1

Theorem :-

There does not exist any rational number such that

$$P^2 = 2$$

Proof :-

Let there exist a rational "P" such that

$$P^2 = 2 \quad \text{and} \quad P = \frac{m}{n}$$

$$\left(\frac{m}{n}\right)^2 = 2$$

$$m^2 = 2n^2$$

Note :- $\begin{cases} (m, n) = 1 \\ n \neq 0 \\ m, n \in \mathbb{Z} \end{cases}$

Which shows that "m^2" is even. So that "m" is also even. But "m^2" is divisible by 4, which possibly if "n^2" is even, i.e

n is even.

Which contradict our assumption that (m,n) are relatively Prime.

Hence, there does not exist any rational number whose square is 2.

Theorem :-

$$A = \{ P \in Q^+ / P^2 < 2 \}$$

$$B = \{ P \in Q^+ / P^2 > 2 \}$$

Show that A contain no largest and B contain no smallest.

Proof :- To Prove the result, we need to show that for every P in A, we can find a rational q

in A such that $P<q$ and for every P in B, we can find a rational q in B such that $q<P$.

To do this, we associate with each rational $P>0$

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$$q = P - \frac{P^2-2}{P+2} \Rightarrow \frac{2P+2}{P+2} \quad \text{--- (I)}$$

$$q^2-2 = \frac{2(P^2-2)}{(P+2)^2} \quad \text{--- (II)}$$

Now, if $P \in A$ then

$$P^2<2 \quad \text{i.e} \quad P^2-2<0$$

Eq (I) $\Rightarrow$

$$q>P$$

Also, Eq (II)

$$q^2-2<0$$

$$q^2<2 \quad \text{i.e} \quad q \in A$$

Therefore, A does not contain any larger.

But if $p \in B$ then $p^2 > 2$

i.e $p^2 - 2 > 0$

$$q < p$$

$$q^2 - 2 > 0$$

i.e $q^2 > 2$

Hence, B contain no smallest.

Order Set :-

Let S be a non-empty set. An order on S is a relation denoted by "$<$" with the following two Properties:

(i) If $x \in S$, $y \in S$ then either

$$x < y \quad \text{or} \quad y < x$$

or $x = y$ is true.

(ii) If $x, y, z \in S$

$x < y$ and $y < z$

$\Rightarrow x < z$

The set "S" together with a relation "$<$" is known as an ordered set.

Supremium Of Set :-

Suppose S is an order set, $E \subset S$ and E is bounded above. Let there exist an $\alpha \in S$ with the following Properties

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E. Then α is called least upper bound or Supremium.

$$\alpha = \operatorname{Sup} E$$

Similarly, if E is bounded below and α is the greatest lower bound or infimum of E.

$$\alpha = \inf E$$

Which means that α is a lower bound of E and that no β is greater then α is a lower bound of E.

Theorem :-

Suppose S is an order set with least upper bound property $B \subset S$, B is not empty and B is bounded below.

Let L be the set of all lower bound of B. Then

$$\alpha = \sup L$$

exist in S and $\alpha = \inf B$.

Proof :-

Since B is bounded below and L is not empty. Since L contain exactly those $y \in S$ which satisfy the inequality $y \leq x$ for every $x \in B$. We see that for every $x \in B$ is an upper bound of L. Thus L is bounded Above. Our hypothesis about S implies that L has a superimum in S call it α

$$\alpha = \sup L$$

If $\gamma < \alpha$ then γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$

If $\alpha < \beta$, then $\beta \notin L$, since α is an upper bound of L.

We have shown that $\alpha \in L$ but $\beta \notin L$

In other words α is a lower bound of B and β is not.

$$\alpha = \inf B$$

Proposition 1.14 :-

(i) If $x+y = x+z$ then $y = z$

$y + 0 = y$ (Additive Identity)

$y + x - x = y$ (Inverse Prop.)

$x + y - x = y$

$x + z - x = y$ (Real No. Property)

$x - x + z = y$

$$\boxed{z = y}$$

(ii) If $x+y=x$ then $y=0$

$y+0=y$ (Additive Identity)

$y+x-x=y$ (Inverse Property)

$x+y-x=y$ (Real No. Property)

$x-x=y$ (Inverse)

$$\boxed{0=y}$$

(iii) If $x+y=0$ then $y=-x$

$y+0=0\,y$

$y+x-x=0\,y$

$x+y-x=y$

$0-x=y$

$$\boxed{y=-x}$$

(iv) $-(-x)=x$

$x-x=0$

Using (iii)

$x+y=0$ then $y=-x$

$$\boxed{x = -(-x)}$$

Proved.

Preposition 1.15 :-

(i) $x \neq 0$ and $xy = xz$ then $y = z$

$\because$ $x \neq 0$ then $\frac{1}{x} \neq 0$

$y = 1.y$ (Multiplicative Identity)

$y = (x)(\frac{1}{x})\, y$

$y = (xy)\, \frac{1}{x}$ (Associative Prop)

$y = (xz)\, \frac{1}{x}$

$y = (x)(\frac{1}{x})\, z$ (Associative Prop)

$y = 1\, z$ (Multiplicative Identity)

$$\boxed{y = z}$$

(ii) $x \neq 0$ and $xy = x$ the $y = 1$

$\because$ $x \neq 0$ then $\frac{1}{x} \neq 0$

$y = 1.y$

(iii)

$\because$

(iv)

$\because$

$$y = (x)\left(\frac{1}{x}\right) y$$

$$y = (xy) \cdot \frac{1}{x}$$

$$y = x \cdot \frac{1}{x}$$

$$\boxed{y = 1}$$

(iii) $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$

if $x \neq 0$ then $\frac{1}{x} \neq 0$

$$y = 1 \cdot y$$

$$y = (x)\left(\frac{1}{x}\right) y$$

$$y = (xy) \cdot \frac{1}{x}$$

$$y = 1 \cdot \frac{1}{x}$$

$$\boxed{y = \frac{1}{x}}$$

(iv) $x \neq 0$ then $\frac{1}{1/x} = x$

if $x \neq 0$ then $\frac{1}{x} \neq 0$

$$x = 1 \cdot x$$

$$x = \frac{1}{1/x} \cdot \frac{1}{x} \cdot x$$

$$x = \left(\frac{1}{1/x}\right) \cdot x \cdot \frac{1}{x}$$

$$x = \frac{1}{1/x} \cdot 1$$

$$\boxed{x = \frac{1}{1/x}}$$

Proposition 1.16 :-

(i) $\quad 0x = 0$

$$0x + 0x = 0x$$

$$x(0+0) = 0x$$

if $x + y = x$ then $y = 0$

$$\boxed{0x = 0}$$

(ii) $\quad x \neq 0$ and $y \neq 0$ then $xy \neq 0$

Suppose $\quad xy = 0$

$$\Rightarrow x \neq 0 \;,\; y \neq 0$$

$$1 = \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)(xy)$$

$$1 = \left(\frac{1}{xy}\right)(0)$$

A contradiction. Hence (b) holds.

(iii) $(-x)y = -(xy) = x(-y)$

Consider

$$(-x)y + xy = (-x+x)y = 0y$$

So,

$$(-x)y = -(xy)$$

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Similarly

$$x(-y) + xy = 0$$

$$x(-y+y) = 0$$

$$0 = 0$$

$$x(-y) = -(xy)$$

Hence

$$x(-y) = -(xy) = (-x)y$$

(iv) $(-x)(-y) = xy$

$-[x(-y)]$ using (iii)

$-[-(xy)] =$

$-[-(xy)] = xy$ using 1.14(d)

Proposition 1.18 :-

(i) $x > 0$ then $-x < 0$

if $x > 0$

$$-x + x > -x + 0$$

$$0 > -x$$

$$-x < 0$$

(ii) $x > 0$ and $y < z$ then $xy < xz$

Since

$$z > y$$

$$z - y > y - y$$

$$z - y > 0$$

$$x(z - y) > 0$$

$$x(z - y) + xy > 0 + xy$$

$xz > xy$

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(iii) By using (ii)

$z - y > 0$

$(-x)(z-y) > 0$

$(-x)(z-y) - xy > 0 - xy$

$-xz > -xy$

$xz < xy$

(iv) $x \neq 0$ then $x^2 > 0$. In particular

$1 > 0$

if $x \neq 0$ then either

$x > 0$ or $x < 0$

if $x > 0 \Rightarrow x^2 > 0$

if $x < 0 \Rightarrow -x > 0 \Rightarrow (-x)^2 > 0$

Since $1^2 = 1$

$1 > 0$

(v) $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$

if $y>0$ and $v<0$

then $yv<0$

But $y \cdot \frac{1}{y} = 1 > 0$

Hence $\frac{1}{y} > 0$

Similarly $\frac{1}{x} > 0$

$$0 < x < y$$

$$\frac{1}{xy}(0) < \frac{1}{xy}(x) < \frac{1}{xy}(y)$$

$$0 < \frac{1}{y} < \frac{1}{x}$$

Theorem 1.20 :-

(a) Archimedian Property of R

$x \in R$, $y \in R$ and $x > 0$

then there is a positive integar n such that

$$nx > y$$

Proof :-

Let

$$A = \{ nx \in \mathbb{R}^+ / n \in \mathbb{Z}^+ \}$$

if A is false.

$$A = \{ nx \in \mathbb{R}^+ / x \in \mathbb{R}^+ , n \in \mathbb{Z}^+ \ \forall \ nx < y \}$$

Then y would be upper bound of A. But, then A has a least upper bound in R.

$$\alpha = \operatorname{Sup} A$$

Since

$$x > 0$$

$$-x < 0$$

$$\alpha - x < \alpha$$

$\alpha - x$ is not an upper bound of A.

Hence,

$$\alpha - x < mx$$

For some positive integer m, $mx \in A$

But then

$$\alpha - x < mx$$

$$\alpha < mx + x$$

$$\alpha < x(m+1)$$

Which is impossible. Since α is an upper bound of A.

(b) $\mathbb{Q}$ dense in $\mathbb{R}$

let $x, y \in \mathbb{R}$

Since $0 < x < y$

$$y - x > 0$$

There exist a positive integer n such that

$$n(y-x) > 1$$

$$ny - nx > 1$$

$$ny > 1 + nx \quad \text{—— (1)}$$

Consider,

$$m_1, m_2 \in \mathbb{Z}^+$$

Applying $\quad nx > y$

Here,

$$n = 1 \quad , \quad x = m_1 \quad , \quad y = nx$$

$$m_1 > nx$$

and

$$m_2 > -nx$$

$$nx > -m_2$$

Combining both

$$-m_2 < nx < m_1$$

Hence there is an integer m with

$$-m_2 \leq m \leq m_1$$

$$m-1 \le nx < m$$

$\Rightarrow$ $nx < m$ —— (ii)

$\Rightarrow$ $m-1 \le nx$ ——

$m \le 1 + nx$ —— (iii)

Combining these in-equalities

$$nx < m \le nx+1 < ny$$

$$nx < m < ny$$

$$x < \frac{m}{n} < y$$

This Proves that

$$P = m/n$$

Theorem 1.21 :-

For every real $x > 0$ and every $n > 0$ there is one and only one real such that

$$y^n = x$$

Proof :- There is at most one such y. Since

$$0 < y_1 < y_2$$

$$\Rightarrow \quad y_1^n < y_2^n$$

Let E be the set of all positive real number t such that $t^n < x$.

Since, E is bounded above and is a subset of real no. which has least upper bound property. Therefore, $\sup(E)$ exist (say)

$$y = \sup E$$

To prove the result

$$y^n = x$$

We will show that $y^n > x$ and $y^n < x$ leads to a contradiction.

⇒ Binomial Expansion

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots\cdots + a^{n-1})$$

if $a = b^n$ then

$$b^n - a^n < (b-a)\, n b^{n-1} \quad \text{——} \quad (1)$$

⇒ Assume $y^n < x$

So that $0 < h < 1$

$$h < \frac{x - y^n}{n(y+1)^{n+1}}$$

Put $a = y$, $b = y+h$ then

$$(y+h)^n - y^n < nh(y+h)^{n-1} < nh(y+1)^{n-1} < x - y^n$$

Thus

$$y^n < x \Rightarrow (y+h)^n < x$$

$y+h \in E$.

Since $y+h > y$

This contradict the fact y is an upper bound of E.

⇒ Assume $y^n > x$

Put

$$K = \frac{y^n - x}{ny^{n-1}}$$

Then

$$0 < K < y$$

if

$$t \geq y - K$$

$$y^n - t^n \leq y^n - (y-K)^n < Kny^{n-1} = y^n - x$$

Thus $t^n > x$ and $t \notin E$

It follows that $y-K$ is an upper bound of E, which contradict the fact y is the least upper bound of E.

Hence $y^n = x$

Proved

Corollary :- If a and b are positive real number and n is a positive integer, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}$$

Proof :-

let

$$\alpha = a^{1/n} \Rightarrow \alpha^n = a$$

$$\beta = b^{1/n} \Rightarrow \beta^n = b$$

Since

$$\alpha^n \beta^n = (\alpha\beta)^n = ab$$

$$\alpha\beta = (ab)^{1/n}$$

$$a^{1/n} . b^{1/n} = (ab)^{1/n}$$

Hence, we proved this.

Schwarz Inequality :-

If $a_1, a_2, \ldots\ldots, a_n$ and

$b_1, b_2, \ldots\ldots, b_n$ are complex number

then
$$\sum_{i=1}^{k} a_i b_i \leq \sum_{i=1}^{k} \|a_i\|^2 \sum_{i=1}^{k} \|b_i\|^2$$

This inequality is also true for real number.

Euclidean Space's :-

Theorem 1.37 :-

(a) $\|x\| > 0$

$\Rightarrow (x_i . x_i)^{1/2} \qquad \because x_i^2 \geq 0$

$\Rightarrow \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$

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$\|x\| \geq 0$

(b) $\|x\| = 0$ if and only if $x = 0$

$$\left(\sum_{i=1}^{k} x_i^2 \right)^{1/2} = 0$$

$$\Leftrightarrow \sum_{i=1}^{k} x_i^2 = 0$$

$$\Leftrightarrow x_i^2 = 0$$

$$x_i = 0$$

$$\|x\| = 0$$

(c) $\|\alpha x\| = |\alpha| \, \|x\|$

$$= (\alpha x . \alpha x)^{1/2}$$

$$= \left[\alpha^2 \left(\sum_{i=1}^{k} x_i^2 \right) \right]^{1/2}$$

$$= |\alpha| \left[\sum_{i=1}^{k} x_i \right]^{1/2}$$

$$= |\alpha| \, \| x \|$$

(d) $\| x . y \| \leq \| x \| \, \| y \|$

if $\lambda \in \mathbb{R}$

$$0 \leq \| x - \lambda y \|^2$$

$$0 = \left[(x - \lambda y) . (x - \lambda y) \right]^{2 \times 1/2}$$

$$0 = x.x - \lambda x y - \lambda x.y + \lambda^2 y.y \quad \text{—— (1)}$$

$$\because (x.x)^{1/2} = \| x \|$$

Taking square on both side

$$x.x = \| x \|^2$$

Eq (1) $\Rightarrow$

$$= \| x \|^2 - 2\lambda x.y + \lambda^2 \| y \|^2$$

Choose

$$\lambda = \frac{x \cdot y}{\|y\|^2}$$

$$0 \leq \|x\|^2 - 2\,\frac{(x \cdot y)(x \cdot y)}{\|y\|^2} + \frac{(x \cdot y)^2}{\|y\|^4} \cdot \|y\|^2$$

$$0 \leq \|x\|^2 - \frac{(x \cdot y)^2}{\|y\|^2}$$

$$0 \leq \|x\|^2 \|y\|^2 - (x \cdot y)^2$$

$$(x \cdot y)^2 \leq \|x\|^2 \|y\|^2$$

$$\|x \cdot y\| \leq \|x\| \, \|y\|$$

(e)

$$\|x+y\| \leq \|x\| + \|y\|$$

$$0 \leq \|x+y\|^2$$

$$= (x+y) \cdot (x+y)$$

$$= \|x\|^2 + 2x \cdot y + \|y\|^2$$

$$\leq \|x\|^2 + 2\,|x \cdot y| + \|y\|^2$$

$$\leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2.$$

$\|x+y\|^2 \le (\|x\| + \|y\|)^2$

$\|x+y\| \le \|x\| + \|y\|$

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(f)

$\|x-z\| \le \|x-y\| + \|y-z\|$

$\|x-z\| = \|x-y+y-z\|$

Through (e)

$\le \|x-y\| + \|y-z\|$

$CH \# 01$

Question No. 1

If r is rational and x is irrational, Prove that $r+x$ and rx are irrational.

Solution :- Assume that $r+x$ is rational

Then,

$$r+x = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}$$
$$q \neq 0$$

Then we have,

$$x = \frac{p}{q} - r$$

$$x = \frac{p - rq}{q}$$

Since

$$p - rq \in \mathbb{Z} \quad \text{and} \quad \frac{p - rq}{q} \in \mathbb{Q}$$

$\Rightarrow$ $x \in \mathbb{Q}$ i.e x is rational

This is contradiction, because x was assumed to be irrational.

$\Rightarrow$ Hence $\gamma + x$ is irrational.

Similarly, Assume that γx is rational.

Then,

$$\gamma x = \frac{m}{n} \quad \text{where } m, n \in \mathbb{Z}$$

$$n \neq 0$$

Then we have

$$x = \frac{m}{n} \cdot \frac{1}{\gamma} \Rightarrow \frac{m}{n\gamma}$$

$$\Rightarrow \frac{m}{n\gamma} \in \mathbb{Q}$$

$x \in \mathbb{Q}$ i.e x is rational.

This is contradiction, because x was assumed to be irrational.

Hence γx is irrational.

Question No. 2

Prove that there is rational number whose square is 12.

Solution :-

Suppose that there is a rational number r whose square is 12

$$r = \frac{x}{y} \quad , \quad y \neq 0$$

Here, x and y are two co-prime numbers.

$$(r)^2 = \left(\frac{x}{y}\right)^2$$

$$\Rightarrow \frac{x^2}{y^2} = 12$$

$$\Rightarrow x^2 = 12y^2$$

$$\Rightarrow x^2 = 3.4y^2$$

$$\Rightarrow x^2 = 3(2y)^2 \quad \text{—— (1)}$$

This implies that x is multiple of 3. i.e

$$x = 3K \quad \text{—— (11)}$$

Using (1) and (11)

$$(3K)^2 = 3(2y)^2$$

$$\Rightarrow \quad 9k^2 = 12y^2$$

$$\Rightarrow \quad 3k^2 = 4y^2$$

This implies that y is also multiple of 3. So, our assumption was wrong that x and y are co-prime. There is no such rational number whose square is 12.

Question No. 4

Let E be a non-empty subset of ordered set, Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$

Solution :- E is a non-empty set and α is a lower bound of E

Then

$$x \geq \alpha \quad \text{—— (I)} \quad \forall\, x \in E$$

Similarly, β is an upper bound of E. Then

$$x \leq \beta \quad \text{—— (II)} \quad \forall\, x \in E$$

From (I) and (II)

$$\alpha \leq x \leq \beta$$

Now by transitivity of ordered set

$$\alpha \leq \beta$$

Question NO. 5

Let A be a non-empty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$ where $x \in A$. Prove that

$$\inf(A) = -\operatorname{Sup}(-A)$$

Solution :-

We need to prove that $-\sup(-A)$ is the greatest lower bound of A. $\inf(A) = -\sup(-A)$

We need to show that

$$\alpha \leq x \quad \forall \; x \in A$$

$$\alpha \geq \beta \quad \text{if } \beta \text{ is any lower bound of } A.$$

Suppose,

$$x \in A$$

$$-x \in -A$$

Hence,

$$-x \leq +\sup(-A)$$

$$x \geq -\sup(-A)$$

$$x \geq \alpha$$

Thus α is a lower bound of A.

Now, Let β be any lower bound of A. This means

$$\beta \le x \quad \forall \quad x \in A$$

$$-x \le -\beta \quad \forall \quad x \in A$$

Which says,

$$y \le -\beta \text{ for all } y \in -A$$

This means $-\beta$ is an upper bound of $-A$. Hence,

$$-\beta \geq \sup(-A)$$

$$\beta \le -\sup(-A)$$

$$\Rightarrow \quad \beta \le \alpha$$

Hence, $-\sup(-A)$ is the greatest lower bound of A i.e

$$\inf(A) = -\sup(-A)$$

Question NO. 6

(a) Fix $b > 1$

If m, n, p, q are integers, $n > 0, q > 0$

$r = \frac{m}{n} = \frac{p}{q}$. Prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

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Solution :-

Let $K = mq = np$

Since there is only one positive real no.s

c such that

$$c^{nq} = b^k$$

Putting value of c

$$(b^{m/n})^{nq} = b^{mq} = b^k \quad \text{--- (i)}$$

Similarly, Let $(b^{p/q}) = d$

$$(d)^{nq} \Rightarrow (b^{p/q})^{nq} = b^{np} = b^k \quad \text{--- (ii)}$$

Comparing (i) and (ii)

$$((b^m)^{1/n})^{nq} = ((b^p)^{1/q})^{nq}$$

$$\Rightarrow \quad (b^m)^{1/n} = (b^p)^{1/q}$$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Solution :-

Let

$$r = \frac{m}{n} \text{ and } s = \frac{v}{w}$$

$$r+s = \frac{m}{n} + \frac{v}{w} \Rightarrow \frac{mw+vn}{nw}$$

and

$$b^{r+s} = \left(b^{(mw+vn)}\right)^{\frac{1}{nw}} \Rightarrow \left((b^{mw})(b^{vn})\right)^{\frac{1}{nw}}$$

$$b^{r+s} = b^{\frac{mw}{nw}} \cdot b^{\frac{vn}{nw}}$$

$$b^{r+s} = b^r b^s$$

Proved.

(c) If x is real, define $B(x)$ to be set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \operatorname{Sup} B(r)$$

Hence, it makes sense to define

$$b^x = \operatorname{Sup} B(x)$$

Solution :-
According to given condition

for $x \in \mathbb{R}$ we can define $B(x)$ such that

$$B(x) = \{ b^t, t \in Q, t \leq x \} \quad \text{--- (1)}$$

$$b^x = \operatorname{Sup} B(x)$$

$$\therefore b^t \leq b^x \quad \forall\ b^t \in B(x).$$

If for any $b^t \in B(x)$, we have

$$b^t = b^x$$

then x will be rational.

Let, this rational $x = r$

So, set (1) becomes

$$B(r) = \{ b^t, t \in Q, t \leq r \}$$

$$\Rightarrow b^r = \operatorname{Sup} B(r)$$

This leads us to define for every $x \in \mathbb{R}$ there is $B(x)$ such that

$$B(x) = \{ b^t, t \in \mathbb{R}, t \leq x \}$$

$$b^x = \operatorname{Sup} B(x)$$

$$\because \; b^t \leq b^x \quad \forall \; b^t \in B(x)$$

(d) Prove that

$$b^{x+y} = b^x b^y$$

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where

$$x, y \in \mathbb{R}$$

Solution :- Let A and B be two non-empty set of positive real numbers.

A and B are both bounded Above

Then Using Part (c)

$$A = B(x) = \{ b^r : r \in Q , r \leq x \}$$

$$\Rightarrow \operatorname{Sup}(A) = \operatorname{Sup} B(x) = b^x$$

$$B = B(y) = \{ b^s : s \in Q : s \leq y \}$$

$$\Rightarrow \operatorname{Sup}(B) = \operatorname{Sup} B(y) = b^y$$

Let C be another set defined by

$$C = \{ ab : a \in A , b \in B \}$$

$\Rightarrow \sup(C) = \sup(A) . \sup(B) = b^x . b^y$

$\Rightarrow \quad b^x b^y = \sup B(x) . \sup B(y)$

$b^x b^y = \sup\{b^{r+s} : r \in Q, s \in Q, r \leq x, s \leq y\}$

Let $r+s = t$

$\because \quad r \leq x$ and $s \leq y$

$\Rightarrow \quad r+s \leq x+y$

$\Rightarrow \quad t \leq x+y$

$b^x b^y = \sup\{b^t : t \in Q, t \leq x+y\}$

$\Rightarrow \quad b^x b^y = \sup B(x+y)$

$b^x b^y = b^{x+y}$

Question NO. 17

Prove that

$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$

Solution :- $|x+y|^2 + |x-y|^2$

$= (x.y).(x+y) + (x-y)(x-y)$

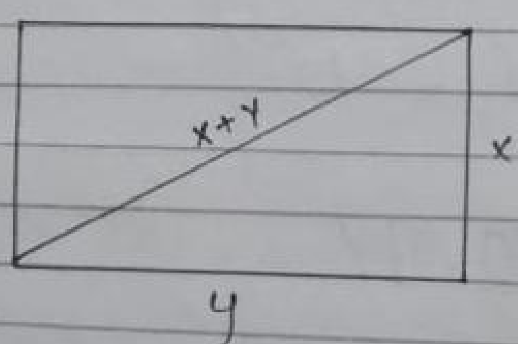
$= (x.x) + (x.y) + (y.x) + (y.y) + (x.x) - (x.y)$
$\qquad - (y.x) + (y.y)$

$= \|x\|^2 + 2(x.y) + \|y\|^2 + \|x\|^2 - 2(x.y) + \|y\|^2$

$= 2\|x\|^2 + 2\|y\|^2$

$\Rightarrow$ This shows that sum of square on the diagonals of a parallelogram equal the sum of square on the sides.

Figure :-



Question NO. 18

If $k \geq 2$ and $x \in R^k$, prove that there exist $y \in R^k$ such that $y \neq 0$ but

$x.y = 0$

$k =$

Solution :-

If x

$x = (x,$

$= (0$

Then we

$y =$

satisfy

$x.y =$

$x.y =$

$x.y$

$\Rightarrow$

then w

non-ze

$x \cdot y = 0$. Is this also true if

$K = 1$

Solution :- (i) For $K \geq 2$

If x has all zero component

$x = (x_1, x_2, \ldots\ldots, x_K)$

$= (0, 0, \ldots\ldots, 0)$

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Then we can have a non zero y component

$y = (y_1, y_2, \ldots\ldots, y_K)$

satisfying condition $x \cdot y = 0$

$x \cdot y = (0, 0, \ldots\ldots, 0) \cdot (y_1, y_2, \ldots\ldots, y_K)$

$x \cdot y = (0 + 0 + \ldots\ldots + 0)$

$x \cdot y = 0$

$\Rightarrow$ If x has any zero components

Then we can take corresponding component

non-zero and all other components

zero satisfy $x.y = 0, \; y \neq 0$

$x = (x_1, 0, 0, x_4, x_5, \dots\dots, x_k)$

$y = (0, y_2, y_3, 0, 0, \dots\dots, 0)$

$x.y = (x_1.0 + 0.y_2 + 0.y_3 + \dots\dots + x_k.0)$

$x.y = 0 + 0 + 0 + \dots\dots + 0$

$x.y = 0$

⇒ If all component of x are non-zero

then we can take $y = (-x_2, x_1, 0, 0, \dots\dots, 0)$

to satisfy $x.y = 0$ and $y \neq 0$

$x = (x_1, x_2, x_3, \dots\dots, x_k)$

$x.y = (x_1, x_2, x_3, \dots\dots, x_k).(x_1, -x_1, 0, \dots, 0)$

$x.y = -x_1x_2 + x_1x_2 + 0 + 0 + \dots\dots + 0$

$x.y = 0$

Hence Proved for $x \in R^k$ $\exists$ $y \in R^k$

such that $y \neq 0$ but $xy = 0$

for $K \geq 2$

(ii) For $K = 1$

Now, $x \in \mathbb{R}$ and $y \in \mathbb{R}$

The only value of x satisfying

$x.y = 0$ and $y \neq 0$ is $x = 0$

Because, we know that product of any two non-zero real number is always non-zero.

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Question NO. 19

Suppose $a \in R^k$, $b \in R^k$. Find $c \in R^k$ and $r > 0$ such that

$$|x-a| = 2|x-b|$$

if and only if

$$|x-c| = r$$

Solution :-

$$|x-a| = 2|x-b|$$

Equivalent to

$$|x-c| = r$$

$$|x - \frac{4}{3}b + \frac{1}{3}a| = \frac{2}{3}|b-a|$$

Taking square on both side

$$(x-\frac{4}{3}b+\frac{1}{3}a).(x-\frac{4}{3}b+\frac{1}{3}a) = \frac{4}{9}(b^2+a^2-2ba)$$

$$|x|^2 - \frac{4}{3}bx + \frac{1}{3}xa - \frac{4}{3}bx + \frac{16}{9}|b|^2 - \frac{4}{9}ba + \frac{1}{3}ax - \frac{4}{9}ba + \frac{1}{9}|a|^2 = \frac{4}{9}(|b|^2+|a|^2-2|ba|)$$

$$|x|^2 - \frac{8}{3}bx + \frac{2}{3}ax + \frac{16}{9}|b|^2 - \cancel{\frac{8}{9}ba} + \frac{1}{9}|a|^2 = \frac{4}{9}b^2 + \frac{4}{9}a^2 - \cancel{\frac{8}{9}ba}$$

$$|x|^2 - \frac{8}{3}bx + \frac{2}{3}ax + \frac{16}{9}|b|^2 + \frac{1}{9}|a|^2 - \frac{4}{9}a^2 - \frac{4}{9}b^2 = 0$$

$$9|x|^2 - 24bx + 6ax + 16|b|^2 + |a|^2 - 4|a^2| - 4|b^2| = 0$$

$$3|x^2| - 8bx + 2ax + 4|b|^2 - |a^2| = 0$$

Taking Square on both side

$$(|x-a|)^2 = (2|x-b|)^2$$

$$(x-a).(x-a) = 4(x-b).(x-b)$$

$$x^2 - a.x - x.a + a^2 = 4(x^2 - x.b - b.x + b^2)$$

$$x^2 - 2(a.x) + a^2 = 4x^2 - 8x.b + 4b^2$$

$$-3|x^2| - 2|a.x| + 8(x.b) + |a|^2 - 4|b^2| = 0$$

$$3|x^2| + 2|a.x| - 8(x.b) + 4|b^2| - |a|^2 = 0$$

So, both are Equivalent.

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