

$$\text{c. } g_3(x) = \left( \frac{x+3}{x^2+2} \right)^{1/2} \qquad \text{d. } g_4(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$$

2. a. Perform four iterations, if possible, on each of the functions  $g$  defined in Exercise 1. Let  $p_0 = 1$  and  $p_{n+1} = g(p_n)$ , for  $n = 0, 1, 2, 3$ .
- b. Which function do you think gives the best approximation to the solution?
3. The following four methods are proposed to compute  $21^{1/3}$ . Rank them in order, based on their apparent speed of convergence, assuming  $p_0 = 1$ .
- $$\text{a. } p_n = \frac{20p_{n-1} + 21/p_{n-1}^2}{21} \qquad \text{b. } p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$$
- $$\text{c. } p_n = p_{n-1} - \frac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21} \qquad \text{d. } p_n = \left( \frac{21}{p_{n-1}} \right)^{1/2}$$
4. The following four methods are proposed to compute  $7^{1/5}$ . Rank them in order, based on their apparent speed of convergence, assuming  $p_0 = 1$ .
- $$\text{a. } p_n = p_{n-1} \left( 1 + \frac{7 - p_{n-1}^5}{p_{n-1}^2} \right)^3 \qquad \text{b. } p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{p_{n-1}^2}$$
- $$\text{c. } p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4} \qquad \text{d. } p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$$
5. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-2}$  for  $x^4 - 3x^2 - 3 = 0$  on  $[1, 2]$ . Use  $p_0 = 1$ .
6. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-2}$  for  $x^3 - x - 1 = 0$  on  $[1, 2]$ . Use  $p_0 = 1$ .
7. Use Theorem 2.3 to show that  $g(x) = \pi + 0.5 \sin(x/2)$  has a unique fixed point on  $[0, 2\pi]$ . Use fixed-point iteration to find an approximation to the fixed point that is accurate to within  $10^{-2}$ . Use Corollary 2.5 to estimate the number of iterations required to achieve  $10^{-2}$  accuracy, and compare this theoretical estimate to the number actually needed.
8. Use Theorem 2.3 to show that  $g(x) = 2^{-x}$  has a unique fixed point on  $[\frac{1}{3}, 1]$ . Use fixed-point iteration to find an approximation to the fixed point accurate to within  $10^{-4}$ . Use Corollary 2.5 to estimate the number of iterations required to achieve  $10^{-4}$  accuracy, and compare this theoretical estimate to the number actually needed.
9. Use a fixed-point iteration method to find an approximation to  $\sqrt{3}$  that is accurate to within  $10^{-4}$ . Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.
10. Use a fixed-point iteration method to find an approximation to  $\sqrt[3]{25}$  that is accurate to within  $10^{-4}$ . Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.
11. For each of the following equations, determine an interval  $[a, b]$  on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within  $10^{-5}$ , and perform the calculations.
- $$\text{a. } x = \frac{2 - e^x + x^2}{3} \qquad \text{b. } x = \frac{5}{x^2} + 2$$
- $$\text{c. } x = (e^x/3)^{1/2} \qquad \text{d. } x = 5^{-x}$$
- $$\text{e. } x = 6^{-x} \qquad \text{f. } x = 0.5(\sin x + \cos x)$$
12. For each of the following equations, use the given interval or determine an interval  $[a, b]$  on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within  $10^{-5}$ , and perform the calculations.
- $$\text{a. } 2 + \sin x - x = 0 \quad \text{use } [2, 3] \qquad \text{b. } x^3 - 2x - 5 = 0 \quad \text{use } [2, 3]$$
- $$\text{c. } 3x^2 - e^x = 0 \qquad \text{d. } x - \cos x = 0$$
13. Find all the zeros of  $f(x) = x^2 + 10 \cos x$  by using the fixed-point iteration method for an appropriate iteration function  $g$ . Find the zeros accurate to within  $10^{-4}$ .

14. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-4}$  for  $x = \tan x$ , for  $x$  in  $[4, 5]$ .
15. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-2}$  for  $2 \sin \pi x + x = 0$  on  $[1, 2]$ . Use  $p_0 = 1$ .
16. Let  $A$  be a given positive constant and  $g(x) = 2x - Ax^2$ .
- Show that if fixed-point iteration converges to a nonzero limit, then the limit is  $p = 1/A$ , so the inverse of a number can be found using only multiplications and subtractions.
  - Find an interval about  $1/A$  for which fixed-point iteration converges, provided  $p_0$  is in that interval.
17. Find a function  $g$  defined on  $[0, 1]$  that satisfies none of the hypotheses of Theorem 2.3 but still has a unique fixed point on  $[0, 1]$ .
18.
  - Show that Theorem 2.2 is true if the inequality  $|g'(x)| \leq k$  is replaced by  $g'(x) \leq k$ , for all  $x \in (a, b)$ . [Hint: Only uniqueness is in question.]
  - Show that Theorem 2.3 may not hold if inequality  $|g'(x)| \leq k$  is replaced by  $g'(x) \leq k$ . [Hint: Show that  $g(x) = 1 - x^2$ , for  $x$  in  $[0, 1]$ , provides a counterexample.]
19.
  - Use Theorem 2.4 to show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to  $\sqrt{2}$  whenever  $x_0 > \sqrt{2}$ .

- Use the fact that  $0 < (x_0 - \sqrt{2})^2$  whenever  $x_0 \neq \sqrt{2}$  to show that if  $0 < x_0 < \sqrt{2}$ , then  $x_1 > \sqrt{2}$ .
  - Use the results of parts (a) and (b) to show that the sequence in (a) converges to  $\sqrt{2}$  whenever  $x_0 > 0$ .
20.
  - Show that if  $A$  is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to  $\sqrt{A}$  whenever  $x_0 > 0$ .

- What happens if  $x_0 < 0$ ?
21. Replace the assumption in Theorem 2.4 that “a positive number  $k < 1$  exists with  $|g'(x)| \leq k$ ” with “ $g$  satisfies a Lipschitz condition on the interval  $[a, b]$  with Lipschitz constant  $L < 1$ .” (See Exercise 27, Section 1.1.) Show that the conclusions of this theorem are still valid.
22. Suppose that  $g$  is continuously differentiable on some interval  $(c, d)$  that contains the fixed point  $p$  of  $g$ . Show that if  $|g'(p)| < 1$ , then there exists a  $\delta > 0$  such that if  $|p_0 - p| \leq \delta$ , then the fixed-point iteration converges.
23. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass  $m$  is dropped from a height  $s_0$  and that the height of the object after  $t$  seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2g}{k^2}(1 - e^{-kt/m}),$$

where  $g = 32.17 \text{ ft/s}^2$  and  $k$  represents the coefficient of air resistance in lb-s/ft. Suppose  $s_0 = 300 \text{ ft}$ ,  $m = 0.25 \text{ lb}$ , and  $k = 0.1 \text{ lb-s/ft}$ . Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.

24. Let  $g \in C^1[a, b]$  and  $p$  be in  $(a, b)$  with  $g(p) = p$  and  $|g'(p)| > 1$ . Show that there exists a  $\delta > 0$  such that if  $0 < |p_0 - p| < \delta$ , then  $|p_0 - p| < |p_1 - p|$ . Thus, no matter how close the initial approximation  $p_0$  is to  $p$ , the next iterate  $p_1$  is farther away, so the fixed-point iteration does not converge if  $p_0 \neq p$ .

## 2.3 Newton's Method and Its Extensions

Isaac Newton (1641–1727) was one of the most brilliant scientists of all time. The late 17th century was a vibrant period for science and mathematics and Newton's work touched nearly every aspect of mathematics. His method for solving was introduced to find a root of the equation  $y^3 - 2y - 5 = 0$ . Although he demonstrated the method only for polynomials, it is clear that he realized its broader applications.

**Newton's** (or the *Newton-Raphson*) **method** is one of the most powerful and well-known numerical methods for solving a root-finding problem. There are many ways of introducing Newton's method.

### Newton's Method

If we only want an algorithm, we can consider the technique graphically, as is often done in calculus. Another possibility is to derive Newton's method as a technique to obtain faster convergence than offered by other types of functional iteration, as is done in Section 2.4. A third means of introducing Newton's method, which is discussed next, is based on Taylor polynomials. We will see there that this particular derivation produces not only the method, but also a bound for the error of the approximation.

Suppose that  $f \in C^2[a, b]$ . Let  $p_0 \in [a, b]$  be an approximation to  $p$  such that  $f'(p_0) \neq 0$  and  $|p - p_0|$  is "small." Consider the first Taylor polynomial for  $f(x)$  expanded about  $p_0$  and evaluated at  $x = p$ .

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p)),$$

where  $\xi(p)$  lies between  $p$  and  $p_0$ . Since  $f(p) = 0$ , this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2}f''(\xi(p)).$$

Newton's method is derived by assuming that since  $|p - p_0|$  is small, the term involving  $(p - p_0)^2$  is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0).$$

Solving for  $p$  gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

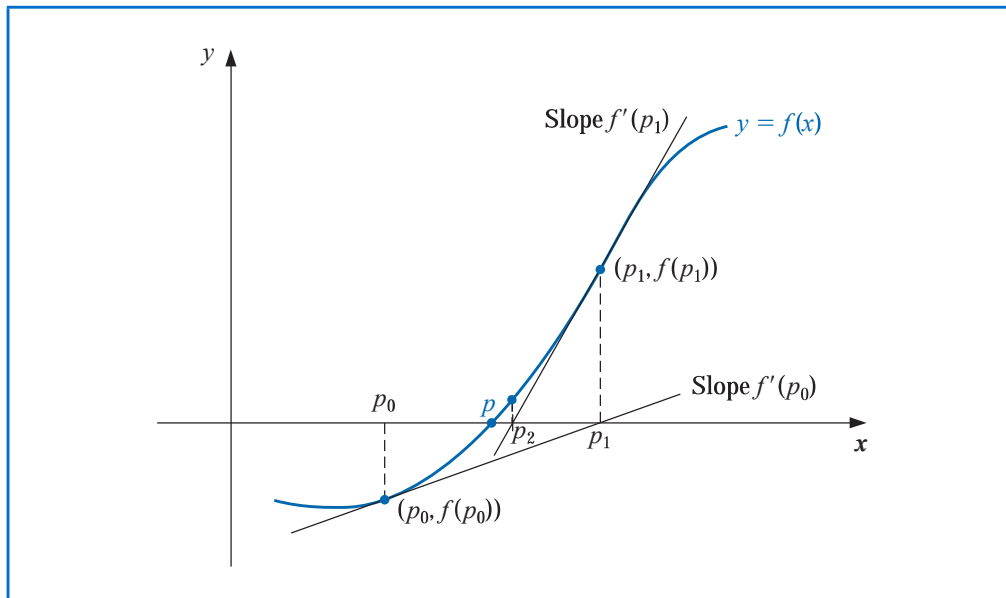
This sets the stage for Newton's method, which starts with an initial approximation  $p_0$  and generates the sequence  $\{p_n\}_{n=0}^{\infty}$  by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \quad (2.7)$$

Figure 2.8 on page 68 illustrates how the approximations are obtained using successive tangents. (Also see Exercise 15.) Starting with the initial approximation  $p_0$ , the approximation  $p_1$  is the  $x$ -intercept of the tangent line to the graph of  $f$  at  $(p_0, f(p_0))$ . The approximation  $p_2$  is the  $x$ -intercept of the tangent line to the graph of  $f$  at  $(p_1, f(p_1))$  and so on. Algorithm 2.3 follows this procedure.

Joseph Raphson (1648–1715) gave a description of the method attributed to Isaac Newton in 1690, acknowledging Newton as the source of the discovery. Neither Newton nor Raphson explicitly used the derivative in their description since both considered only polynomials. Other mathematicians, particularly James Gregory (1636–1675), were aware of the underlying process at or before this time.

Figure 2.8


**ALGORITHM**  
**2.3**

### Newton's

To find a solution to  $f(x) = 0$  given an initial approximation  $p_0$ :

**INPUT** initial approximation  $p_0$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $i = 1$ .

**Step 2** While  $i \leq N_0$  do Steps 3–6.

**Step 3** Set  $p = p_0 - f(p_0)/f'(p_0)$ . (Compute  $p_i$ .)

**Step 4** If  $|p - p_0| < TOL$  then  
**OUTPUT** ( $p$ ); (The procedure was successful.)  
**STOP**.

**Step 5** Set  $i = i + 1$ .

**Step 6** Set  $p_0 = p$ . (Update  $p_0$ .)

**Step 7** **OUTPUT** ('The method failed after  $N_0$  iterations,  $N_0 =$ ',  $N_0$ );  
(The procedure was unsuccessful.)  
**STOP**. ■

The stopping-technique inequalities given with the Bisection method are applicable to Newton's method. That is, select a tolerance  $\varepsilon > 0$ , and construct  $p_1, \dots, p_N$  until

$$|p_N - p_{N-1}| < \varepsilon, \quad (2.8)$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0, \quad (2.9)$$

or

$$|f(p_N)| < \varepsilon. \quad (2.10)$$

A form of Inequality (2.8) is used in Step 4 of Algorithm 2.3. Note that none of the inequalities (2.8), (2.9), or (2.10) give precise information about the actual error  $|p_N - p|$ . (See Exercises 16 and 17 in Section 2.1.)

Newton's method is a functional iteration technique with  $p_n = g(p_{n-1})$ , for which

$$g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \tag{2.11}$$

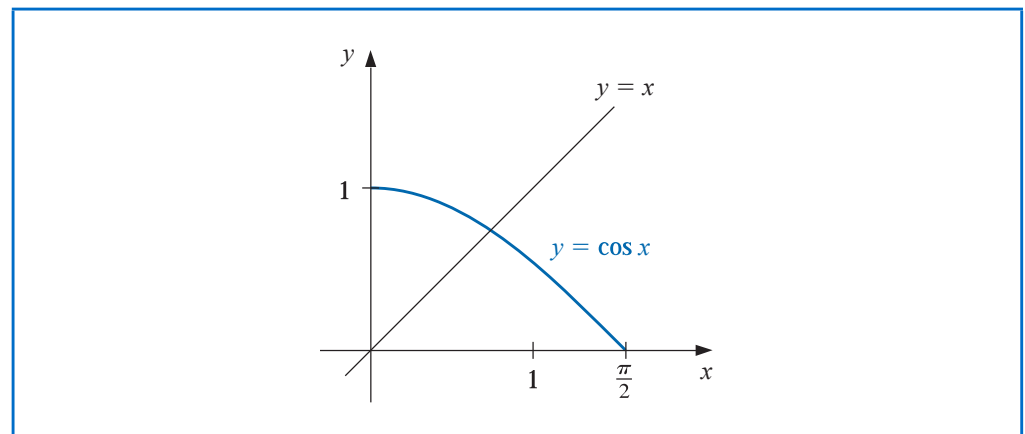
In fact, this is the functional iteration technique that was used to give the rapid convergence we saw in column (e) of Table 2.2 in Section 2.2.

It is clear from Equation (2.7) that Newton's method cannot be continued if  $f'(p_{n-1}) = 0$  for some  $n$ . In fact, we will see that the method is most effective when  $f'$  is bounded away from zero near  $p$ .

**Example 1** Consider the function  $f(x) = \cos x - x = 0$ . Approximate a root of  $f$  using (a) a fixed-point method, and (b) Newton's Method

**Solution** (a) A solution to this root-finding problem is also a solution to the fixed-point problem  $x = \cos x$ , and the graph in Figure 2.9 implies that a single fixed-point  $p$  lies in  $[0, \pi/2]$ .

Figure 2.9



Note that the variable in the trigonometric function is in radian measure, not degrees. This will always be the case unless specified otherwise.

Table 2.3

$n$	$p_n$
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Table 2.3 shows the results of fixed-point iteration with  $p_0 = \pi/4$ . The best we could conclude from these results is that  $p \approx 0.74$ .

(b) To apply Newton's method to this problem we need  $f'(x) = -\sin x - 1$ . Starting again with  $p_0 = \pi/4$ , we generate the sequence defined, for  $n \geq 1$ , by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}.$$

This gives the approximations in Table 2.4. An excellent approximation is obtained with  $n = 3$ . Because of the agreement of  $p_3$  and  $p_4$  we could reasonably expect this result to be accurate to the places listed. ■

Table 2.4

**Newton's Method**

$n$	$p_n$
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

**Convergence using Newton's Method**

Example 1 shows that Newton's method can provide extremely accurate approximations with very few iterations. For that example, only one iteration of Newton's method was needed to give better accuracy than 7 iterations of the fixed-point method. It is now time to examine Newton's method more carefully to discover why it is so effective.

The Taylor series derivation of Newton's method at the beginning of the section points out the importance of an accurate initial approximation. The crucial assumption is that the term involving  $(p - p_0)^2$  is, by comparison with  $|p - p_0|$ , so small that it can be deleted. This will clearly be false unless  $p_0$  is a good approximation to  $p$ . If  $p_0$  is not sufficiently close to the actual root, there is little reason to suspect that Newton's method will converge to the root. However, in some instances, even poor initial approximations will produce convergence. (Exercises 20 and 21 illustrate some of these possibilities.)

The following convergence theorem for Newton's method illustrates the theoretical importance of the choice of  $p_0$ .

**Theorem 2.6** Let  $f \in C^2[a, b]$ . If  $p \in (a, b)$  is such that  $f(p) = 0$  and  $f'(p) \neq 0$ , then there exists a  $\delta > 0$  such that Newton's method generates a sequence  $\{p_n\}_{n=1}^{\infty}$  converging to  $p$  for any initial approximation  $p_0 \in [p - \delta, p + \delta]$ . ■

**Proof** The proof is based on analyzing Newton's method as the functional iteration scheme  $p_n = g(p_{n-1})$ , for  $n \geq 1$ , with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Let  $k$  be in  $(0, 1)$ . We first find an interval  $[p - \delta, p + \delta]$  that  $g$  maps into itself and for which  $|g'(x)| \leq k$ , for all  $x \in (p - \delta, p + \delta)$ .

Since  $f'$  is continuous and  $f'(p) \neq 0$ , part (a) of Exercise 29 in Section 1.1 implies that there exists a  $\delta_1 > 0$ , such that  $f'(x) \neq 0$  for  $x \in [p - \delta_1, p + \delta_1] \subseteq [a, b]$ . Thus  $g$  is defined and continuous on  $[p - \delta_1, p + \delta_1]$ . Also

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for  $x \in [p - \delta_1, p + \delta_1]$ , and, since  $f \in C^2[a, b]$ , we have  $g \in C^1[p - \delta_1, p + \delta_1]$ .

By assumption,  $f(p) = 0$ , so

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Since  $g'$  is continuous and  $0 < k < 1$ , part (b) of Exercise 29 in Section 1.1 implies that there exists a  $\delta$ , with  $0 < \delta < \delta_1$ , and

$$|g'(x)| \leq k, \quad \text{for all } x \in [p - \delta, p + \delta].$$

It remains to show that  $g$  maps  $[p - \delta, p + \delta]$  into  $[p - \delta, p + \delta]$ . If  $x \in [p - \delta, p + \delta]$ , the Mean Value Theorem implies that for some number  $\xi$  between  $x$  and  $p$ ,  $|g(x) - g(p)| = |g'(\xi)||x - p|$ . So

$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)||x - p| \leq k|x - p| < |x - p|.$$

Since  $x \in [p - \delta, p + \delta]$ , it follows that  $|x - p| < \delta$  and that  $|g(x) - p| < \delta$ . Hence,  $g$  maps  $[p - \delta, p + \delta]$  into  $[p - \delta, p + \delta]$ .

All the hypotheses of the Fixed-Point Theorem 2.4 are now satisfied, so the sequence  $\{p_n\}_{n=1}^{\infty}$ , defined by

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1,$$

converges to  $p$  for any  $p_0 \in [p - \delta, p + \delta]$ . ■ ■ ■

Theorem 2.6 states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen. It also implies that the constant  $k$  that bounds the derivative of  $g$ , and, consequently, indicates the speed of convergence of the method, decreases to 0 as the procedure continues. This result is important for the theory of Newton's method, but it is seldom applied in practice because it does not tell us how to determine  $\delta$ .

In a practical application, an initial approximation is selected and successive approximations are generated by Newton's method. These will generally either converge quickly to the root, or it will be clear that convergence is unlikely.

### The Secant Method

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of  $f$  at each approximation. Frequently,  $f'(x)$  is far more difficult and needs more arithmetic operations to calculate than  $f(x)$ .

To circumvent the problem of the derivative evaluation in Newton's method, we introduce a slight variation. By definition,

$$f'(p_{n-1}) = \lim_{x \rightarrow p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

If  $p_{n-2}$  is close to  $p_{n-1}$ , then

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

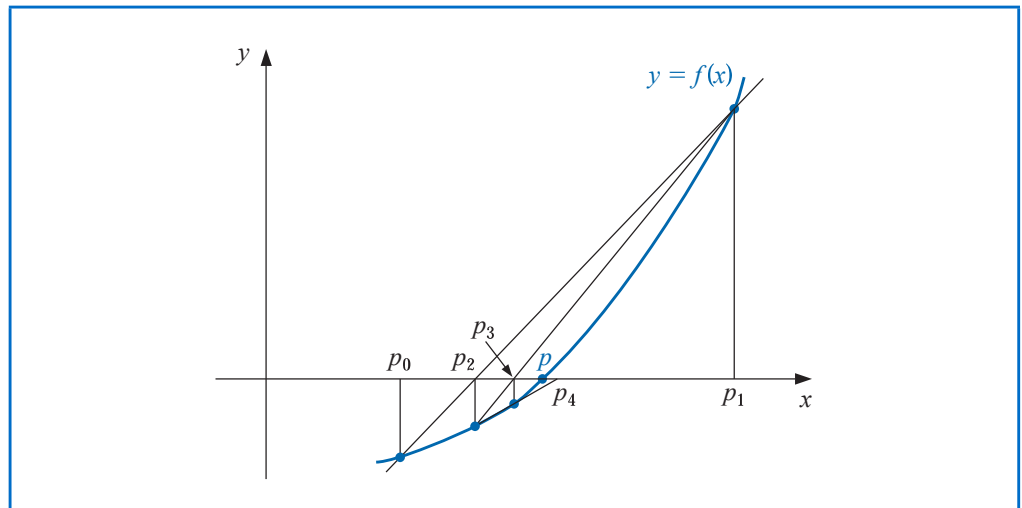
Using this approximation for  $f'(p_{n-1})$  in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}. \tag{2.12}$$

This technique is called the **Secant method** and is presented in Algorithm 2.4. (See Figure 2.10.) Starting with the two initial approximations  $p_0$  and  $p_1$ , the approximation  $p_2$  is the  $x$ -intercept of the line joining  $(p_0, f(p_0))$  and  $(p_1, f(p_1))$ . The approximation  $p_3$  is the  $x$ -intercept of the line joining  $(p_1, f(p_1))$  and  $(p_2, f(p_2))$ , and so on. Note that only one function evaluation is needed per step for the Secant method after  $p_2$  has been determined. In contrast, each step of Newton's method requires an evaluation of both the function and its derivative.

The word secant is derived from the Latin word *secan*, which means to cut. The secant method uses a secant line, a line joining two points that cut the curve, to approximate a root.

Figure 2.10



**ALGORITHM**  
**2.4**

### Secant

To find a solution to  $f(x) = 0$  given initial approximations  $p_0$  and  $p_1$ :

**INPUT** initial approximations  $p_0, p_1$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $i = 2$ ;  
 $q_0 = f(p_0)$ ;  
 $q_1 = f(p_1)$ .

**Step 2** While  $i \leq N_0$  do Steps 3–6.

**Step 3** Set  $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$ . (Compute  $p_i$ .)

**Step 4** If  $|p - p_1| < TOL$  then  
**OUTPUT** ( $p$ ); (The procedure was successful.)  
**STOP**.

**Step 5** Set  $i = i + 1$ .

**Step 6** Set  $p_0 = p_1$ ; (Update  $p_0, q_0, p_1, q_1$ .)  
 $q_0 = q_1$ ;  
 $p_1 = p$ ;  
 $q_1 = f(p)$ .

**Step 7** **OUTPUT** ('The method failed after  $N_0$  iterations,  $N_0 =$ ,  $N_0$ );  
(The procedure was unsuccessful.)  
**STOP**.

The next example involves a problem considered in Example 1, where we used Newton's method with  $p_0 = \pi/4$ .

**Example 2** Use the Secant method to find a solution to  $x = \cos x$ , and compare the approximations with those given in Example 1 which applied Newton's method.

**Table 2.5**

Secant	
$n$	$p_n$
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

Newton	
$n$	$p_n$
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

**Solution** In Example 1 we compared fixed-point iteration and Newton's method starting with the initial approximation  $p_0 = \pi/4$ . For the Secant method we need two initial approximations. Suppose we use  $p_0 = 0.5$  and  $p_1 = \pi/4$ . Succeeding approximations are generated by the formula

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2.$$

These give the results in Table 2.5.

Comparing the results in Table 2.5 from the Secant method and Newton's method, we see that the Secant method approximation  $p_5$  is accurate to the tenth decimal place, whereas Newton's method obtained this accuracy by  $p_3$ . For this example, the convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method. This is generally the case. (See Exercise 14 of Section 2.4.)

Newton's method or the Secant method is often used to refine an answer obtained by another technique, such as the Bisection method, since these methods require good first approximations but generally give rapid convergence.



## The Method of False Position

Each successive pair of approximations in the Bisection method brackets a root  $p$  of the equation; that is, for each positive integer  $n$ , a root lies between  $a_n$  and  $b_n$ . This implies that, for each  $n$ , the Bisection method iterations satisfy

$$|p_n - p| < \frac{1}{2}|a_n - b_n|,$$

which provides an easily calculated error bound for the approximations.

Root bracketing is not guaranteed for either Newton's method or the Secant method. In Example 1, Newton's method was applied to  $f(x) = \cos x - x$ , and an approximate root was found to be 0.7390851332. Table 2.5 shows that this root is not bracketed by either  $p_0$  and  $p_1$  or  $p_1$  and  $p_2$ . The Secant method approximations for this problem are also given in Table 2.5. In this case the initial approximations  $p_0$  and  $p_1$  bracket the root, but the pair of approximations  $p_3$  and  $p_4$  fail to do so.

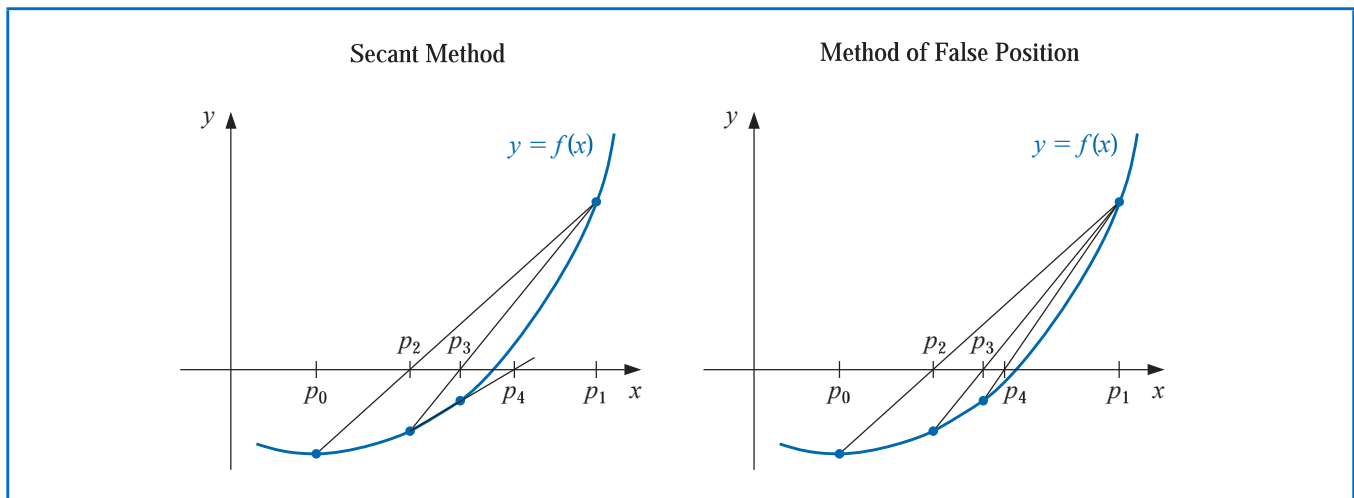
The **method of False Position** (also called *Regula Falsi*) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is always bracketed between successive iterations. Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.

First choose initial approximations  $p_0$  and  $p_1$  with  $f(p_0) \cdot f(p_1) < 0$ . The approximation  $p_2$  is chosen in the same manner as in the Secant method, as the  $x$ -intercept of the line joining  $(p_0, f(p_0))$  and  $(p_1, f(p_1))$ . To decide which secant line to use to compute  $p_3$ , consider  $f(p_2) \cdot f(p_1)$ , or more correctly  $\text{sgn } f(p_2) \cdot \text{sgn } f(p_1)$ .

- If  $\text{sgn } f(p_2) \cdot \text{sgn } f(p_1) < 0$ , then  $p_1$  and  $p_2$  bracket a root. Choose  $p_3$  as the  $x$ -intercept of the line joining  $(p_1, f(p_1))$  and  $(p_2, f(p_2))$ .
- If not, choose  $p_3$  as the  $x$ -intercept of the line joining  $(p_0, f(p_0))$  and  $(p_2, f(p_2))$ , and then interchange the indices on  $p_0$  and  $p_1$ .

In a similar manner, once  $p_3$  is found, the sign of  $f(p_3) \cdot f(p_2)$  determines whether we use  $p_2$  and  $p_3$  or  $p_3$  and  $p_1$  to compute  $p_4$ . In the latter case a relabeling of  $p_2$  and  $p_1$  is performed. The relabeling ensures that the root is bracketed between successive iterations. The process is described in Algorithm 2.5, and Figure 2.11 shows how the iterations can differ from those of the Secant method. In this illustration, the first three approximations are the same, but the fourth approximations differ.

Figure 2.11



**ALGORITHM**  
**2.5**

### False Position

To find a solution to  $f(x) = 0$  given the continuous function  $f$  on the interval  $[p_0, p_1]$  where  $f(p_0)$  and  $f(p_1)$  have opposite signs:

**INPUT** initial approximations  $p_0, p_1$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $i = 2$ ;

$$q_0 = f(p_0);$$

$$q_1 = f(p_1).$$

**Step 2** While  $i \leq N_0$  do Steps 3–7.

**Step 3** Set  $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$ . (Compute  $p_i$ .)

**Step 4** If  $|p - p_1| < TOL$  then

OUTPUT ( $p$ ); (The procedure was successful.)

STOP.

**Step 5** Set  $i = i + 1$ ;

$$q = f(p).$$

**Step 6** If  $q \cdot q_1 < 0$  then set  $p_0 = p_1$ ;

$$q_0 = q_1.$$

**Step 7** Set  $p_1 = p$ ;

$$q_1 = q.$$

**Step 8** OUTPUT ('Method failed after  $N_0$  iterations,  $N_0 =$ ',  $N_0$ );

(The procedure unsuccessful.)

STOP.

**Example 3** Use the method of False Position to find a solution to  $x = \cos x$ , and compare the approximations with those given in Example 1 which applied fixed-point iteration and Newton's method, and to those found in Example 2 which applied the Secant method.

**Solution** To make a reasonable comparison we will use the same initial approximations as in the Secant method, that is,  $p_0 = 0.5$  and  $p_1 = \pi/4$ . Table 2.6 shows the results of the method of False Position applied to  $f(x) = \cos x - x$  together with those we obtained using the Secant and Newton's methods. Notice that the False Position and Secant approximations agree through  $p_3$  and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method.

**Table 2.6**

	False Position	Secant	Newton
$n$	$p_n$	$p_n$	$p_n$
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

The added insurance of the method of False Position commonly requires more calculation than the Secant method, just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations. Further examples of the positive and negative features of these methods can be seen by working Exercises 17 and 18.

Maple has Newton's method, the Secant method, and the method of False Position implemented in its *NumericalAnalysis* package. The options that were available for the Bisection method are also available for these techniques. For example, to generate the results in Tables 2.4, 2.5, and 2.6 we could use the commands

*with(Student[NumericalAnalysis])*

$f := \cos(x) - x$

*Newton*  $\left(f, x = \frac{\pi}{4.0}, \text{tolerance} = 10^{-8}, \text{output} = \text{sequence}, \text{maxiterations} = 20\right)$

*Secant*  $\left(f, x = \left[0.5, \frac{\pi}{4.0}\right], \text{tolerance} = 10^{-8}, \text{output} = \text{sequence}, \text{maxiterations} = 20\right)$

and

*FalsePosition*  $\left(f, x = \left[0.5, \frac{\pi}{4.0}\right], \text{tolerance} = 10^{-8}, \text{output} = \text{sequence}, \text{maxiterations} = 20\right)$

## EXERCISE SET 2.3

- Let  $f(x) = x^2 - 6$  and  $p_0 = 1$ . Use Newton's method to find  $p_2$ .
- Let  $f(x) = -x^3 - \cos x$  and  $p_0 = -1$ . Use Newton's method to find  $p_2$ . Could  $p_0 = 0$  be used?
- Let  $f(x) = x^2 - 6$ . With  $p_0 = 3$  and  $p_1 = 2$ , find  $p_3$ .
  - Use the Secant method.
  - Use the method of False Position.
  - Which of **a.** or **b.** is closer to  $\sqrt{6}$ ?
- Let  $f(x) = -x^3 - \cos x$ . With  $p_0 = -1$  and  $p_1 = 0$ , find  $p_3$ .
  - Use the Secant method.
  - Use the method of False Position.
- Use Newton's method to find solutions accurate to within  $10^{-4}$  for the following problems.
 

<b>a.</b> $x^3 - 2x^2 - 5 = 0, [1, 4]$	<b>b.</b> $x^3 + 3x^2 - 1 = 0, [-3, -2]$
<b>c.</b> $x - \cos x = 0, [0, \pi/2]$	<b>d.</b> $x - 0.8 - 0.2 \sin x = 0, [0, \pi/2]$
- Use Newton's method to find solutions accurate to within  $10^{-5}$  for the following problems.
  - $e^x + 2^{-x} + 2 \cos x - 6 = 0$  for  $1 \leq x \leq 2$
  - $\ln(x - 1) + \cos(x - 1) = 0$  for  $1.3 \leq x \leq 2$
  - $2x \cos 2x - (x - 2)^2 = 0$  for  $2 \leq x \leq 3$  and  $3 \leq x \leq 4$
  - $(x - 2)^2 - \ln x = 0$  for  $1 \leq x \leq 2$  and  $e \leq x \leq 4$
  - $e^x - 3x^2 = 0$  for  $0 \leq x \leq 1$  and  $3 \leq x \leq 5$
  - $\sin x - e^{-x} = 0$  for  $0 \leq x \leq 1$ ,  $3 \leq x \leq 4$  and  $6 \leq x \leq 7$
- Repeat Exercise 5 using the Secant method.
- Repeat Exercise 6 using the Secant method.
- Repeat Exercise 5 using the method of False Position.
- Repeat Exercise 6 using the method of False Position.
- Use all three methods in this Section to find solutions to within  $10^{-5}$  for the following problems.
  - $3xe^x = 0$  for  $1 \leq x \leq 2$
  - $2x + 3 \cos x - e^x = 0$  for  $0 \leq x \leq 1$

12. Use all three methods in this Section to find solutions to within  $10^{-7}$  for the following problems.
- $x^2 - 4x + 4 - \ln x = 0$  for  $1 \leq x \leq 2$  and for  $2 \leq x \leq 4$
  - $x + 1 - 2 \sin \pi x = 0$  for  $0 \leq x \leq 1/2$  and for  $1/2 \leq x \leq 1$
13. Use Newton's method to approximate, to within  $10^{-4}$ , the value of  $x$  that produces the point on the graph of  $y = x^2$  that is closest to  $(1, 0)$ . [Hint: Minimize  $[d(x)]^2$ , where  $d(x)$  represents the distance from  $(x, x^2)$  to  $(1, 0)$ .]
14. Use Newton's method to approximate, to within  $10^{-4}$ , the value of  $x$  that produces the point on the graph of  $y = 1/x$  that is closest to  $(2, 1)$ .
15. The following describes Newton's method graphically: Suppose that  $f'(x)$  exists on  $[a, b]$  and that  $f'(x) \neq 0$  on  $[a, b]$ . Further, suppose there exists one  $p \in [a, b]$  such that  $f(p) = 0$ , and let  $p_0 \in [a, b]$  be arbitrary. Let  $p_1$  be the point at which the tangent line to  $f$  at  $(p_0, f(p_0))$  crosses the  $x$ -axis. For each  $n \geq 1$ , let  $p_n$  be the  $x$ -intercept of the line tangent to  $f$  at  $(p_{n-1}, f(p_{n-1}))$ . Derive the formula describing this method.
16. Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x \sin x - \frac{1}{2} \cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$

Iterate using Newton's method until an accuracy of  $10^{-5}$  is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with  $p_0 = 5\pi$  and  $p_0 = 10\pi$ .

17. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in  $[-1, 0]$  and the other in  $[0, 1]$ . Attempt to approximate these zeros to within  $10^{-6}$  using the

- Method of False Position
- Secant method
- Newton's method

Use the endpoints of each interval as the initial approximations in (a) and (b) and the midpoints as the initial approximation in (c).

18. The function  $f(x) = \tan \pi x - 6$  has a zero at  $(1/\pi) \arctan 6 \approx 0.447431543$ . Let  $p_0 = 0$  and  $p_1 = 0.48$ , and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?
- Bisection method
  - Method of False Position
  - Secant method
19. The iteration equation for the Secant method can be written in the simpler form

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in Algorithm 2.4.

20. The equation  $x^2 - 10 \cos x = 0$  has two solutions,  $\pm 1.3793646$ . Use Newton's method to approximate the solutions to within  $10^{-5}$  with the following values of  $p_0$ .
- |                 |                |                |
|-----------------|----------------|----------------|
| a. $p_0 = -100$ | b. $p_0 = -50$ | c. $p_0 = -25$ |
| d. $p_0 = 25$   | e. $p_0 = 50$  | f. $p_0 = 100$ |
21. The equation  $4x^2 - e^x - e^{-x} = 0$  has two positive solutions  $x_1$  and  $x_2$ . Use Newton's method to approximate the solution to within  $10^{-5}$  with the following values of  $p_0$ .

- |                |               |               |
|----------------|---------------|---------------|
| a. $p_0 = -10$ | b. $p_0 = -5$ | c. $p_0 = -3$ |
| d. $p_0 = -1$  | e. $p_0 = 0$  | f. $p_0 = 1$  |
| g. $p_0 = 3$   | h. $p_0 = 5$  | i. $p_0 = 10$ |

22. Use Maple to determine how many iterations of Newton's method with  $p_0 = \pi/4$  are needed to find a root of  $f(x) = \cos x - x$  to within  $10^{-100}$ .
23. The function described by  $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$  has an infinite number of zeros.
- Determine, within  $10^{-6}$ , the only negative zero.
  - Determine, within  $10^{-6}$ , the four smallest positive zeros.
  - Determine a reasonable initial approximation to find the  $n$ th smallest positive zero of  $f$ . [Hint: Sketch an approximate graph of  $f$ .]
  - Use part (c) to determine, within  $10^{-6}$ , the 25th smallest positive zero of  $f$ .
24. Find an approximation for  $\lambda$ , accurate to within  $10^{-4}$ , for the population equation

$$1,564,000 = 1,000,000e^\lambda + \frac{435,000}{\lambda}(e^\lambda - 1),$$

discussed in the introduction to this chapter. Use this value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435,000 individuals per year.

25. The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within  $10^{-4}$ .
26. The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation*,

$$A = \frac{P}{i}[(1 + i)^n - 1].$$

In this equation,  $A$  is the amount in the account,  $P$  is the amount regularly deposited, and  $i$  is the rate of interest per period for the  $n$  deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

27. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1 + i)^{-n}],$$

known as an *ordinary annuity equation*. In this equation,  $A$  is the amount of the mortgage,  $P$  is the amount of each payment, and  $i$  is the interest rate per period for the  $n$  payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

28. A drug administered to a patient produces a concentration in the blood stream given by  $c(t) = Ate^{-t/3}$  milligrams per milliliter,  $t$  hours after  $A$  units have been injected. The maximum safe concentration is 1 mg/mL.
- What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?
  - An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/mL. Determine, to the nearest minute, when this second injection should be given.
  - Assume that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection. When is it time for the third injection?
29. Let  $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$ .
- Use the Maple commands *solve* and *fsolve* to try to find all roots of  $f$ .
  - Plot  $f(x)$  to find initial approximations to roots of  $f$ .

- c. Use Newton's method to find roots of  $f$  to within  $10^{-16}$ .
- d. Find the exact solutions of  $f(x) = 0$  without using Maple.
30. Repeat Exercise 29 using  $f(x) = 2x^2 - 3 \cdot 7^{x+1}$ .
31. The logistic population growth model is described by an equation of the form

$$P(t) = \frac{P_L}{1 - ce^{-kt}},$$

where  $P_L$ ,  $c$ , and  $k > 0$  are constants, and  $P(t)$  is the population at time  $t$ .  $P_L$  represents the limiting value of the population since  $\lim_{t \rightarrow \infty} P(t) = P_L$ . Use the census data for the years 1950, 1960, and 1970 listed in the table on page 105 to determine the constants  $P_L$ ,  $c$ , and  $k$  for a logistic growth model. Use the logistic model to predict the population of the United States in 1980 and in 2010, assuming  $t = 0$  at 1950. Compare the 1980 prediction to the actual value.

32. The Gompertz population growth model is described by

$$P(t) = P_L e^{-ce^{-kt}},$$

where  $P_L$ ,  $c$ , and  $k > 0$  are constants, and  $P(t)$  is the population at time  $t$ . Repeat Exercise 31 using the Gompertz growth model in place of the logistic model.

33. Player A will shut out (win by a score of 21–0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2} \left( \frac{p}{1-p+p^2} \right)^{21},$$

where  $p$  denotes the probability A will win any specific rally (independent of the server). (See [Keller, J], p. 267.) Determine, to within  $10^{-3}$ , the minimal value of  $p$  that will ensure that A will shut out B in at least half the matches they play.

34. In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called *hang-up failure* and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground.

The accompanying figure, adapted from [Bek], shows the components associated with the nose-in failure of a vehicle. In that reference it is shown that the maximum angle  $\alpha$  that can be negotiated by a vehicle when  $\beta$  is the maximum angle at which hang-up failure does *not* occur satisfies the equation

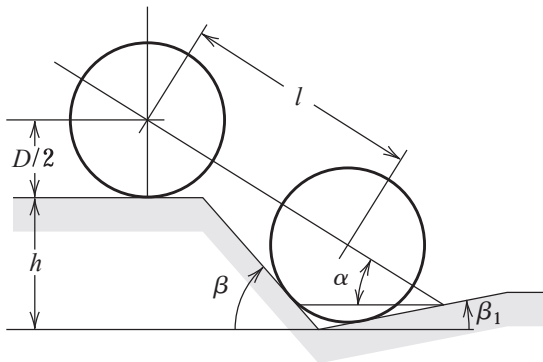
$$A \sin \alpha \cos \alpha + B \sin^2 \alpha - C \cos \alpha - E \sin \alpha = 0,$$

where

$$A = l \sin \beta_1, \quad B = l \cos \beta_1, \quad C = (h + 0.5D) \sin \beta_1 - 0.5D \tan \beta_1,$$

$$\text{and } E = (h + 0.5D) \cos \beta_1 - 0.5D.$$

- a. It is stated that when  $l = 89$  in.,  $h = 49$  in.,  $D = 55$  in., and  $\beta_1 = 11.5^\circ$ , angle  $\alpha$  is approximately  $33^\circ$ . Verify this result.
- b. Find  $\alpha$  for the situation when  $l$ ,  $h$ , and  $\beta_1$  are the same as in part (a) but  $D = 30$  in.



## 2.4 Error Analysis for Iterative Methods

In this section we investigate the order of convergence of functional iteration schemes and, as a means of obtaining rapid convergence, rediscover Newton's method. We also consider ways of accelerating the convergence of Newton's method in special circumstances. First, however, we need a new procedure for measuring how rapidly a sequence converges.

### Order of Convergence

**Definition 2.7** Suppose  $\{p_n\}_{n=0}^{\infty}$  is a sequence that converges to  $p$ , with  $p_n \neq p$  for all  $n$ . If positive constants  $\lambda$  and  $\alpha$  exist with

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda,$$

then  $\{p_n\}_{n=0}^{\infty}$  **converges to  $p$  of order  $\alpha$ , with asymptotic error constant  $\lambda$** . ■

An iterative technique of the form  $p_n = g(p_{n-1})$  is said to be of *order  $\alpha$*  if the sequence  $\{p_n\}_{n=0}^{\infty}$  converges to the solution  $p = g(p)$  of order  $\alpha$ .

In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order. The asymptotic constant affects the speed of convergence but not to the extent of the order. Two cases of order are given special attention.

- (i) If  $\alpha = 1$  (and  $\lambda < 1$ ), the sequence is **linearly convergent**.
- (ii) If  $\alpha = 2$ , the sequence is **quadratically convergent**.

The next illustration compares a linearly convergent sequence to one that is quadratically convergent. It shows why we try to find methods that produce higher-order convergent sequences.

**Illustration** Suppose that  $\{p_n\}_{n=0}^{\infty}$  is linearly convergent to 0 with

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1}|}{|p_n|} = 0.5$$

and that  $\{\tilde{p}_n\}_{n=0}^{\infty}$  is quadratically convergent to 0 with the same asymptotic error constant,

$$\lim_{n \rightarrow \infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = 0.5.$$

For simplicity we assume that for each  $n$  we have

$$\frac{|p_{n+1}|}{|p_n|} \approx 0.5 \quad \text{and} \quad \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} \approx 0.5.$$

For the linearly convergent scheme, this means that

$$|p_n - 0| = |p_n| \approx 0.5|p_{n-1}| \approx (0.5)^2|p_{n-2}| \approx \dots \approx (0.5)^n|p_0|,$$

whereas the quadratically convergent procedure has

$$\begin{aligned} |\tilde{p}_n - 0| = |\tilde{p}_n| &\approx 0.5|\tilde{p}_{n-1}|^2 \approx (0.5)[0.5|\tilde{p}_{n-2}|^2]^2 = (0.5)^3|\tilde{p}_{n-2}|^4 \\ &\approx (0.5)^3[(0.5)|\tilde{p}_{n-3}|^2]^4 = (0.5)^7|\tilde{p}_{n-3}|^8 \\ &\approx \dots \approx (0.5)^{2^n-1}|\tilde{p}_0|^{2^n}. \end{aligned}$$

Table 2.7 illustrates the relative speed of convergence of the sequences to 0 if  $|p_0| = |\tilde{p}_0| = 1$ .

**Table 2.7**

$n$	Linear Convergence	Quadratic Convergence
	Sequence $\{p_n\}_{n=0}^{\infty}$ $(0.5)^n$	Sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ $(0.5)^{2^n - 1}$
1	$5.0000 \times 10^{-1}$	$5.0000 \times 10^{-1}$
2	$2.5000 \times 10^{-1}$	$1.2500 \times 10^{-1}$
3	$1.2500 \times 10^{-1}$	$7.8125 \times 10^{-3}$
4	$6.2500 \times 10^{-2}$	$3.0518 \times 10^{-5}$
5	$3.1250 \times 10^{-2}$	$4.6566 \times 10^{-10}$
6	$1.5625 \times 10^{-2}$	$1.0842 \times 10^{-19}$
7	$7.8125 \times 10^{-3}$	$5.8775 \times 10^{-39}$

The quadratically convergent sequence is within  $10^{-38}$  of 0 by the seventh term. At least 126 terms are needed to ensure this accuracy for the linearly convergent sequence.  $\square$

Quadratically convergent sequences are expected to converge much quicker than those that converge only linearly, but the next result implies that an arbitrary technique that generates a convergent sequences does so only linearly.

**Theorem 2.8** Let  $g \in C[a, b]$  be such that  $g(x) \in [a, b]$ , for all  $x \in [a, b]$ . Suppose, in addition, that  $g'$  is continuous on  $(a, b)$  and a positive constant  $k < 1$  exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

If  $g'(p) \neq 0$ , then for any number  $p_0 \neq p$  in  $[a, b]$ , the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

converges only linearly to the unique fixed point  $p$  in  $[a, b]$ .  $\blacksquare$

**Proof** We know from the Fixed-Point Theorem 2.4 in Section 2.2 that the sequence converges to  $p$ . Since  $g'$  exists on  $(a, b)$ , we can apply the Mean Value Theorem to  $g$  to show that for any  $n$ ,

$$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p),$$

where  $\xi_n$  is between  $p_n$  and  $p$ . Since  $\{p_n\}_{n=0}^{\infty}$  converges to  $p$ , we also have  $\{\xi_n\}_{n=0}^{\infty}$  converging to  $p$ . Since  $g'$  is continuous on  $(a, b)$ , we have

$$\lim_{n \rightarrow \infty} g'(\xi_n) = g'(p).$$

Thus

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - p}{p_n - p} = \lim_{n \rightarrow \infty} g'(\xi_n) = g'(p) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|.$$

Hence, if  $g'(p) \neq 0$ , fixed-point iteration exhibits linear convergence with asymptotic error constant  $|g'(p)|$ .  $\blacksquare \blacksquare \blacksquare$



Theorem 2.8 implies that higher-order convergence for fixed-point methods of the form  $g(p) = p$  can occur only when  $g'(p) = 0$ . The next result describes additional conditions that ensure the quadratic convergence we seek.

**Theorem 2.9** Let  $p$  be a solution of the equation  $x = g(x)$ . Suppose that  $g'(p) = 0$  and  $g''$  is continuous with  $|g''(x)| < M$  on an open interval  $I$  containing  $p$ . Then there exists a  $\delta > 0$  such that, for  $p_0 \in [p - \delta, p + \delta]$ , the sequence defined by  $p_n = g(p_{n-1})$ , when  $n \geq 1$ , converges at least quadratically to  $p$ . Moreover, for sufficiently large values of  $n$ ,

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2. \quad \blacksquare$$

**Proof** Choose  $k$  in  $(0, 1)$  and  $\delta > 0$  such that on the interval  $[p - \delta, p + \delta]$ , contained in  $I$ , we have  $|g'(x)| \leq k$  and  $g''$  continuous. Since  $|g'(x)| \leq k < 1$ , the argument used in the proof of Theorem 2.6 in Section 2.3 shows that the terms of the sequence  $\{p_n\}_{n=0}^{\infty}$  are contained in  $[p - \delta, p + \delta]$ . Expanding  $g(x)$  in a linear Taylor polynomial for  $x \in [p - \delta, p + \delta]$  gives

$$g(x) = g(p) + g'(p)(x - p) + \frac{g''(\xi)}{2}(x - p)^2,$$

where  $\xi$  lies between  $x$  and  $p$ . The hypotheses  $g(p) = p$  and  $g'(p) = 0$  imply that

$$g(x) = p + \frac{g''(\xi)}{2}(x - p)^2.$$

In particular, when  $x = p_n$ ,

$$p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n - p)^2,$$

with  $\xi_n$  between  $p_n$  and  $p$ . Thus,

$$p_{n+1} - p = \frac{g''(\xi_n)}{2}(p_n - p)^2.$$

Since  $|g'(x)| \leq k < 1$  on  $[p - \delta, p + \delta]$  and  $g$  maps  $[p - \delta, p + \delta]$  into itself, it follows from the Fixed-Point Theorem that  $\{p_n\}_{n=0}^{\infty}$  converges to  $p$ . But  $\xi_n$  is between  $p$  and  $p_n$  for each  $n$ , so  $\{\xi_n\}_{n=0}^{\infty}$  also converges to  $p$ , and

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2}.$$

This result implies that the sequence  $\{p_n\}_{n=0}^{\infty}$  is quadratically convergent if  $g''(p) \neq 0$  and of higher-order convergence if  $g''(p) = 0$ .

Because  $g''$  is continuous and strictly bounded by  $M$  on the interval  $[p - \delta, p + \delta]$ , this also implies that, for sufficiently large values of  $n$ ,

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2. \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

Theorems 2.8 and 2.9 tell us that our search for quadratically convergent fixed-point methods should point in the direction of functions whose derivatives are zero at the fixed point. That is:

- For a fixed point method to converge quadratically we need to have both  $g(p) = p$ , and  $g'(p) = 0$ .

The easiest way to construct a fixed-point problem associated with a root-finding problem  $f(x) = 0$  is to add or subtract a multiple of  $f(x)$  from  $x$ . Consider the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geq 1,$$

for  $g$  in the form

$$g(x) = x - \phi(x)f(x),$$

where  $\phi$  is a differentiable function that will be chosen later.

For the iterative procedure derived from  $g$  to be quadratically convergent, we need to have  $g'(p) = 0$  when  $f(p) = 0$ . Because

$$g'(x) = 1 - \phi'(x)f(x) - f'(x)\phi(x),$$

and  $f(p) = 0$ , we have

$$g'(p) = 1 - \phi'(p)f(p) - f'(p)\phi(p) = 1 - \phi'(p) \cdot 0 - f'(p)\phi(p) = 1 - f'(p)\phi(p),$$

and  $g'(p) = 0$  if and only if  $\phi(p) = 1/f'(p)$ .

If we let  $\phi(x) = 1/f'(x)$ , then we will ensure that  $\phi(p) = 1/f'(p)$  and produce the quadratically convergent procedure

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

This, of course, is simply Newton's method. Hence

- If  $f(p) = 0$  and  $f'(p) \neq 0$ , then for starting values sufficiently close to  $p$ , Newton's method will converge at least quadratically.

## Multiple Roots

In the preceding discussion, the restriction was made that  $f'(p) \neq 0$ , where  $p$  is the solution to  $f(x) = 0$ . In particular, Newton's method and the Secant method will generally give problems if  $f'(p) = 0$  when  $f(p) = 0$ . To examine these difficulties in more detail, we make the following definition.

**Definition 2.10** A solution  $p$  of  $f(x) = 0$  is a **zero of multiplicity**  $m$  of  $f$  if for  $x \neq p$ , we can write  $f(x) = (x - p)^m q(x)$ , where  $\lim_{x \rightarrow p} q(x) \neq 0$ . ■

For polynomials,  $p$  is a zero of multiplicity  $m$  of  $f$  if  $f(x) = (x - p)^m q(x)$ , where  $q(p) \neq 0$ .

In essence,  $q(x)$  represents that portion of  $f(x)$  that does not contribute to the zero of  $f$ . The following result gives a means to easily identify **simple** zeros of a function, those that have multiplicity one.

**Theorem 2.11** The function  $f \in C^1[a, b]$  has a simple zero at  $p$  in  $(a, b)$  if and only if  $f(p) = 0$ , but  $f'(p) \neq 0$ . ■

**Proof** If  $f$  has a simple zero at  $p$ , then  $f(p) = 0$  and  $f(x) = (x - p)q(x)$ , where  $\lim_{x \rightarrow p} q(x) \neq 0$ . Since  $f \in C^1[a, b]$ ,

$$f'(p) = \lim_{x \rightarrow p} f'(x) = \lim_{x \rightarrow p} [q(x) + (x - p)q'(x)] = \lim_{x \rightarrow p} q(x) \neq 0.$$

Conversely, if  $f(p) = 0$ , but  $f'(p) \neq 0$ , expand  $f$  in a zeroth Taylor polynomial about  $p$ . Then

$$f(x) = f(p) + f'(\xi(x))(x - p) = (x - p)f'(\xi(x)),$$

where  $\xi(x)$  is between  $x$  and  $p$ . Since  $f \in C^1[a, b]$ ,

$$\lim_{x \rightarrow p} f'(\xi(x)) = f'(\lim_{x \rightarrow p} \xi(x)) = f'(p) \neq 0.$$

Letting  $q = f' \circ \xi$  gives  $f(x) = (x - p)q(x)$ , where  $\lim_{x \rightarrow p} q(x) \neq 0$ . Thus  $f$  has a simple zero at  $p$ . ■ ■ ■

The following generalization of Theorem 2.11 is considered in Exercise 12.

**Theorem 2.12** The function  $f \in C^m[a, b]$  has a zero of multiplicity  $m$  at  $p$  in  $(a, b)$  if and only if

$$0 = f(p) = f'(p) = f''(p) = \dots = f^{(m-1)}(p), \quad \text{but} \quad f^{(m)}(p) \neq 0. \quad \blacksquare$$

The result in Theorem 2.12 implies that an interval about  $p$  exists where Newton's method converges quadratically to  $p$  for any initial approximation  $p_0 = p$ , provided that  $p$  is a simple zero. The following example shows that quadratic convergence might not occur if the zero is not simple.

**Example 1** Let  $f(x) = e^x - x - 1$ . (a) Show that  $f$  has a zero of multiplicity 2 at  $x = 0$ . (b) Show that Newton's method with  $p_0 = 1$  converges to this zero but not quadratically.

**Table 2.8**

$n$	$p_n$
0	1.0
1	0.58198
2	0.31906
3	0.16800
4	0.08635
5	0.04380
6	0.02206
7	0.01107
8	0.005545
9	$2.7750 \times 10^{-3}$
10	$1.3881 \times 10^{-3}$
11	$6.9411 \times 10^{-4}$
12	$3.4703 \times 10^{-4}$
13	$1.7416 \times 10^{-4}$
14	$8.8041 \times 10^{-5}$
15	$4.2610 \times 10^{-5}$
16	$1.9142 \times 10^{-6}$

**Solution** (a) We have

$$f(x) = e^x - x - 1, \quad f'(x) = e^x - 1 \quad \text{and} \quad f''(x) = e^x,$$

so

$$f(0) = e^0 - 0 - 1 = 0, \quad f'(0) = e^0 - 1 = 0 \quad \text{and} \quad f''(0) = e^0 = 1.$$

Theorem 2.12 implies that  $f$  has a zero of multiplicity 2 at  $x = 0$ .

(b) The first two terms generated by Newton's method applied to  $f$  with  $p_0 = 1$  are

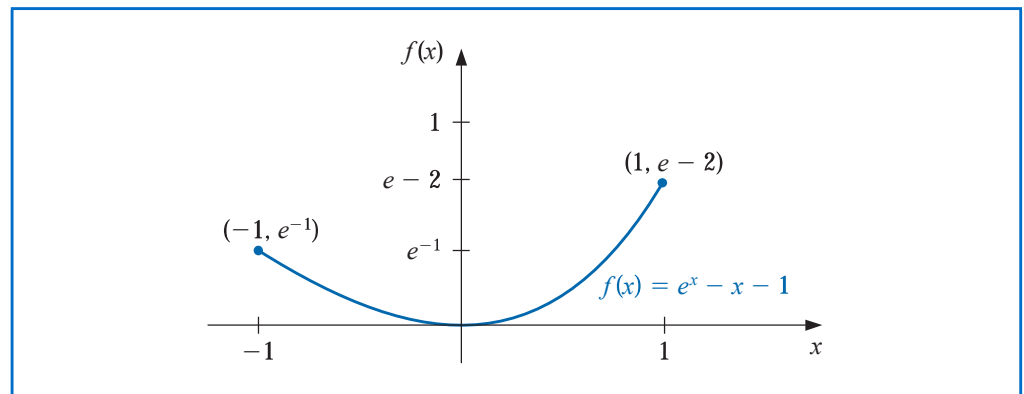
$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{e - 2}{e - 1} \approx 0.58198,$$

and

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} \approx 0.58198 - \frac{0.20760}{0.78957} \approx 0.31906.$$

The first sixteen terms of the sequence generated by Newton's method are shown in Table 2.8. The sequence is clearly converging to 0, but not quadratically. The graph of  $f$  is shown in Figure 2.12. ■

**Figure 2.12**



One method of handling the problem of multiple roots of a function  $f$  is to define

$$\mu(x) = \frac{f(x)}{f'(x)}.$$

If  $p$  is a zero of  $f$  of multiplicity  $m$  with  $f(x) = (x - p)^m q(x)$ , then

$$\begin{aligned}\mu(x) &= \frac{(x - p)^m q(x)}{m(x - p)^{m-1} q(x) + (x - p)^m q'(x)} \\ &= (x - p) \frac{q(x)}{mq(x) + (x - p)q'(x)}\end{aligned}$$

also has a zero at  $p$ . However,  $q(p) \neq 0$ , so

$$\frac{q(p)}{mq(p) + (p - p)q'(p)} = \frac{1}{m} \neq 0,$$

and  $p$  is a simple zero of  $\mu(x)$ . Newton's method can then be applied to  $\mu(x)$  to give

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)/f'(x)}{\{[f'(x)]^2 - [f(x)][f''(x)]\}/[f'(x)]^2}$$

which simplifies to

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}. \quad (2.13)$$

If  $g$  has the required continuity conditions, functional iteration applied to  $g$  will be quadratically convergent regardless of the multiplicity of the zero of  $f$ . Theoretically, the only drawback to this method is the additional calculation of  $f''(x)$  and the more laborious procedure of calculating the iterates. In practice, however, multiple roots can cause serious round-off problems because the denominator of (2.13) consists of the difference of two numbers that are both close to 0.

**Example 2** In Example 1 it was shown that  $f(x) = e^x - x - 1$  has a zero of multiplicity 2 at  $x = 0$  and that Newton's method with  $p_0 = 1$  converges to this zero but not quadratically. Show that the modification of Newton's method as given in Eq. (2.13) improves the rate of convergence.

**Solution** Modified Newton's method gives

$$p_1 = p_0 - \frac{f(p_0)f'(p_0)}{f'(p_0)^2 - f(p_0)f''(p_0)} = 1 - \frac{(e - 2)(e - 1)}{(e - 1)^2 - (e - 2)e} \approx -2.3421061 \times 10^{-1}.$$

This is considerably closer to 0 than the first term using Newton's method, which was 0.58918. Table 2.9 lists the first five approximations to the double zero at  $x = 0$ . The results were obtained using a system with ten digits of precision. The relative lack of improvement in the last two entries is due to the fact that using this system both the numerator and the denominator approach 0. Consequently there is a loss of significant digits of accuracy as the approximations approach 0. ■

The following illustrates that the modified Newton's method converges quadratically even when in the case of a simple zero.

**Illustration** In Section 2.2 we found that a zero of  $f(x) = x^3 + 4x^2 - 10 = 0$  is  $p = 1.36523001$ . Here we will compare convergence for a simple zero using both Newton's method and the modified Newton's method listed in Eq. (2.13). Let

**Table 2.9**

$n$	$p_n$
1	$-2.3421061 \times 10^{-1}$
2	$-8.4582788 \times 10^{-3}$
3	$-1.1889524 \times 10^{-5}$
4	$-6.8638230 \times 10^{-6}$
5	$-2.8085217 \times 10^{-7}$

$$(i) \quad p_n = p_{n-1} - \frac{p_{n-1}^3 + 4p_{n-1}^2 - 10}{3p_{n-1}^2 + 8p_{n-1}}, \quad \text{from Newton's method}$$

and, from the Modified Newton's method given by Eq. (2.13),

$$(ii) \quad p_n = p_{n-1} - \frac{(p_{n-1}^3 + 4p_{n-1}^2 - 10)(3p_{n-1}^2 + 8p_{n-1})}{(3p_{n-1}^2 + 8p_{n-1})^2 - (p_{n-1}^3 + 4p_{n-1}^2 - 10)(6p_{n-1} + 8)}.$$

With  $p_0 = 1.5$ , we have

#### Newton's method

$$p_1 = 1.37333333, \quad p_2 = 1.36526201, \quad \text{and} \quad p_3 = 1.36523001.$$

#### Modified Newton's method

$$p_1 = 1.35689898, \quad p_2 = 1.36519585, \quad \text{and} \quad p_3 = 1.36523001.$$

Both methods are rapidly convergent to the actual zero, which is given by both methods as  $p_3$ . Note, however, that in the case of a simple zero the original Newton's method requires substantially less computation.  $\square$

Maple contains Modified Newton's method as described in Eq. (2.13) in its *Numerical-Analysis* package. The options for this command are the same as those for the Bisection method. To obtain results similar to those in Table 2.9 we can use

`with(Student[NumericalAnalysis])`

$$f := e^x - x - 1$$

`ModifiedNewton(f, x = 1.0, tolerance = 10-10, output = sequence, maxiterations = 20)`

Remember that there is sensitivity to round-off error in these calculations, so you might need to reset *Digits* in Maple to get the exact values in Table 2.9.

## EXERCISE SET 2.4

- Use Newton's method to find solutions accurate to within  $10^{-5}$  to the following problems.
  - $x^2 - 2xe^{-x} + e^{-2x} = 0$ , for  $0 \leq x \leq 1$
  - $\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0$ , for  $-2 \leq x \leq -1$
  - $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) - 8^{-x} = 0$ , for  $0 \leq x \leq 1$
  - $e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3 = 0$ , for  $-1 \leq x \leq 0$
- Use Newton's method to find solutions accurate to within  $10^{-5}$  to the following problems.
  - $1 - 4x \cos x + 2x^2 + \cos 2x = 0$ , for  $0 \leq x \leq 1$
  - $x^2 + 6x^5 + 9x^4 - 2x^3 - 6x^2 + 1 = 0$ , for  $-3 \leq x \leq -2$
  - $\sin 3x + 3e^{-2x} \sin x - 3e^{-x} \sin 2x - e^{-3x} = 0$ , for  $3 \leq x \leq 4$
  - $e^{3x} - 27x^6 + 27x^4 e^x - 9x^2 e^{2x} = 0$ , for  $3 \leq x \leq 5$
- Repeat Exercise 1 using the modified Newton's method described in Eq. (2.13). Is there an improvement in speed or accuracy over Exercise 1?

4. Repeat Exercise 2 using the modified Newton's method described in Eq. (2.13). Is there an improvement in speed or accuracy over Exercise 2?
5. Use Newton's method and the modified Newton's method described in Eq. (2.13) to find a solution accurate to within  $10^{-5}$  to the problem

$$e^{6x} + 1.441e^{2x} - 2.079e^{4x} - 0.3330 = 0, \quad \text{for } -1 \leq x \leq 0.$$

This is the same problem as 1(d) with the coefficients replaced by their four-digit approximations. Compare the solutions to the results in 1(d) and 2(d).

6. Show that the following sequences converge linearly to  $p = 0$ . How large must  $n$  be before  $|p_n - p| \leq 5 \times 10^{-2}$ ?
  - a.  $p_n = \frac{1}{n}, \quad n \geq 1$
  - b.  $p_n = \frac{1}{n^2}, \quad n \geq 1$
7.
  - a. Show that for any positive integer  $k$ , the sequence defined by  $p_n = 1/n^k$  converges linearly to  $p = 0$ .
  - b. For each pair of integers  $k$  and  $m$ , determine a number  $N$  for which  $1/N^k < 10^{-m}$ .
8.
  - a. Show that the sequence  $p_n = 10^{-2^n}$  converges quadratically to 0.
  - b. Show that the sequence  $p_n = 10^{-n^k}$  does not converge to 0 quadratically, regardless of the size of the exponent  $k > 1$ .
9.
  - a. Construct a sequence that converges to 0 of order 3.
  - b. Suppose  $\alpha > 1$ . Construct a sequence that converges to 0 zero of order  $\alpha$ .
10. Suppose  $p$  is a zero of multiplicity  $m$  of  $f$ , where  $f^{(m)}$  is continuous on an open interval containing  $p$ . Show that the following fixed-point method has  $g'(p) = 0$ :

$$g(x) = x - \frac{mf(x)}{f'(x)}.$$

11. Show that the Bisection Algorithm 2.1 gives a sequence with an error bound that converges linearly to 0.
12. Suppose that  $f$  has  $m$  continuous derivatives. Modify the proof of Theorem 2.11 to show that  $f$  has a zero of multiplicity  $m$  at  $p$  if and only if

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \quad \text{but } f^{(m)}(p) \neq 0.$$

13. The iterative method to solve  $f(x) = 0$ , given by the fixed-point method  $g(x) = x$ , where

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} - \frac{f''(p_{n-1})}{2f'(p_{n-1})} \left[ \frac{f(p_{n-1})}{f'(p_{n-1})} \right]^2, \quad \text{for } n = 1, 2, 3, \dots,$$

has  $g'(p) = g''(p) = 0$ . This will generally yield cubic ( $\alpha = 3$ ) convergence. Expand the analysis of Example 1 to compare quadratic and cubic convergence.

14. It can be shown (see, for example, [DaB], pp. 228–229) that if  $\{p_n\}_{n=0}^{\infty}$  are convergent Secant method approximations to  $p$ , the solution to  $f(x) = 0$ , then a constant  $C$  exists with  $|p_{n+1} - p| \approx C|p_n - p||p_{n-1} - p|$  for sufficiently large values of  $n$ . Assume  $\{p_n\}$  converges to  $p$  of order  $\alpha$ , and show that  $\alpha = (1 + \sqrt{5})/2$ . (Note: This implies that the order of convergence of the Secant method is approximately 1.62).

## 2.5 Accelerating Convergence

Theorem 2.8 indicates that it is rare to have the luxury of quadratic convergence. We now consider a technique called **Aitken's  $\Delta^2$  method** that can be used to accelerate the convergence of a sequence that is linearly convergent, regardless of its origin or application.

Alexander Aitken (1895-1967) used this technique in 1926 to accelerate the rate of convergence of a series in a paper on algebraic equations [Ai]. This process is similar to one used much earlier by the Japanese mathematician Takakazu Seki Kowa (1642-1708).

### Aitken's $\Delta^2$ Method

Suppose  $\{p_n\}_{n=0}^{\infty}$  is a linearly convergent sequence with limit  $p$ . To motivate the construction of a sequence  $\{\hat{p}_n\}_{n=0}^{\infty}$  that converges more rapidly to  $p$  than does  $\{p_n\}_{n=0}^{\infty}$ , let us first assume that the signs of  $p_n - p$ ,  $p_{n+1} - p$ , and  $p_{n+2} - p$  agree and that  $n$  is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Then

$$(p_{n+1} - p)^2 \approx (p_{n+2} - p)(p_n - p),$$

so

$$p_{n+1}^2 - 2p_{n+1}p + p^2 \approx p_{n+2}p_n - (p_n + p_{n+2})p + p^2$$

and

$$(p_{n+2} + p_n - 2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2.$$

Solving for  $p$  gives

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Adding and subtracting the terms  $p_n^2$  and  $2p_n p_{n+1}$  in the numerator and grouping terms appropriately gives

$$\begin{aligned} p &\approx \frac{p_n p_{n+2} - 2p_n p_{n+1} + p_n^2 - p_{n+1}^2 + 2p_n p_{n+1} - p_n^2}{p_{n+2} - 2p_{n+1} + p_n} \\ &= \frac{p_n(p_{n+2} - 2p_{n+1} + p_n) - (p_{n+1}^2 - 2p_n p_{n+1} + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n} \\ &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}. \end{aligned}$$

**Table 2.10**

$n$	$p_n$	$\hat{p}_n$
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

**Aitken's  $\Delta^2$  method** is based on the assumption that the sequence  $\{\hat{p}_n\}_{n=0}^{\infty}$ , defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}, \quad (2.14)$$

converges more rapidly to  $p$  than does the original sequence  $\{p_n\}_{n=0}^{\infty}$ .

**Example 1** The sequence  $\{p_n\}_{n=1}^{\infty}$ , where  $p_n = \cos(1/n)$ , converges linearly to  $p = 1$ . Determine the first five terms of the sequence given by Aitken's  $\Delta^2$  method.

**Solution** In order to determine a term  $\hat{p}_n$  of the Aitken's  $\Delta^2$  method sequence we need to have the terms  $p_n$ ,  $p_{n+1}$ , and  $p_{n+2}$  of the original sequence. So to determine  $\hat{p}_5$  we need the first 7 terms of  $\{p_n\}$ . These are given in Table 2.10. It certainly appears that  $\{\hat{p}_n\}_{n=1}^{\infty}$  converges more rapidly to  $p = 1$  than does  $\{p_n\}_{n=1}^{\infty}$ . ■

The  $\Delta$  notation associated with this technique has its origin in the following definition.

**Definition 2.13** For a given sequence  $\{p_n\}_{n=0}^\infty$ , the **forward difference**  $\Delta p_n$  (read “delta  $p_n$ ”) is defined by

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator  $\Delta$  are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2. \quad \blacksquare$$

The definition implies that

$$\Delta^2 p_n = \Delta(p_{n+1} - p_n) = \Delta p_{n+1} - \Delta p_n = (p_{n+2} - p_{n+1}) - (p_{n+1} - p_n).$$

So  $\Delta^2 p_n = p_{n+2} - 2p_{n+1} + p_n$ , and the formula for  $\hat{p}_n$  given in Eq. (2.14) can be written as

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \quad \text{for } n \geq 0. \quad (2.15)$$

To this point in our discussion of Aitken’s  $\Delta^2$  method, we have stated that the sequence  $\{\hat{p}_n\}_{n=0}^\infty$  converges to  $p$  more rapidly than does the original sequence  $\{p_n\}_{n=0}^\infty$ , but we have not said what is meant by the term “more rapid” convergence. Theorem 2.14 explains and justifies this terminology. The proof of this theorem is considered in Exercise 16.

**Theorem 2.14** Suppose that  $\{p_n\}_{n=0}^\infty$  is a sequence that converges linearly to the limit  $p$  and that

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - p}{p_n - p} < 1.$$

Then the Aitken’s  $\Delta^2$  sequence  $\{\hat{p}_n\}_{n=0}^\infty$  converges to  $p$  faster than  $\{p_n\}_{n=0}^\infty$  in the sense that

$$\lim_{n \rightarrow \infty} \frac{\hat{p}_n - p}{p_n - p} = 0. \quad \blacksquare$$

### Steffensen’s Method

Johan Frederik Steffensen (1873–1961) wrote an influential book entitled *Interpolation* in 1927.

By applying a modification of Aitken’s  $\Delta^2$  method to a linearly convergent sequence obtained from fixed-point iteration, we can accelerate the convergence to quadratic. This procedure is known as Steffensen’s method and differs slightly from applying Aitken’s  $\Delta^2$  method directly to the linearly convergent fixed-point iteration sequence. Aitken’s  $\Delta^2$  method constructs the terms in order:

$$p_0, \quad p_1 = g(p_0), \quad p_2 = g(p_1), \quad \hat{p}_0 = \{\Delta^2\}(p_0), \\ p_3 = g(p_2), \quad \hat{p}_1 = \{\Delta^2\}(p_1), \dots,$$

where  $\{\Delta^2\}$  indicates that Eq. (2.15) is used. Steffensen’s method constructs the same first four terms,  $p_0, p_1, p_2$ , and  $\hat{p}_0$ . However, at this step we assume that  $\hat{p}_0$  is a better approximation to  $p$  than is  $p_2$  and apply fixed-point iteration to  $\hat{p}_0$  instead of  $p_2$ . Using this notation, the sequence is

$$p_0^{(0)}, \quad p_1^{(0)} = g(p_0^{(0)}), \quad p_2^{(0)} = g(p_1^{(0)}), \quad p_0^{(1)} = \{\Delta^2\}(p_0^{(0)}), \quad p_1^{(1)} = g(p_0^{(1)}), \dots$$

Every third term of the Steffensen sequence is generated by Eq. (2.15); the others use fixed-point iteration on the previous term. The process is described in Algorithm 2.6.



**ALGORITHM**  
**2.6**
**Steffensen's**

To find a solution to  $p = g(p)$  given an initial approximation  $p_0$ :

**INPUT** initial approximation  $p_0$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $i = 1$ .

**Step 2** While  $i \leq N_0$  do Steps 3–6.

**Step 3** Set  $p_1 = g(p_0)$ ; (Compute  $p_1^{(i-1)}$ .)  
 $p_2 = g(p_1)$ ; (Compute  $p_2^{(i-1)}$ .)  
 $p = p_0 - (p_1 - p_0)^2 / (p_2 - 2p_1 + p_0)$ . (Compute  $p_0^{(i)}$ .)

**Step 4** If  $|p - p_0| < TOL$  then  
**OUTPUT** ( $p$ ); (Procedure completed successfully.)  
**STOP**.

**Step 5** Set  $i = i + 1$ .

**Step 6** Set  $p_0 = p$ . (Update  $p_0$ .)

**Step 7** **OUTPUT** ('Method failed after  $N_0$  iterations,  $N_0 =$ ,  $N_0$ );  
(Procedure completed unsuccessfully.)  
**STOP**. ■

Note that  $\Delta^2 p_n$  might be 0, which would introduce a 0 in the denominator of the next iterate. If this occurs, we terminate the sequence and select  $p_2^{(n-1)}$  as the best approximation.

**Illustration** To solve  $x^3 + 4x^2 - 10 = 0$  using Steffensen's method, let  $x^3 + 4x^2 = 10$ , divide by  $x + 4$ , and solve for  $x$ . This procedure produces the fixed-point method

$$g(x) = \left( \frac{10}{x + 4} \right)^{1/2}.$$

We considered this fixed-point method in Table 2.2 column (d) of Section 2.2.

Applying Steffensen's procedure with  $p_0 = 1.5$  gives the values in Table 2.11. The iterate  $p_0^{(2)} = 1.365230013$  is accurate to the ninth decimal place. In this example, Steffensen's method gave about the same accuracy as Newton's method applied to this polynomial. These results can be seen in the Illustration at the end of Section 2.4. □

**Table 2.11**

$k$	$p_0^{(k)}$	$p_1^{(k)}$	$p_2^{(k)}$
0	1.5	1.348399725	1.367376372
1	1.365265224	1.365225534	1.365230583
2	1.365230013		

From the Illustration, it appears that Steffensen's method gives quadratic convergence without evaluating a derivative, and Theorem 2.14 states that this is the case. The proof of this theorem can be found in [He2], pp. 90–92, or [IK], pp. 103–107.

**Theorem 2.15** Suppose that  $x = g(x)$  has the solution  $p$  with  $g'(p) \neq 1$ . If there exists a  $\delta > 0$  such that  $g \in C^3[p - \delta, p + \delta]$ , then Steffensen's method gives quadratic convergence for any  $p_0 \in [p - \delta, p + \delta]$ . ■

Steffensen's method can be implemented in Maple with the *NumericalAnalysis* package. For example, after entering the function

$$g := \sqrt{\frac{10}{x+4}}$$

the Maple command

*Steffensen(fixedpointiterator = g, x = 1.5, tolerance = 10<sup>-8</sup>, output = information, maxiterations = 20)*

produces the results in Table 2.11, as well as an indication that the final approximation has a relative error of approximately  $7.32 \times 10^{-10}$ .

## EXERCISE SET 2.5

- The following sequences are linearly convergent. Generate the first five terms of the sequence  $\{\hat{p}_n\}$  using Aitken's  $\Delta^2$  method.
  - $p_0 = 0.5, \quad p_n = (2 - e^{p_{n-1}} + p_{n-1}^2)/3, \quad n \geq 1$
  - $p_0 = 0.75, \quad p_n = (e^{p_{n-1}}/3)^{1/2}, \quad n \geq 1$
  - $p_0 = 0.5, \quad p_n = 3^{-p_{n-1}}, \quad n \geq 1$
  - $p_0 = 0.5, \quad p_n = \cos p_{n-1}, \quad n \geq 1$
- Consider the function  $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$ . Use Newton's method with  $p_0 = 0$  to approximate a zero of  $f$ . Generate terms until  $|p_{n+1} - p_n| < 0.0002$ . Construct the sequence  $\{\hat{p}_n\}$ . Is the convergence improved?
- Let  $g(x) = \cos(x - 1)$  and  $p_0^{(0)} = 2$ . Use Steffensen's method to find  $p_0^{(1)}$ .
- Let  $g(x) = 1 + (\sin x)^2$  and  $p_0^{(0)} = 1$ . Use Steffensen's method to find  $p_0^{(1)}$  and  $p_0^{(2)}$ .
- Steffensen's method is applied to a function  $g(x)$  using  $p_0^{(0)} = 1$  and  $p_2^{(0)} = 3$  to obtain  $p_0^{(1)} = 0.75$ . What is  $p_1^{(0)}$ ?
- Steffensen's method is applied to a function  $g(x)$  using  $p_0^{(0)} = 1$  and  $p_1^{(0)} = \sqrt{2}$  to obtain  $p_0^{(1)} = 2.7802$ . What is  $p_2^{(0)}$ ?
- Use Steffensen's method to find, to an accuracy of  $10^{-4}$ , the root of  $x^3 - x - 1 = 0$  that lies in  $[1, 2]$ , and compare this to the results of Exercise 6 of Section 2.2.
- Use Steffensen's method to find, to an accuracy of  $10^{-4}$ , the root of  $x - 2^{-x} = 0$  that lies in  $[0, 1]$ , and compare this to the results of Exercise 8 of Section 2.2.
- Use Steffensen's method with  $p_0 = 2$  to compute an approximation to  $\sqrt{3}$  accurate to within  $10^{-4}$ . Compare this result with those obtained in Exercise 9 of Section 2.2 and Exercise 12 of Section 2.1.
- Use Steffensen's method with  $p_0 = 3$  to compute an approximation to  $\sqrt[3]{25}$  accurate to within  $10^{-4}$ . Compare this result with those obtained in Exercise 10 of Section 2.2 and Exercise 13 of Section 2.1.
- Use Steffensen's method to approximate the solutions of the following equations to within  $10^{-5}$ .
  - $x = (2 - e^x + x^2)/3$ , where  $g$  is the function in Exercise 11(a) of Section 2.2.
  - $x = 0.5(\sin x + \cos x)$ , where  $g$  is the function in Exercise 11(f) of Section 2.2.
  - $x = (e^x/3)^{1/2}$ , where  $g$  is the function in Exercise 11(c) of Section 2.2.
  - $x = 5^{-x}$ , where  $g$  is the function in Exercise 11(d) of Section 2.2.
- Use Steffensen's method to approximate the solutions of the following equations to within  $10^{-5}$ .
  - $2 + \sin x - x = 0$ , where  $g$  is the function in Exercise 12(a) of Section 2.2.
  - $x^3 - 2x - 5 = 0$ , where  $g$  is the function in Exercise 12(b) of Section 2.2.



Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time, proved the Fundamental Theorem of Algebra in his doctoral dissertation and published it in 1799. He published different proofs of this result throughout his lifetime, in 1815, 1816, and as late as 1848. The result had been stated, without proof, by Albert Girard (1595–1632), and partial proofs had been given by Jean d’Alembert (1717–1783), Euler, and Lagrange.

To determine the zeros of  $x^2 - 4x + 13$  we use the quadratic formula in its standard form, which gives the complex zeros

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i.$$

Hence the third-degree polynomial  $P(x)$  has three zeros,  $x_1 = 1$ ,  $x_2 = 2 - 3i$ , and  $x_3 = 2 + 3i$ . ■

In the preceding example we found that the third-degree polynomial had three distinct zeros. An important consequence of the Fundamental Theorem of Algebra is the following corollary. It states that this is always the case, provided that when the zeros are not distinct we count the number of zeros according to their multiplicities.

### Corollary 2.17

If  $P(x)$  is a polynomial of degree  $n \geq 1$  with real or complex coefficients, then there exist unique constants  $x_1, x_2, \dots, x_k$ , possibly complex, and unique positive integers  $m_1, m_2, \dots, m_k$ , such that  $\sum_{i=1}^k m_i = n$  and

$$P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k}. \quad \blacksquare$$

By Corollary 2.17 the collection of zeros of a polynomial is unique and, if each zero  $x_i$  is counted as many times as its multiplicity  $m_i$ , a polynomial of degree  $n$  has exactly  $n$  zeros.

The following corollary of the Fundamental Theorem of Algebra is used often in this section and in later chapters.

### Corollary 2.18

Let  $P(x)$  and  $Q(x)$  be polynomials of degree at most  $n$ . If  $x_1, x_2, \dots, x_k$ , with  $k > n$ , are distinct numbers with  $P(x_i) = Q(x_i)$  for  $i = 1, 2, \dots, k$ , then  $P(x) = Q(x)$  for all values of  $x$ . ■

This result implies that to show that two polynomials of degree less than or equal to  $n$  are the same, we only need to show that they agree at  $n + 1$  values. This will be frequently used, particularly in Chapters 3 and 8.

## Horner’s Method

To use Newton’s method to locate approximate zeros of a polynomial  $P(x)$ , we need to evaluate  $P(x)$  and  $P'(x)$  at specified values. Since  $P(x)$  and  $P'(x)$  are both polynomials, computational efficiency requires that the evaluation of these functions be done in the nested manner discussed in Section 1.2. Horner’s method incorporates this nesting technique, and, as a consequence, requires only  $n$  multiplications and  $n$  additions to evaluate an arbitrary  $n$ th-degree polynomial.

### Theorem 2.19 (Horner’s Method)

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Define  $b_n = a_n$  and

$$b_k = a_k + b_{k+1} x_0, \quad \text{for } k = n - 1, n - 2, \dots, 1, 0.$$

William Horner (1786–1837) was a child prodigy who became headmaster of a school in Bristol at age 18. Horner’s method for solving algebraic equations was published in 1819 in the *Philosophical Transactions of the Royal Society*.

Then  $b_0 = P(x_0)$ . Moreover, if

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1,$$

then

$$P(x) = (x - x_0)Q(x) + b_0. \quad \blacksquare$$

Paolo Ruffini (1765–1822) had described a similar method which won him the gold medal from the Italian Mathematical Society for Science. Neither Ruffini nor Horner was the first to discover this method; it was known in China at least 500 years earlier.

**Proof** By the definition of  $Q(x)$ ,

$$\begin{aligned} (x - x_0)Q(x) + b_0 &= (x - x_0)(b_n x^{n-1} + \cdots + b_2 x + b_1) + b_0 \\ &= (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x) \\ &\quad - (b_n x_0 x^{n-1} + \cdots + b_2 x_0 x + b_1 x_0) + b_0 \\ &= b_n x^n + (b_{n-1} - b_n x_0) x^{n-1} + \cdots + (b_1 - b_2 x_0) x + (b_0 - b_1 x_0). \end{aligned}$$

By the hypothesis,  $b_n = a_n$  and  $b_k - b_{k+1} x_0 = a_k$ , so

$$(x - x_0)Q(x) + b_0 = P(x) \quad \text{and} \quad b_0 = P(x_0). \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

**Example 2** Use Horner's method to evaluate  $P(x) = 2x^4 - 3x^2 + 3x - 4$  at  $x_0 = -2$ .

**Solution** When we use hand calculation in Horner's method, we first construct a table, which suggests the *synthetic division* name that is often applied to the technique. For this problem, the table appears as follows:

	Coefficient of $x^4$	Coefficient of $x^3$	Coefficient of $x^2$	Coefficient of $x$	Constant term
$x_0 = -2$	$a_4 = 2$	$a_3 = 0$	$a_2 = -3$	$a_1 = 3$	$a_0 = -4$
		$b_4 x_0 = -4$	$b_3 x_0 = 8$	$b_2 x_0 = -10$	$b_1 x_0 = 14$
	$b_4 = 2$	$b_3 = -4$	$b_2 = 5$	$b_1 = -7$	$b_0 = 10$

So,

$$P(x) = (x + 2)(2x^3 - 4x^2 + 5x - 7) + 10. \quad \blacksquare$$

An additional advantage of using the Horner (or synthetic-division) procedure is that, since

$$P(x) = (x - x_0)Q(x) + b_0,$$

where

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1,$$

differentiating with respect to  $x$  gives

$$P'(x) = Q(x) + (x - x_0)Q'(x) \quad \text{and} \quad P'(x_0) = Q(x_0). \quad (2.16)$$

When the Newton-Raphson method is being used to find an approximate zero of a polynomial,  $P(x)$  and  $P'(x)$  can be evaluated in the same manner.

The word synthetic has its roots in various languages. In standard English it generally provides the sense of something that is "false" or "substituted". But in mathematics it takes the form of something that is "grouped together". Synthetic geometry treats shapes as whole, rather than as individual objects, which is the style in analytic geometry. In synthetic division of polynomials, the various powers of the variables are not explicitly given but kept grouped together.

**Example 3** Find an approximation to a zero of

$$P(x) = 2x^4 - 3x^2 + 3x - 4,$$

using Newton's method with  $x_0 = -2$  and synthetic division to evaluate  $P(x_n)$  and  $P'(x_n)$  for each iterate  $x_n$ .

**Solution** With  $x_0 = -2$  as an initial approximation, we obtained  $P(-2)$  in Example 1 by

$$x_0 = -2 \quad \begin{array}{r|rrrrr} 2 & 2 & 0 & -3 & 3 & -4 \\ & & -4 & 8 & -10 & 14 \\ \hline & 2 & -4 & 5 & -7 & 10 & = P(-2). \end{array}$$

Using Theorem 2.19 and Eq. (2.16),

$$Q(x) = 2x^3 - 4x^2 + 5x - 7 \quad \text{and} \quad P'(-2) = Q(-2),$$

so  $P'(-2)$  can be found by evaluating  $Q(-2)$  in a similar manner:

$$x_0 = -2 \quad \begin{array}{r|rrrr} 2 & 2 & -4 & 5 & -7 \\ & & -4 & 16 & -42 \\ \hline & 2 & -8 & 21 & -49 & = Q(-2) = P'(-2) \end{array}$$

and

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = x_0 - \frac{P(x_0)}{Q(x_0)} = -2 - \frac{10}{-49} \approx -1.796.$$

Repeating the procedure to find  $x_2$  gives

$$\begin{array}{r|rrrrr} -1.796 & 2 & 0 & -3 & 3 & -4 \\ & & -3.592 & 6.451 & -6.197 & 5.742 \\ \hline & 2 & -3.592 & 3.451 & -3.197 & 1.742 & = P(x_1) \\ & & -3.592 & 12.902 & -29.368 & & \\ \hline & 2 & -7.184 & 16.353 & -32.565 & = Q(x_1) & = P'(x_1). \end{array}$$

So  $P(-1.796) = 1.742$ ,  $P'(-1.796) = Q(-1.796) = -32.565$ , and

$$x_2 = -1.796 - \frac{1.742}{-32.565} \approx -1.7425.$$

In a similar manner,  $x_3 = -1.73897$ , and an actual zero to five decimal places is  $-1.73896$ .

Note that the polynomial  $Q(x)$  depends on the approximation being used and changes from iterate to iterate. ■

Algorithm 2.7 computes  $P(x_0)$  and  $P'(x_0)$  using Horner's method.



ALGORITHM  
2.7

**Horner's**

To evaluate the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = (x - x_0)Q(x) + b_0$$

and its derivative at  $x_0$ :

**INPUT** degree  $n$ ; coefficients  $a_0, a_1, \dots, a_n; x_0$ .

**OUTPUT**  $y = P(x_0); z = P'(x_0)$ .

**Step 1** Set  $y = a_n$ ; (Compute  $b_n$  for  $P$ .)  
 $z = a_n$ . (Compute  $b_{n-1}$  for  $Q$ .)

**Step 2** For  $j = n - 1, n - 2, \dots, 1$   
set  $y = x_0 y + a_j$ ; (Compute  $b_j$  for  $P$ .)  
 $z = x_0 z + y$ . (Compute  $b_{j-1}$  for  $Q$ .)

**Step 3** Set  $y = x_0 y + a_0$ . (Compute  $b_0$  for  $P$ .)

**Step 4** OUTPUT  $(y, z)$ ;  
STOP.

If the  $N$ th iterate,  $x_N$ , in Newton's method is an approximate zero for  $P$ , then

$$P(x) = (x - x_N)Q(x) + b_0 = (x - x_N)Q(x) + P(x_N) \approx (x - x_N)Q(x),$$

so  $x - x_N$  is an approximate factor of  $P(x)$ . Letting  $\hat{x}_1 = x_N$  be the approximate zero of  $P$  and  $Q_1(x) \equiv Q(x)$  be the approximate factor gives

$$P(x) \approx (x - \hat{x}_1)Q_1(x).$$

We can find a second approximate zero of  $P$  by applying Newton's method to  $Q_1(x)$ .

If  $P(x)$  is an  $n$ th-degree polynomial with  $n$  real zeros, this procedure applied repeatedly will eventually result in  $(n - 2)$  approximate zeros of  $P$  and an approximate quadratic factor  $Q_{n-2}(x)$ . At this stage,  $Q_{n-2}(x) = 0$  can be solved by the quadratic formula to find the last two approximate zeros of  $P$ . Although this method can be used to find all the approximate zeros, it depends on repeated use of approximations and can lead to inaccurate results.

The procedure just described is called **deflation**. The accuracy difficulty with deflation is due to the fact that, when we obtain the approximate zeros of  $P(x)$ , Newton's method is used on the reduced polynomial  $Q_k(x)$ , that is, the polynomial having the property that

$$P(x) \approx (x - \hat{x}_1)(x - \hat{x}_2) \cdots (x - \hat{x}_k)Q_k(x).$$

An approximate zero  $\hat{x}_{k+1}$  of  $Q_k$  will generally not approximate a root of  $P(x) = 0$  as well as it does a root of the reduced equation  $Q_k(x) = 0$ , and inaccuracy increases as  $k$  increases. One way to eliminate this difficulty is to use the reduced equations to find approximations  $\hat{x}_2, \hat{x}_3, \dots, \hat{x}_k$  to the zeros of  $P$ , and then improve these approximations by applying Newton's method to the original polynomial  $P(x)$ .

**Complex Zeros: Müller's Method**

One problem with applying the Secant, False Position, or Newton's method to polynomials is the possibility of the polynomial having complex roots even when all the coefficients are

real numbers. If the initial approximation is a real number, all subsequent approximations will also be real numbers. One way to overcome this difficulty is to begin with a complex initial approximation and do all the computations using complex arithmetic. An alternative approach has its basis in the following theorem.

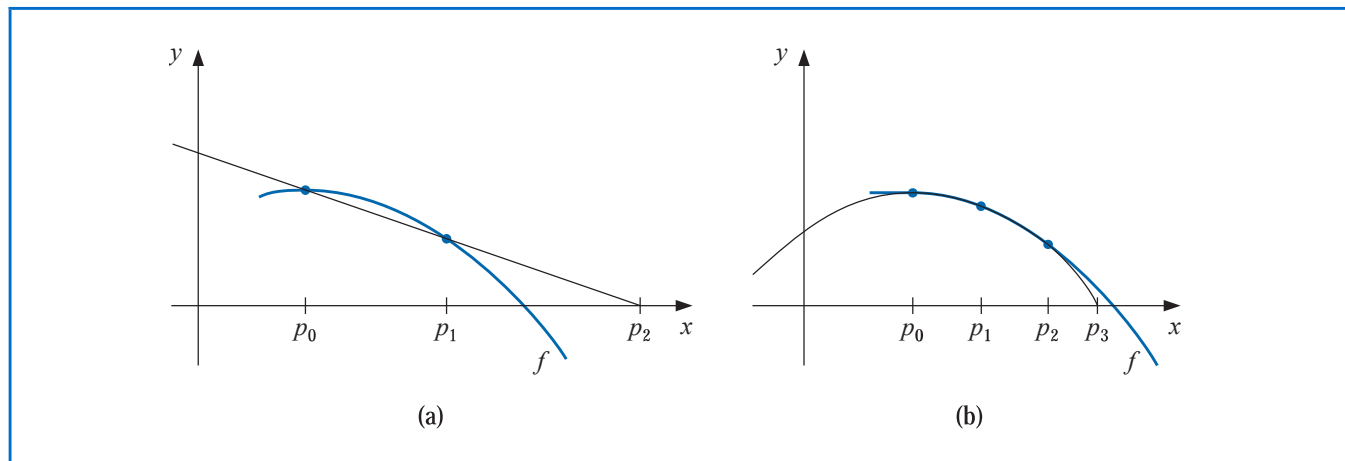
**Theorem 2.20** If  $z = a + bi$  is a complex zero of multiplicity  $m$  of the polynomial  $P(x)$  with real coefficients, then  $\bar{z} = a - bi$  is also a zero of multiplicity  $m$  of the polynomial  $P(x)$ , and  $(x^2 - 2ax + a^2 + b^2)^m$  is a factor of  $P(x)$ . ■

Müller's method is similar to the Secant method. But whereas the Secant method uses a line through two points on the curve to approximate the root, Müller's method uses a parabola through three points on the curve for the approximation.

A synthetic division involving quadratic polynomials can be devised to approximately factor the polynomial so that one term will be a quadratic polynomial whose complex roots are approximations to the roots of the original polynomial. This technique was described in some detail in our second edition [BFR]. Instead of proceeding along these lines, we will now consider a method first presented by D. E. Müller [Mu]. This technique can be used for any root-finding problem, but it is particularly useful for approximating the roots of polynomials.

The Secant method begins with two initial approximations  $p_0$  and  $p_1$  and determines the next approximation  $p_2$  as the intersection of the  $x$ -axis with the line through  $(p_0, f(p_0))$  and  $(p_1, f(p_1))$ . (See Figure 2.13(a).) Müller's method uses three initial approximations,  $p_0, p_1$ , and  $p_2$ , and determines the next approximation  $p_3$  by considering the intersection of the  $x$ -axis with the parabola through  $(p_0, f(p_0))$ ,  $(p_1, f(p_1))$ , and  $(p_2, f(p_2))$ . (See Figure 2.13(b).)

Figure 2.13



The derivation of Müller's method begins by considering the quadratic polynomial

$$P(x) = a(x - p_2)^2 + b(x - p_2) + c$$

that passes through  $(p_0, f(p_0))$ ,  $(p_1, f(p_1))$ , and  $(p_2, f(p_2))$ . The constants  $a$ ,  $b$ , and  $c$  can be determined from the conditions

$$f(p_0) = a(p_0 - p_2)^2 + b(p_0 - p_2) + c, \quad (2.17)$$

$$f(p_1) = a(p_1 - p_2)^2 + b(p_1 - p_2) + c, \quad (2.18)$$

and

$$f(p_2) = a \cdot 0^2 + b \cdot 0 + c = c \quad (2.19)$$



to be

$$c = f(p_2), \quad (2.20)$$

$$b = \frac{(p_0 - p_2)^2[f(p_1) - f(p_2)] - (p_1 - p_2)^2[f(p_0) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}, \quad (2.21)$$

and

$$a = \frac{(p_1 - p_2)[f(p_0) - f(p_2)] - (p_0 - p_2)[f(p_1) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}. \quad (2.22)$$

To determine  $p_3$ , a zero of  $P$ , we apply the quadratic formula to  $P(x) = 0$ . However, because of round-off error problems caused by the subtraction of nearly equal numbers, we apply the formula in the manner prescribed in Eq (1.2) and (1.3) of Section 1.2:

$$p_3 - p_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}.$$

This formula gives two possibilities for  $p_3$ , depending on the sign preceding the radical term. In Müller's method, the sign is chosen to agree with the sign of  $b$ . Chosen in this manner, the denominator will be the largest in magnitude and will result in  $p_3$  being selected as the closest zero of  $P$  to  $p_2$ . Thus

$$p_3 = p_2 - \frac{2c}{b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}},$$

where  $a$ ,  $b$ , and  $c$  are given in Eqs. (2.20) through (2.22).

Once  $p_3$  is determined, the procedure is reinitialized using  $p_1$ ,  $p_2$ , and  $p_3$  in place of  $p_0$ ,  $p_1$ , and  $p_2$  to determine the next approximation,  $p_4$ . The method continues until a satisfactory conclusion is obtained. At each step, the method involves the radical  $\sqrt{b^2 - 4ac}$ , so the method gives approximate complex roots when  $b^2 - 4ac < 0$ . Algorithm 2.8 implements this procedure.

### ALGORITHM 2.8

### Müller's

To find a solution to  $f(x) = 0$  given three approximations,  $p_0$ ,  $p_1$ , and  $p_2$ :

**INPUT**  $p_0, p_1, p_2$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $h_1 = p_1 - p_0$ ;  
 $h_2 = p_2 - p_1$ ;  
 $\delta_1 = (f(p_1) - f(p_0))/h_1$ ;  
 $\delta_2 = (f(p_2) - f(p_1))/h_2$ ;  
 $d = (\delta_2 - \delta_1)/(h_2 + h_1)$ ;  
 $i = 3$ .

**Step 2** While  $i \leq N_0$  do Steps 3–7.

**Step 3**  $b = \delta_2 + h_2d$ ;  
 $D = (b^2 - 4f(p_2)d)^{1/2}$ . (Note: May require complex arithmetic.)

**Step 4** If  $|b - D| < |b + D|$  then set  $E = b + D$   
 else set  $E = b - D$ .

**Step 5** Set  $h = -2f(p_2)/E$ ;  
 $p = p_2 + h$ .



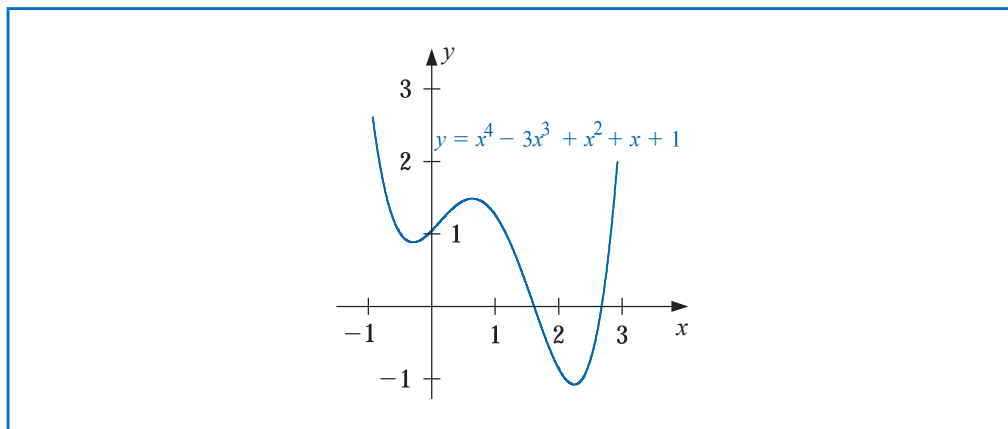
**Step 6** If  $|h| < TOL$  then  
 OUTPUT ( $p$ ); (*The procedure was successful.*)  
 STOP.

**Step 7** Set  $p_0 = p_1$ ; (*Prepare for next iteration.*)  
 $p_1 = p_2$ ;  
 $p_2 = p$ ;  
 $h_1 = p_1 - p_0$ ;  
 $h_2 = p_2 - p_1$ ;  
 $\delta_1 = (f(p_1) - f(p_0))/h_1$ ;  
 $\delta_2 = (f(p_2) - f(p_1))/h_2$ ;  
 $d = (\delta_2 - \delta_1)/(h_2 + h_1)$ ;  
 $i = i + 1$ .

**Step 8** OUTPUT ('Method failed after  $N_0$  iterations,  $N_0 =$ ',  $N_0$ );  
 (*The procedure was unsuccessful.*)  
 STOP.

**Illustration** Consider the polynomial  $f(x) = x^4 - 3x^3 + x^2 + x + 1$ , part of whose graph is shown in Figure 2.14.

**Figure 2.14**



Three sets of three initial points will be used with Algorithm 2.8 and  $TOL = 10^{-5}$  to approximate the zeros of  $f$ . The first set will use  $p_0 = 0.5$ ,  $p_1 = -0.5$ , and  $p_2 = 0$ . The parabola passing through these points has complex roots because it does not intersect the  $x$ -axis. Table 2.12 gives approximations to the corresponding complex zeros of  $f$ .

**Table 2.12**

$p_0 = 0.5, p_1 = -0.5, p_2 = 0$		
$i$	$p_i$	$f(p_i)$
3	$-0.100000 + 0.888819i$	$-0.01120000 + 3.014875548i$
4	$-0.492146 + 0.447031i$	$-0.1691201 - 0.7367331502i$
5	$-0.352226 + 0.484132i$	$-0.1786004 + 0.0181872213i$
6	$-0.340229 + 0.443036i$	$0.01197670 - 0.0105562185i$
7	$-0.339095 + 0.446656i$	$-0.0010550 + 0.000387261i$
8	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$
9	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$

Table 2.13 gives the approximations to the two real zeros of  $f$ . The smallest of these uses  $p_0 = 0.5$ ,  $p_1 = 1.0$ , and  $p_2 = 1.5$ , and the largest root is approximated when  $p_0 = 1.5$ ,  $p_1 = 2.0$ , and  $p_2 = 2.5$ .

Table 2.13

$p_0 = 0.5, p_1 = 1.0, p_2 = 1.5$			$p_0 = 1.5, p_1 = 2.0, p_2 = 2.5$		
$i$	$p_i$	$f(p_i)$	$i$	$p_i$	$f(p_i)$
3	1.40637	-0.04851	3	2.24733	-0.24507
4	1.38878	0.00174	4	2.28652	-0.01446
5	1.38939	0.00000	5	2.28878	-0.00012
6	1.38939	0.00000	6	2.28880	0.00000
			7	2.28879	0.00000

The values in the tables are accurate approximations to the places listed. □

We used Maple to generate the results in Table 2.12. To find the first result in the table, define  $f(x)$  with

$$f := x \rightarrow x^4 - 3x^3 + x^2 + x + 1$$

Then enter the initial approximations with

$$p0 := 0.5; p1 := -0.5; p2 := 0.0$$

and evaluate the function at these points with

$$f0 := f(p0); f1 := f(p1); f2 := f(p2)$$

To determine the coefficients  $a$ ,  $b$ ,  $c$ , and the approximate solution, enter

$$c := f2;$$

$$b := \frac{((p0 - p2)^2 \cdot (f1 - f2) - (p1 - p2)^2 \cdot (f0 - f2))}{(p0 - p2) \cdot (p1 - p2) \cdot (p0 - p1)}$$

$$a := \frac{((p1 - p2) \cdot (f0 - f2) - (p0 - p2) \cdot (f1 - f2))}{(p0 - p2) \cdot (p1 - p2) \cdot (p0 - p1)}$$

$$p3 := p2 - \frac{2c}{b + \left(\frac{b}{\text{abs}(b)}\right) \sqrt{b^2 - 4a \cdot c}}$$

This produces the final Maple output

$$-0.1000000000 + 0.8888194418I$$

and evaluating at this approximation gives  $f(p3)$  as

$$-0.0112000001 + 3.014875548I$$

This is our first approximation, as seen in Table 2.12.

The illustration shows that Müller's method can approximate the roots of polynomials with a variety of starting values. In fact, Müller's method generally converges to the root of a polynomial for any initial approximation choice, although problems can be constructed for

which convergence will not occur. For example, suppose that for some  $i$  we have  $f(p_i) = f(p_{i+1}) = f(p_{i+2}) \neq 0$ . The quadratic equation then reduces to a nonzero constant function and never intersects the  $x$ -axis. This is not usually the case, however, and general-purpose software packages using Müller's method request only one initial approximation per root and will even supply this approximation as an option.

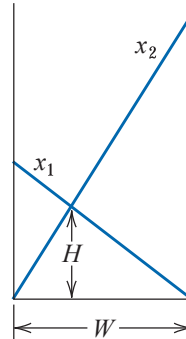
## EXERCISE SET 2.6

- Find the approximations to within  $10^{-4}$  to all the real zeros of the following polynomials using Newton's method.
  - $f(x) = x^3 - 2x^2 - 5$
  - $f(x) = x^3 + 3x^2 - 1$
  - $f(x) = x^3 - x - 1$
  - $f(x) = x^4 + 2x^2 - x - 3$
  - $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$
  - $f(x) = x^5 - x^4 + 2x^3 - 3x^2 + x - 4$
- Find approximations to within  $10^{-5}$  to all the zeros of each of the following polynomials by first finding the real zeros using Newton's method and then reducing to polynomials of lower degree to determine any complex zeros.
  - $f(x) = x^4 + 5x^3 - 9x^2 - 85x - 136$
  - $f(x) = x^4 - 2x^3 - 12x^2 + 16x - 40$
  - $f(x) = x^4 + x^3 + 3x^2 + 2x + 2$
  - $f(x) = x^5 + 11x^4 - 21x^3 - 10x^2 - 21x - 5$
  - $f(x) = 16x^4 + 88x^3 + 159x^2 + 76x - 240$
  - $f(x) = x^4 - 4x^2 - 3x + 5$
  - $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4$
  - $f(x) = x^3 - 7x^2 + 14x - 6$
- Repeat Exercise 1 using Müller's method.
- Repeat Exercise 2 using Müller's method.
- Use Newton's method to find, within  $10^{-3}$ , the zeros and critical points of the following functions. Use this information to sketch the graph of  $f$ .
  - $f(x) = x^3 - 9x^2 + 12$
  - $f(x) = x^4 - 2x^3 - 5x^2 + 12x - 5$
- $f(x) = 10x^3 - 8.3x^2 + 2.295x - 0.21141 = 0$  has a root at  $x = 0.29$ . Use Newton's method with an initial approximation  $x_0 = 0.28$  to attempt to find this root. Explain what happens.
- Use Maple to find a real zero of the polynomial  $f(x) = x^3 + 4x - 4$ .
- Use Maple to find a real zero of the polynomial  $f(x) = x^3 - 2x - 5$ .
- Use each of the following methods to find a solution in  $[0.1, 1]$  accurate to within  $10^{-4}$  for

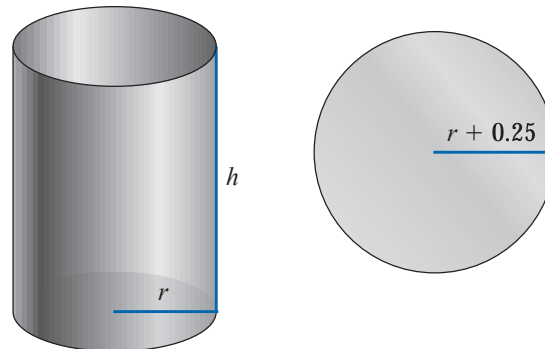
$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

- |                     |                             |                    |
|---------------------|-----------------------------|--------------------|
| a. Bisection method | c. Secant method            | e. Müller's method |
| b. Newton's method  | d. method of False Position |                    |

10. Two ladders crisscross an alley of width  $W$ . Each ladder reaches from the base of one wall to some point on the opposite wall. The ladders cross at a height  $H$  above the pavement. Find  $W$  given that the lengths of the ladders are  $x_1 = 20$  ft and  $x_2 = 30$  ft, and that  $H = 8$  ft.



11. A can in the shape of a right circular cylinder is to be constructed to contain  $1000 \text{ cm}^3$ . The circular top and bottom of the can must have a radius of  $0.25 \text{ cm}$  more than the radius of the can so that the excess can be used to form a seal with the side. The sheet of material being formed into the side of the can must also be  $0.25 \text{ cm}$  longer than the circumference of the can so that a seal can be formed. Find, to within  $10^{-4}$ , the minimal amount of material needed to construct the can.



12. In 1224, Leonardo of Pisa, better known as Fibonacci, answered a mathematical challenge of John of Palermo in the presence of Emperor Frederick II: find a root of the equation  $x^3 + 2x^2 + 10x = 20$ . He first showed that the equation had no rational roots and no Euclidean irrational root—that is, no root in any of the forms  $a \pm \sqrt{b}$ ,  $\sqrt{a} \pm \sqrt{b}$ ,  $\sqrt{a} \pm \sqrt{b}$ , or  $\sqrt{\sqrt{a} \pm \sqrt{b}}$ , where  $a$  and  $b$  are rational numbers. He then approximated the only real root, probably using an algebraic technique of Omar Khayyam involving the intersection of a circle and a parabola. His answer was given in the base-60 number system as

$$1 + 22 \left(\frac{1}{60}\right) + 7 \left(\frac{1}{60}\right)^2 + 42 \left(\frac{1}{60}\right)^3 + 33 \left(\frac{1}{60}\right)^4 + 4 \left(\frac{1}{60}\right)^5 + 40 \left(\frac{1}{60}\right)^6.$$

How accurate was his approximation?

## 2.7 Survey of Methods and Software

In this chapter we have considered the problem of solving the equation  $f(x) = 0$ , where  $f$  is a given continuous function. All the methods begin with initial approximations and generate a sequence that converges to a root of the equation, if the method is successful. If  $[a, b]$  is an interval on which  $f(a)$  and  $f(b)$  are of opposite sign, then the Bisection method and the method of False Position will converge. However, the convergence of these methods might be slow. Faster convergence is generally obtained using the Secant method or Newton's method. Good initial approximations are required for these methods, two for the Secant method and one for Newton's method, so the root-bracketing techniques such as Bisection or the False Position method can be used as starter methods for the Secant or Newton's method.

Müller's method will give rapid convergence without a particularly good initial approximation. It is not quite as efficient as Newton's method; its order of convergence near a root is approximately  $\alpha = 1.84$ , compared to the quadratic,  $\alpha = 2$ , order of Newton's method. However, it is better than the Secant method, whose order is approximately  $\alpha = 1.62$ , and it has the added advantage of being able to approximate complex roots.

Deflation is generally used with Müller's method once an approximate root of a polynomial has been determined. After an approximation to the root of the deflated equation has been determined, use either Müller's method or Newton's method in the original polynomial with this root as the initial approximation. This procedure will ensure that the root being approximated is a solution to the true equation, not to the deflated equation. We recommend Müller's method for finding all the zeros of polynomials, real or complex. Müller's method can also be used for an arbitrary continuous function.

Other high-order methods are available for determining the roots of polynomials. If this topic is of particular interest, we recommend that consideration be given to Laguerre's method, which gives cubic convergence and also approximates complex roots (see [Ho], pp. 176–179 for a complete discussion), the Jenkins-Traub method (see [JT]), and Brent's method (see [Bre]).

Another method of interest, Cauchy's method, is similar to Müller's method but avoids the failure problem of Müller's method when  $f(x_i) = f(x_{i+1}) = f(x_{i+2})$ , for some  $i$ . For an interesting discussion of this method, as well as more detail on Müller's method, we recommend [YG], Sections 4.10, 4.11, and 5.4.

Given a specified function  $f$  and a tolerance, an efficient program should produce an approximation to one or more solutions of  $f(x) = 0$ , each having an absolute or relative error within the tolerance, and the results should be generated in a reasonable amount of time. If the program cannot accomplish this task, it should at least give meaningful explanations of why success was not obtained and an indication of how to remedy the cause of failure.

IMSL has subroutines that implement Müller's method with deflation. Also included in this package is a routine due to R. P. Brent that uses a combination of linear interpolation, an inverse quadratic interpolation similar to Müller's method, and the Bisection method. Laguerre's method is also used to find zeros of a real polynomial. Another routine for finding the zeros of real polynomials uses a method of Jenkins-Traub, which is also used to find zeros of a complex polynomial.

The NAG library has a subroutine that uses a combination of the Bisection method, linear interpolation, and extrapolation to approximate a real zero of a function on a given interval. NAG also supplies subroutines to approximate all zeros of a real polynomial or complex polynomial, respectively. Both subroutines use a modified Laguerre method.

The netlib library contains a subroutine that uses a combination of the Bisection and Secant method developed by T. J. Dekker to approximate a real zero of a function in the interval. It requires specifying an interval that contains a root and returns an interval with a width that is within a specified tolerance. Another subroutine uses a combination of the bisection method, interpolation, and extrapolation to find a real zero of the function on the interval.

MATLAB has a routine to compute all the roots, both real and complex, of a polynomial, and one that computes a zero near a specified initial approximation to within a specified tolerance.

Notice that in spite of the diversity of methods, the professionally written packages are based primarily on the methods and principles discussed in this chapter. You should be able to use these packages by reading the manuals accompanying the packages to better understand the parameters and the specifications of the results that are obtained.

There are three books that we consider to be classics on the solution of nonlinear equations: those by Traub [Tr], by Ostrowski [Os], and by Householder [Ho]. In addition, the book by Brent [Bre] served as the basis for many of the currently used root-finding methods.