C_0\left[1 - \frac{2(2+1)}{2!}x^2\right]$$

$$= C_0\left[1 - \frac{2(3)}{2}x^2\right]$$

$$P_2(x) = C_0\left[1 - 3x^2\right]$$

As $P_2(1) = 1$ Therefore

$$1 = C_0\left[1 - 3(1)^2\right]$$

$$1 = C_0(1-3)$$

$$1 = -2C_0 \implies C_0 = -1/2.$$

So required Legendre polynomial is

$$P_2(x) = -\frac{1}{2}\left[1 - 3x^2\right]$$

$$P_2(x) = -\frac{1}{2}\left[1 - 3x^2\right]$$

An. for $n = 3$

$$P_3(x) = c_1\left[x - \frac{(3-1)}{3!}(3+2)x^3\right]$$

$$P_3(x) = c_1\left[x - \frac{2(5)}{3!}x^3\right]$$

$$P_3(x) = c_1\left[x - \frac{5}{3}x^3\right]$$

$$P_3(1) = c_1\left[1 - \frac{5}{3}\right]$$

as

$$P_3(1) = 1$$

Therefore

$$1 = c_1(-2/3) \Rightarrow c_1 = -3/2$$

Hence

$$P_3(x) = -\frac{3}{2}\left[x - \frac{5}{3}x^3\right]$$

Find the Legender polynomial for $n = 4$

$$P_4(x) = C_0\left[1 - \frac{4(4+1)}{2!}x^2 + \frac{4(4-2)(4+1)(4+3)}{4!}x^4\right]$$

$$P_4(x) = C_0\left[1 - 10x^2 + \frac{35}{3}x^4\right]$$

$$P_4(1) = C_0\left[1 - 10 + \frac{35}{3}\right]$$

$$P_4(1) = C_0\left[8/3\right]$$

For $P_4(x)$

As
$$P_4(x) = C_0 + C_2 x^2 + C_4 x^4.$$

As we know that
$$C_n = \frac{(2n)!}{2^n (n!)^2}$$

put $n = 4$

$$C_4 = \frac{8!}{2^4 (4!)^2}$$

$$C_4 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{16 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$C_4 = \frac{35}{8}$$

Now

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j$$

put $j = 2$

$$C_4 = \frac{(2-4)(2+4+1)}{(2+1)(2+2)} C_2$$

$$\frac{35}{8} = \frac{(-2)(7)}{(3)(4)} C_2$$

put $j = 0$

$$C_2 = \frac{(0-4)(0+4+1)}{(0+1)(0+2)} C_0$$

$$-\frac{15}{4} = -\frac{\cancel{20}^{10}}{\cancel{2}} C_0$$

$$C_0 = -\frac{\cancel{15}^{3}}{4} \times -\frac{1}{\cancel{10}_{2}} = +3/8$$

$$C_0 = 3/8$$

So

$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

$P_5(x)$

As

$$P_5(x) = \cancel{C_1 + x} \quad C_1 x + C_3 x^3 + C_5 x^5$$

Now

$$C_5 = \frac{(10!)}{2^5 \cdot (5!)^2}$$

$$C_5 = \frac{\cancel{10}\cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{32} \cdot 8 \cdot 4 \cdot \cancel{3} \cdot 2 \cdot 1 \cdot \cancel{5} \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

Now we know that

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j.$$

for $j = 3$,

$$C_5 = \frac{(3-5)(3+5+1)}{(3+1)(3+2)} C_3.$$

$$C_5 = \frac{(-2)(9)}{(4)(5)} C_3$$

$$\frac{63}{8} = \frac{(-2)(9)}{(4)(5)} C_3$$

$$C_3 = -\frac{\cancel{63}^{7}}{\cancel{8}_{2}} \frac{(\cancel{4}^{1})(5)}{(2)(\cancel{9})}$$

$$C_3 = -\frac{35}{4}.$$

for $j = 1$

$$C_3 = \frac{(1-5)(1+5+1)}{(1+1)(1+2)} C_1$$

$$-\frac{35}{4} = \frac{(-4)(7)}{(2)(3)} C_1$$

Available at
mathcity.org

So
$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

$$P_8(x)$$
So The Legendre eqn is

$$P_8(x) = C_0 + C_2x^2 + C_4x^4 + C_6x^6 + C_8x^8$$

$$C_8 = \frac{(16!)}{2^8 \cdot (8!)(8!)}$$

$$C_8 = \frac{16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8!}{256\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$C_8 = \frac{715}{32}.$$

Now we know That
$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)}C_j$$

$$C_6 = -\frac{\cancel{(715)}^{143}}{\cancel{(32)}_{4}} \cdot \frac{(7)\cancel{(8)}^{1}}{(2)\cancel{(18)}_{3}} = -\frac{143 \times 7}{4 \times 2 \times 3}.$$

$$C_6 = \cancel{\frac{2275}{24}}$$

$$C_6 = -\frac{1001}{24}.$$

Put $r = 4$.

$$C_6 = \frac{(4-8)(4+8+1)}{(4+1)(4+2)} C_4.$$

$$-\frac{1001}{24} = \frac{(-4)(13)}{(5)(6)} C_4$$

$$C_4 = \frac{1001}{\cancel{24}_{4}} \cdot \frac{(5)\cancel{(6)}^{1}}{(4)(13)}$$

$$C_4 = \frac{5005}{208}$$

Put $j = 2$

$$C_4 = \frac{(2-8)\ (2+8+1)}{(2+1)\ (2+2)} C_2$$

$$\frac{5005}{208} = \frac{(-6)(11)}{(3)(4)} C_2$$

$$C_2 = -\frac{5005}{\cancel{208}_{104}} \frac{(\cancel{3}^{1})(\cancel{4}^{\cancel{2}\,1})}{(\cancel{6}_{2})(11)}$$

$$C_2 = -\frac{5005}{1144}$$

Now Put $j = 0$

$$C_2 = \frac{(-8)(9)}{(1)(2)} C_0$$

$$C_0 = -\frac{5005}{1144} \frac{(-1)(\cancel{2})}{(\cancel{8}_{4})(9)}$$

$$C_0 = +\frac{5005}{41184}$$

Now

(H8) Generating function :-

Suppose we have a function

$$F(x,t) = \sum_{m=0}^{\infty} f_m(x)\, t^m$$

we called $F(x,t)$ a generating function for a set of functions $\{f_m(x)\}$ provided the above series converges. For e.g.

Consider

$$\frac{1}{1-xt} = (1-xt)^{-1}$$

$$= 1 + xt + x^2t^2 + x^3t^3 + \cdots$$

$$= \sum_{m=0}^{\infty} x^m t^m$$

$\Rightarrow$

$F(x,t)$ generate a function a set of function $\{x^m\}$

Label.

Q. Show the function is generating ftn. (109)

$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}$ gives $\sum_{n=0}^{x} P_n(x) t^n$.

i.e This function generates Legendre polynomial $P_n(x)$.

Soln:
$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$$

$$= (1-(2xt-t^2))^{-1/2}$$

$$= (1-d)^{-1/2} \text{ where } d = 2xt-t^2.$$

$$= 1 + (-1/2)(-d) + \frac{(-1/2)(-1/2-1)(-d)^2}{2!} + \cdots$$

$$= 1 + d/2 + \frac{3/4}{2!} d^2 + \cdots$$

$$= 1 + d/2 + \frac{(3/4) d^2}{2} + \cdots$$

$$= 1 + \frac{(2xt-t^2)}{2} + \frac{3}{8}(2xt-t^2)^2 + \cdots$$

$(P_1(x) = x)$, $(P_n(1) = 1 \quad \ldots)$

(170)

$$= P_0(x)t^0 + P_1(x)t + P_2(x)t^2 + \cdots$$

$$F(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Hence given function is generating function.

Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

This formula is used to find the Legendre polygonomial of different degree.

---

* Show That $P_n(x)$ is $\begin{cases} \text{even if } n \text{ is even} \\ \text{odd if } n \text{ is odd} \end{cases}$

Proof If n is even.

Put $x = -x$

$$P_n(-x) = C_0 + C_2(-x)^2 + C_4(-x)^4 + \cdots + C_n(-x)^n$$

$$= C_0 + C_2 x^2 + C_4 x^4 + \cdots + C_n x^n$$

$$P_n(-x) = P_n(x)$$

$\Rightarrow$ $P_n(x)$ is even.

If n is odd Then.

$$P_n(x) = C_1 + C_3 x^3 + C_5 x^5 + \cdots + C_n x^n$$

Put $x = -x$.

$$P_n(-x) = C_1 + C_3(-x)^3 + C_5(-x)^5 + \cdots + C_n(-x)^n$$

$$= C_1 - C_3 x^3 - C_5 x^5 + \cdots - C_n(x)^n$$

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(-x) = -P_n(x) \quad \because (-1)^n = -1$$

as n is odd

Qu. Show that

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

Then $P_n(x)$ is even if n is even and $P_n(x)$ is odd if n is odd.

Now as

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

Change 'x' by $-x$, gives

$$\frac{1}{\sqrt{1+2xt+t^2}} = \sum_{n=0}^{\infty} P_n(-x)\,t^n.$$

Now change 't' by '$-t$'

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(-x)\,(-t)^n$$

$$\sum_{n=0}^{\infty} P_n(x)\,t^n = \sum_{n=0}^{\infty} P_n(-x)(-1)^n t^n.$$

$P_0(x)t^0 + P_1(x)t^1 + P_2(x)t^2 + \cdots$

On Comparing the Coefficients of "t^n" we have

$$P_n(x) = P_n(-x)(-1)^n$$

Now if n is even

Then $P_n(x) = P_n(-x)$. $\because (-1)^n = 1$ as n is even.

$\Rightarrow P_n(x)$ is even.

Now if n is odd

$$P_n(x) = P_n(-x)(-1)^n$$

$\Rightarrow P_n(x) = -P_n(-x)$ $\because (-1)^n = -1$ as n is odd.

$\Rightarrow P_n(x)$ is odd.

Q: Show That

$$(2n+1)P_n(x) = \underline{P'_{n+1}(x)} - \underline{P'_{n-1}(x)}$$

Sol:

Since $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

Put $n = n+1$

$$P \ldots = \frac{1}{\ldots} \quad \frac{d^{n+1}}{\ldots}(x^2-1)^{n+1}$$

$$= \frac{1}{2^{n+1}\cdot(n+1)!}\frac{d^n}{dx^n}\left[\frac{d^2}{dx^2}(x^2-1)^{n+1}\right]$$

$$= \frac{1}{2^{n+1}\cdot(n+1)!}\cdot\frac{d^n}{dx^n}\left[\frac{d}{dx}(n+1)(x^2-1)^n 2x\right]$$

$$= \frac{2(n+1)}{2^{n+1}\cdot(n+1)!}\cdot\frac{d^n}{dx^n}\left[n(x^2-1)^{n-1}(x)(2x) + (x^2-1)^n(1)\right]$$

$$= \frac{2(n+1)}{2^{n+1}\cdot(n+1)!}\cdot\frac{d^n}{dx^n}\left[2x^2 n(x^2-1)^{n-1} + (x^2-1)^n\right]$$

$$= \frac{2}{2^{n+1}\cdot n!}\frac{d^n}{dx^n}\left[2x^2 n(x^2-1)^{n-1} + (x^2-1)^n\right]$$

$$= \frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}\left[2x^2 n(x^2-1)^{n-1}\right] + \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P'_{n+1}(x) = \frac{2n}{2^n n!}\frac{d^n}{dx^n}\left(x^2(x^2-1)^{n-1}\right] + P_n(x).$$

Now By Rodring formula

$$P_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$$

By taking derivative

$$P'_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!}\frac{d^{n}}{dx^{n}}(x^2-1)^{n-1} \quad (2)$$

Now subtracting (2) from (1) we get.

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{2n}{2^n . n!}\frac{d^n}{dx^n}\left[x^2(x^2-1)^{n-1}\right] + P_n(x) - \frac{1}{2^n(n-1)!}\frac{d^n}{dx^n}(x^2-1)^{n-1}$$

$$= \frac{2n}{2^n . n!}\frac{d^n}{dx^n}\left[x^2(x^2-1)^{n-1}\right] - \frac{1}{2^n(n-1)!}\frac{d^n}{dx^n}(x^2-1)^{n-1} + P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}\left[\frac{2n\, x^2(x^2-1)^{n-1}}{2n} - (x^2-1)^{n-1}\right] + P_n(x)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{1}{2^{n-1}\cdot(n-1)!}\frac{d^n}{dx^n}(x^2-1)^n + P_n(x)$$

$$= \frac{2n}{2^n(n!)}\frac{d^n}{dx^n}(x^2-1)^n + P_n(x)$$

$$= 2n\,P_n(x) + P_n(x)$$

$$= (2n+1)\,P_n(x)$$

i.e

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)\,P_n(x)$$

Q# $(2n+1)\cancel{x}P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$ for $n = 1, 2, 3)$

Sol: we know that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$$(-1/2)(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$\frac{(x-t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$\frac{x-t}{(1-2xt+t^2)^{1/2}(1-2xt+t^2)} = \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$(x-t) \cdot \frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$(x-t) \sum_{n=0}^{\infty} P_n(x) t^n = (1-2x t+t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$x \sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} P_n(x) \cdot n \cdot t^{n-1} - 2x \sum_{n=0}^{\infty} P_n(x) \cdot n t^n$$

$$\Rightarrow x\sum_{n=0}^{\infty} P_n(x)\, t^n + 2nx \sum_{n=0}^{\infty} P_n(x) t^n$$

$$= \sum_{n=0}^{\infty} P_n(x)\cdot n\cdot t^{n-1} + \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n+1} + \sum_{n=0}^{\infty} P_n(x) t^{n+1}$$

$$\cancel{\Rightarrow (1+2n)\, x \sum_{n=0}^{\infty} P_n(x) t^n}$$

$$\Rightarrow \sum_{n=0}^{\infty} (1+2n)\, x\, P_n(x) t^n$$

$$= \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n-1} + \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n+1} + \sum_{n=0}^{\infty} P_n(x)\, t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+1) x\, P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n-1} + \sum_{n=0}^{\infty} (n+1) P_n(x)\, t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty}(2n+1)x P_n(x) t^n$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)(n+1)t^n \quad \text{By Putting } n=n+1 \text{ \& } n=n-1$$

$$+ \sum_{n=1}^{\infty} n P_{n-1}(x) t^n$$

Now on Comparing Coefficients of t^n for $n=1$ and onward we have

$$\Rightarrow (2n+1)x P_n(x)$$

$$= P_{n+1}(x)(n+1) + n P_{n-1}(x)$$

Hence

$$(2n+1)x P_n(x) = P_{n+1}(x)(n+1) + n P_{n-1}(x).$$

Theorem:

* The hypergeometric function is solution of D.E

$$Z(1-Z)\frac{d^2y}{dZ^2}+\{\gamma-(\alpha+\beta+1)Z\}\frac{dy}{dZ}-\alpha\beta y=0 \qquad (1)$$

which is called Hypergeometric differential equation α, β, γ are constants.

Proof:

we take a trial solution

$$y=\sum_{k=0}^{\infty} C_k Z^{k+\gamma} \qquad (2)$$

$$\Rightarrow \frac{dy}{dZ}=\sum_{k=0}^{\infty} C_k (k+\gamma) Z^{k+\gamma-1}$$

$$\Rightarrow \frac{d^2y}{dZ^2}=\sum_{k=0}^{\infty} C_k (k+\gamma)(k+\gamma-1) Z^{k+\gamma-2}$$

Using in eqn (1)

$$Z(1-Z)\left(\sum_{k=0}^{\infty} C_k (k+\gamma)(k+\gamma-1) Z^{k+\gamma-2}\right)$$

$$+\{\gamma-(\alpha+\beta+1)Z\}\sum_{k=0}^{\infty} C_k (k+\gamma) Z^{k+\gamma-1}$$

$$-\alpha\beta\sum C_k Z^{k+\gamma} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+r)(k+r-1)\, z^{k+r-1}$$

$$- \sum_{k=0}^{\infty} C_k (k+r)(k+r-1)\, z^{k+r}$$

$$+ \Gamma \sum_{k=0}^{\infty} C_k (k+r)\, z^{k+r-1}$$

$$- (\alpha+\beta+1) z \sum_{k=0}^{\infty} C_k (k+r)\, z^{k+r-1}$$

$$- \alpha\beta \sum_{k=0}^{\infty} C_k\, z^{k+r} = 0$$

On Comparing Least power of 'z' i.e z^{r-1} by putting $k=0$. we get

$$C_0\, r(r-1) + \Gamma C_0 (r) = 0$$

$$C_0 r\,(r-1+\Gamma) = 0$$

$$\Rightarrow C_0 \neq 0 \quad \text{so} \quad r(r-1+\Gamma) = 0$$

$$\Rightarrow r = 0 \quad \text{or} \quad r = 1-\Gamma$$

Now Comparing Coefficient of z^{k+r-1}.

Case-I

when $\gamma = 0$

so

$$C_n\left[n(n-1) + \gamma n\right]$$
$$= C_{n-1}\left[(n-1)(n-2) + (\alpha+\beta+1)(n-1) + \alpha\beta\right]$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n-2+\alpha+\beta+1)+\alpha\beta\right]}{n(n-1)+\gamma n.}$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n+\alpha-1+\beta)+\alpha\beta\right]}{n\left[n-1+\gamma\right]}$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n+\alpha-1)+\beta(n-1)+\alpha\beta\right]}{n\left[n-1+\gamma\right]}$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n+\alpha-1)+\beta(n+\alpha-1)\right]}{n(n-1+\gamma)}$$

NOTE 1st ~~the~~ Geometric series is always 1. ∴ $C_0 = 1$

for $k = k-1$

$$C_{k-1} = C_{k-2}\left[\frac{(k-2+\alpha)(k-2+\beta)}{(k-1)(k-2+\gamma)}\right]$$

$$\vdots$$

$$C_2 = C_1\left[\frac{(1+\alpha)(1+\beta)}{(2)(1+\gamma)}\right]$$

and

$$C_1 = C_0\left[\frac{\alpha\cdot\beta}{1\cdot\gamma}\right]$$

$$C_1 = C_0\left(\frac{\alpha\beta}{\gamma}\right)$$

$$C_1 = \frac{\alpha\beta}{\gamma}$$

$$C_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{2\gamma(\gamma+1)}$$

$$C_3 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!\,\gamma(\gamma+1)(\gamma+3)}$$

~~usin~~ Choose $C_0 = 1$

using back subsituation we get

$$C_k = \frac{(k-1+\alpha)(k-2+\alpha)\cdots(k-(k-1)+\alpha)\alpha\cdot(k-1+\beta)(k-2+\beta)\cdots(k-(k-1)+\beta)\beta}{k\cdot(k-1)(k-2)\cdots3\cdot2\cdot1\cdot(k-1+\gamma)(k-2+\gamma)\cdots(k-(k-1)+\gamma)\gamma}$$

$C_k = (\alpha) \ (\beta)$

using in solution we have

$$y = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k!\,(\gamma)_k} z^k \quad \because r=0$$

<u>Case II</u>

when $r = 1-\gamma$
using in equation ③ we get

$$C_k\left[(k+1-\gamma)(k-\gamma)+\gamma(k+1-\gamma)\right]$$

$$= C_{k-1}\left[(k-\gamma)(k-1-\gamma) + (\alpha+\beta+1)(k-\gamma)+\alpha\beta\right]=0$$

$$C_k = \frac{C_{k-1}\left[(k-\gamma)(k-1-\gamma)+(\alpha+1)(k-\gamma) + \beta(k-\gamma) + \alpha\beta\right]}{(k+1-\gamma)(k-\gamma)+\gamma(k+1-\gamma)}$$

$$C_k = C_{k-1}\left[(k-\gamma)\left[k-\cancel{1}-\gamma+k-\gamma+\alpha+\cancel{1}\right]\right.$$

$$C_k = C_{k-1}\left[\frac{(k-r+\alpha)(k-r+\beta)}{(k+1-r)(k)}\right]$$

Put

$k = k-1$

$$C_{k-1} = C_{k-2}\left[\frac{(k-1-r+\alpha)(k-1-r+\beta)}{(k-1+1-r)\,(k-1)}\right.$$

$\vdots$

$$C_2 = C_1\left[\frac{(2-r+\alpha)(2-r+\beta)}{2\,(2-r+1)}\right]$$

$$C_1 = C_0\left[\frac{(1-r+\alpha)(1-r+\beta)}{1\cdot(1-r+1)=(2-r)}\right]$$

Now choose $C_0 = 1$

$$C_k = \frac{(k-r+\alpha)(k-1-r+\alpha)(k-2-r+\alpha)\cdots(1-r+\alpha)\,(k-r+\beta)(k-1-r+\beta)(k-2-r+\beta)\cdots(1-r+\beta)}{k(k-1)(k-2)\cdots 3\cdot2\cdot1\cdot(k+1-r)(k-r)(k-1-r)\cdots(3-r)(2-r)}$$

$$C_k = \frac{(1-r+\alpha)_k\cdot(1-r+\beta)_k}{k!\;(2-r)_\cdot}$$

Bessel Functions:-

Bessel's Differential Equation.

The differential equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad , \; n \geq 0$$

The solution of Bessel's D.E is called Bessel function.

* Find the solution of Bessell's D.E

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \; , \; n \geq 0 \quad \text{(1)}$$

Proof:- Let $y = \sum_{k=0}^{\infty} C_k x^{k+r}$ be a series solution of bessell's D.E. we may write

$$y = \sum_{k=-\infty}^{\infty} C_k x^{k+r} \quad \text{where } C_k = 0 \text{ for } k<0$$

$$\Rightarrow \quad y' = \sum_{k=0}^{\infty} C_k (k+r) x^{k+r-1}$$

$$k+r-2$$

using in eqn (1) we get

$$x^2 \sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r-2} + x \sum_{k=0}^{\infty} C_k (k+r) x^{k+r-1} + (x^2-n^2) \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

$$x^2 \sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r-2} + x \sum_{k=0}^{\infty} C_k (k+r) x^{k+r-1} + x^2 \sum_{k=0}^{\infty} C_k x^{k+r} - n^2 \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

$$\sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} C_k (k+r) x^{k+r}$$

k<0 $$+ \sum_{k=0}^{\infty} C_k x^{k+r+2} - n^2 \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

Comparing cofficent of x^{k+r} we get

$C_k (k+r)(k+r-1) + C_k (k+r)$

$$\Rightarrow \quad C_k\left[(k+r)^2-n^2\right]=-C_{k-2} \quad (2)$$

This is known as recurrence relation or identical relation//

Put $k=0$ in eqn (2)

$$C_0\left[(0+r)^2-n^2\right]=-C_{-2}$$

$$C_0\left[r^2-n^2\right]=0 \quad \because C_{-2}=0 \ \forall k<0$$

Either $C_0=0$ or $r^2-n^2=0$

$k=2$ in eqn (2)

$$C_2\left[(2+r)^2-n^2\right]=-C_0$$

$k=4$

$$C_4\left[(4+r)^2-n^2\right]=-C_2$$

So all C_k for even value of k depends upon C_0. To get a non-trivial solution $C_0 \neq 0$

from eq (2) we get

$$C_k = \frac{(-1)\,C_{k-2}}{(k+r)^2 - n^2} \rightarrow (3)$$

$$C_k = \frac{(-1)\,C_{k-2}}{(k+r+n)(k+r-n)}$$

$$C_k = \frac{(-1)\,C_{k-2}}{(k+r+n)(k+r-n)}$$

Now for

$k = 1$

$$C_1 = \frac{(-1)\,C_{-1}}{(1+r+n)(1+r-n)}$$

$C_1 = 0 \quad \because C_{-1} = 0$

$$C_3 = \frac{(-1)\,C_1}{(3+r+n)(3+r-n)}$$

$C_3 = 0 \quad \because C_1 = 0$

so

p-126
p-136

(190)

$$C_{k-2} = \frac{(-1)\,C_{k-4}}{(k-2)\,(2n+k-2)}$$

$$C_4 = \frac{(-1)\,C_2}{(4)\,(2n+4)}$$

$$C_2 = \frac{(-1)\,C_0}{(2)(2n+2)}$$

let

$$C_0 = \frac{1}{2^n\,\Gamma(n+1)}$$

$$C_2 = \frac{(-1)}{(2)(2n+2)} \cdot \frac{1}{2^n\,\Gamma(n+1)}$$

$$= \frac{(-1)}{2^{n+1}\cdot 2!\,(n+1)\,\Gamma(n+1)} = \frac{(-1)}{2^{n+2}\cdot 1!\,\Gamma(n+2)}$$

$$C_2 = \frac{(-1)}{2^{n+1}\cdot 2!\,\Gamma(n+2)} \qquad \because \Gamma(z+1) = z\cdot\Gamma(z).$$

$$C_4 = \frac{(-1)^2}{4(2n+4)\cdot 2^{n+1}\cdot 2!\,\Gamma(n+2)}$$

$$C_4 = \frac{(-1)^2}{2^2\left(2n+2^2\right)\cdot 2^{n+1}\cdot 2!\,\Gamma(n+2)}$$

$$C_4 = \frac{(-1)^2}{2^{n+4}(n+2)\cdot 2!\,\Gamma(n+2)}$$

$$C_4 = \frac{(-1)^2}{2^{n+4}(2!)\,\Gamma(n+2+1)}$$

$$C_{k=2m} = \frac{(-1)^{2m/2 = m}}{2^{n+2m}(m!)\cdot\Gamma(n+m+1)}$$

using these values of C_k and r in the solution

$$\infty \qquad m \quad 2m+n$$

Case-II

when $r = -n$ put in eqn (3)

$$C_k = \frac{(-1)\, C_{k-2}}{(k-n)^2 - n^2}$$

$$C_k = \frac{(-1)\, C_{k-2}}{k^2 + \cancel{n^2} - 2kn \cancel{+ n^2}} = \frac{(-1)\, C_{k-2}}{k^2 - 2kn}$$

$$C_k = \frac{(-1)\, C_{k-2}}{k(k-2n)}$$

$$C_{k-2} = \frac{(-1)\, C_{k-4}}{(k-2)(k-2-2n)}$$

$$C_4 = \frac{(-1)\, C_2}{(4)(4-2n)}$$

$$C_2 = \frac{(-1)\, C_0}{\cancel{(2)}\,(2-2n)}$$

Let $C_0 = \frac{1}{2^{-n}\Gamma}$ $\quad \because n = -n.$

$$C_2 = \frac{(-1)}{2^2(1-n)} \cdot \frac{1}{2^{-n}\,\Gamma(-n+1)}.$$

$$C_2 = \frac{(-1)}{2^{2-n}\,(1-n)\,\Gamma(-n+1)}$$

$$C_2 = \frac{(-1)}{2^{2-n}\cdot\Gamma(-n)} \qquad \because (1-n)\,\Gamma(-n+1) = \Gamma(-n).$$

$$C_4 = \frac{(-1)^2\, C_2.}{4(4-4n).}$$

$$C_4 = \frac{(-1)^2}{4(4-2n)\cdot 2^{2-n}\,\Gamma(-n+0).}$$

$$C_4 = \frac{(-1)^2}{2^3(2-n)\cdot 2^{2-n}\cdot\Gamma(-n+0)}$$

$$C_4 = \frac{(-1)^2}{\cdots^{4-n}\ (2-n)\,\Gamma(-n+0)}$$

* Find the solution of the Legender equation

$$(1-x^2)\,y'' - 2xy' + n(n+1)y = 0 \quad \text{①}$$

Ans/ we assume a solution of eqn ① as

$$y = \sum_{k=0}^{\infty} C_k x^{k+\beta} \quad \text{②} \quad \text{and } C_k = 0 \text{ for } k < 0$$

$\Rightarrow$

$$y' = \sum_{k=0}^{\infty} C_k (k+\beta)\, x^{k+\beta-1}$$

$$y'' = \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta-2}.$$

using values in equation ① we get

$$(1-x^2)\sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta-2}$$

$$-2x \sum_{k=0}^{\infty} C_k (k+\beta)\, x^{k+\beta-1}$$

$$+\, n(n+1) \sum_{k=0}^{\infty} C_k x^{k+\beta} \Big) = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta-2} - \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta}$$

$$\sum 2 C_k (k+\beta)\, x^{k+\beta} + n(n+1) \sum C_k x^{k+\beta} = 0$$

$$C_{k+2}(k+\beta+2)(k+1+\beta) - C_k(k+\beta)(k+\beta-1)$$
$$-2\,C_k(k+\beta) + n(n+1)\,C_k = 0$$

$$C_{k+2}(k+\beta+2)(k+\beta+1) + C_k\left[n(n+1) - (k+\beta)(k+\beta-1) - 2(k+\beta)\right] = 0$$

$$C_{k+2}(k+\beta+2)(k+\beta+1) + C_k\left[n(n+1) - (k+\beta)\left[k+\beta-1+2\right]\right] = 0$$

$$C_{k+2}(k+\beta+2)(k+\beta+1) + C_k\left[n(n+1) - (k+\beta)(k+\beta+1)\right] = 0 \quad (3)$$

This is known as identical eqn. or recurrence equation.

Put $k = -2$.

$$C_0\,\beta(\beta-1) + C_{-2}\left[n(n+1) - (-2+\beta)(-2+\beta+1)\right] = 0$$

$$C_0\beta(\beta-1) = 0 \quad \because C_{-2} = 0$$

Assum $C_0 \neq 0$

Then either $\beta = 0$ or $\beta - 1 = 0 \Rightarrow \beta = 1$

Case-I put $\beta = 0$ in eqn (3) we get

$$C_{k+2}(k+2)(k+1) + C_k\left(n(n+1) - k(k+1)\right] = 0$$

Put $K=0$

$$C_2 = \frac{-n(n+1)}{1\cdot 2}C_0$$

$$C_2 = \frac{-n(n+1)}{2!}C_0$$

Put $K=1$

$$C_3 = \frac{1(2)-n(n+1)}{2\cdot 3}\cdot C_1$$

$$C_3 = \frac{2-n^2-n}{2\cdot 3}C_1$$

$$C_3 = \frac{2-2n+n-n^2}{2\cdot 3}C_1$$

$$C_3 = \frac{2(1-n)+n(1-n)}{2\cdot 3}C_1$$

$$C_3 = \frac{(2+n)(1-n)}{3!}C_1$$

Put $K=2$.

$$C_4 = \frac{2\cdot 3-n(n+1)}{3\cdot 4}C_2$$

$$= \frac{6-n^2-n}{3\cdot 4}C_2$$

$$C_4 = -\frac{(3+n)(n-2)}{4\cdot 3} C_2.$$

$$C_4 = -\frac{(3+n)(n-2)}{4\cdot 3}\cdot\left(\frac{-n(n+1)}{2!}C_0\right)$$

$$C_4 = \frac{n(n-2)(n+1)(n+3)}{24 = 4!}$$

Put $k = 3$

$$C_5 = \frac{3\cdot 4 - n(n+1)}{4\cdot 5} C_3 = \frac{12 - n^2 - n}{4\cdot 5} C_3$$

$$C_5 = \frac{-n^2 - n + 12}{4\cdot 5} C_3 = -\frac{(n^2+n-12)}{4\cdot 5} C_3$$

$$C_5 = -\frac{(4+n)(n-3)}{5\cdot 4} C_3.$$

$$C_5 = -\frac{(n+4)(n-3)}{5\cdot 4}\cdot -\frac{(n+2)(n-1)}{3!} C_1$$

$$C_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} C_1.$$

using all values in equation (2) we get

$$y = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{}x^4 \cdots\right]$$