Error Estimation of the Iterative Process (11)

Since iterative process is an asympototic process (infinite process). But in practical situation, we have to stop at a certain stage. This gives the error in the approximate root of the iterative equation.

Consider

$$x_{n+p} - x_n = (x_{n+p} - x_{n+p-1}) + (x_{n+p-1} - x_{n+p-2}) + \cdots$$

$$\cdots + (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)$$

taking the modulus on both sides and using the triangular inequality, we get

$$|x_{n+p} - x_n| \le |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \cdots$$

$$\cdots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n|$$

using the result $|x_{n+1} - x_n| \le q^n |x_1 - x_0| \ \forall n \in \mathbb{N}$

We obtain

$$|x_{n+p} - x_n| \le q^{n+p-1}|x_1 - x_0| + q^{n+p-2}|x_1 - x_0| + \cdots$$

$$\cdots + q^{n+1}|x_1 - x_0| + q^n |x_1 - x_0|$$

$$\Rightarrow |x_{n+p} - x_n| \le q^n |x_1 - x_0| \left(q^{p-1} + q^{p-2} + \cdots + q + 1\right)$$

$$\le q^n |x_1 - x_0| \frac{1-q^p}{1-q}$$

$$\Rightarrow |x_{n+p} - x_n| \le \frac{q^n}{1-q} |x_1 - x_0| (1 - q^p) \qquad (1)$$

Now
$$\lim_{p\to\infty} x_{n+p} = \lim_{n+p\to\infty} x_{n+p} = \xi$$

taking the limit on both sides of equation (1), we have

$$\lim_{p\to\infty} |x_{n+p} - x_n| \leq \frac{q^n}{1-q} |x_1 - x_0| \lim_{p\to\infty} (1-q^p)$$

$\because q < 1$

$$|\xi - x_n| \leq \frac{q^n}{1-q} |x_1 - x_0|$$

From the above result it is clear that the convergence of the iteration process will be faster, smaller the number q is.

Convergence of Newton-Raphson method is Quadratic

OR Convergence is of order two for Newton-Raphson Method

(16)

The Newton-Raphson method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{--- (1)}$$

Let the error at nth step be e_n and ξ be the root of the equation $f(x)=0$ i.e $f(\xi)=0$

then

$$x_n - \xi = e_n \quad \text{--- (*)}$$

$$\& \quad x_{n+1} - \xi = e_{n+1} \quad \text{--- (**)}$$

From equation (1)

$$\cancel{\xi} + e_{n+1} = \cancel{\xi} + e_n - \frac{f(e_n+\xi)}{f'(e_n+\xi)}$$

$$e_{n+1} = e_n - \frac{f(e_n+\xi)}{f'(e_n+\xi)}$$

$$= \frac{e_n f'(e_n+\xi) - f(e_n+\xi)}{f'(e_n+\xi)}$$

$$= \frac{e_n\left[f'(\xi) + e_n f''(\xi) + \cdots\right] - \left[f(\xi) + e_n f'(\xi) + \frac{e_n^2}{2!}f''(\xi) \cdots\right]}{f'(\xi) + e_n f''(\xi) + \frac{e_n^2}{2!} f'''(\xi) + \cdots}$$

If e_n is very small we neglected higher powers of e_n

$$e_{n+1} \simeq \frac{1}{2} e_n^2 \frac{f''(\xi)}{f'(\xi)}$$

Let $C = \frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$ = Constant.

$$\Rightarrow e_{n+1} \simeq C e_n^2$$

$$\Rightarrow e_{n+1} \propto e_n^2$$

i.e the error at $(n+1)$th step is proportional to the square of the error at the nth step.

The Convergence is quadratic. This means that the no. of correct decimals is approximately doubled at every iteration atleast. If the factor $\frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$ is not too large.

SOME SPECIAL CASES OF NEWTON'S METHOD

(A) Determination of square roots

Let c be a given number and let $f(x) = x^2 - c$ $(c > 0)$

We want to solve the equation $f(x) = 0$

i.e to Compute $x = \sqrt{c}$

Newton method take the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - c}{2x_n}$$

$$= x_n - \frac{x_n^2}{2x_n} + \frac{c}{2x_n}$$

$$= x_n - \frac{1}{2}x_n + \frac{c}{2x_n}$$

$$= \frac{1}{2}x_n + \frac{c}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right), \quad c > 0, \quad n = 0, 1, 2, \ldots$$

EXERCISE Determine $\sqrt{10}$, with $x_0 = 3$ correct to 8 decimal places.

(B) Finding the roots of arbitrary order

To determine pth root of "c" ($c > 0$). We must solve $x^p = c$ OR $\boxed{f(x) = x^p - c} = 0$

Now the newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^p - c}{p x_n^{p-1}}$$

$$= \frac{p x_n^p - x_n^p - c}{p x_n^{p-1}} = \frac{c + (p-1)x_n^p}{p x_n^{p-1}}$$

$p \geq 2$
$n = 0, 1, 2, 3, \ldots$

EXERCISE

Determine cube root of 7 i.e $\sqrt[3]{7}$ correct to four decimal places with $x_0 = 2$. (19)

(c) Finding reciprocals with division

We want to determine the solution of the equation

$$f(x) = \frac{1}{x} - c = 0, \quad \text{for } c > 0$$

Newton's method yields

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{1/x_n - c}{-1/x_n^2}$$

$$= x_n + x_n - c x_n^2$$

$$x_{n+1} = x_n (2 - c x_n), \quad c > 0$$

$$n = 0, 1, 2, \dots$$

EXERCISE

To calculate e^{-1} with $e = 2.71828183$ starting with $x_0 = 0.3$

We shall now prove a theorem which tells us that "under what conditions the iteration process converges.

THEOREM: Let a function $\phi(x)$ be defined and differentiable in an interval (a, b) with all values $\phi(x) \in [a,b]$, then if there exists a proper fraction q $(0<q<1)$ such that

$$|\phi'(x)| \le q < 1 \quad \forall \quad a < x < b$$

then the process of iteration described above converges irrespective of the choice of initial approximation $x_0 \in [a,b]$, Moreover, the limiting value ξ of x_n is the only root of $x = \phi(x)$ in $[a,b]$.

PROOF :- We know that the equation of iteration is given by

$$x_{n+1} = \phi(x_n) \text{ and } x_n = \phi(x_{n-1})$$

Subtracting, we get $x_{n+1} - x_n = \phi(x_n) - \phi(x_{n-1})$

Applying the mean value theorem to $\phi(x)$ in $[x_{n-1}, x_n]$ we get,

$$x_{n+1} - x_n = (x_n - x_{n-1})\, \phi'(c) \text{ ——— } (A)$$

where $c \in (x_{n-1}, x_n)$

obviously $c \in [a,b]$

taking modulus on both sides of (A), we get

$$|x_{n+1} - x_n| = |x_n - x_{n-1}|\,|\varphi'(c)|$$

Let $|\varphi'(c)| \leq q < 1$ —— (B)

$$\Rightarrow |x_{n+1} - x_n| \leq q\,|x_n - x_{n-1}| \quad \text{—— (C)}$$

putting $n = 1, 2, 3, \ldots$ in (C), we get

$$|x_2 - x_1| \leq q\,|x_1 - x_0|,$$

$$|x_3 - x_2| \leq q\,|x_2 - x_1| \leq q \cdot q\,|x_1 - x_0|$$

$$\Rightarrow |x_3 - x_2| \leq q^2\,|x_1 - x_0|$$

Continuing in this way, we get

$$|x_{n+1} - x_n| \leq q^n\,|x_1 - x_0| \quad \text{—— (D)}$$

Consider now the series

$$x_0 + (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) + (x_{n+1} - x_n) + \cdots \quad \text{—— (E)}$$

For series (E) the $(n+1)$th partial sum equals x_n.

~~There~~ i.e $S_{n+1} = x_n$

Therefore the series (E) is convergent

i.e $\lim\limits_{n \to \infty} S_{n+1} = \lim\limits_{n \to \infty} x_n = \xi.$

Hence the sequence $\{x_n\}$ is convergent

Moreover, $x = \varphi(x)$ is there a function, all of whose values lie in $[a, b]$ and so $\xi \in [a, b]$

applicable.

Now to prove that ξ is a root of equation $x = \varphi(x)$.

taking the limit on both side of

$$x_n = \varphi(x_{n-1})$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \varphi(x_{n-1})$$

$$= \varphi\left(\lim_{n\to\infty} x_{n-1}\right)$$

$$\Rightarrow \xi = \varphi(\xi)$$

Thus $\lim_{n\to\infty} x_n = \xi$ is a root of $x = \varphi(x)$ in $[a,b]$.

Now remain to show that ξ is the only root of $x = \varphi(x)$ in $[a,b]$.

Let $\bar{\xi} \in [a,b]$ be an other root of $x = \varphi(x)$ then $\bar{\xi} = \varphi(\bar{\xi})$. Subtracting this equation from the equation $\xi = \varphi(\xi)$, we get

$$\xi - \bar{\xi} = \varphi(\xi) - \varphi(\bar{\xi})$$

appling the mean value theorem we get

$$\xi - \bar{\xi} = (\xi - \bar{\xi})\,\varphi'(d)$$

where $d \in (\xi, \bar{\xi})$
obviously $d \in [a, b]$

$$(\xi - \bar{\xi})(1 - \varphi'(d)) = 0$$

Sin $|\varphi'(d)| < 1 \quad \forall\ d \in [a,b]$ therefore, we have

$$\xi - \bar{\xi} = 0, \quad \text{Since } 1 - \varphi'(d) \neq 0$$

$$\Rightarrow \xi - \bar{\xi}$$

Hence ξ is only root of $x = \varphi(x)$ in $[a,b]$.