CHAPTER 1

# Infinite Products

**1. Introduction.** Two topics, infinite products and asymptotic series, which are seldom included in standard courses are treated to some extent in short preliminary chapters.

The variables and parameters encountered are to be considered complex except where it is specifically stipulated that they are real.

Exercises are included not only to present the reader with an opportunity to increase his skill but also to make available a few results for which there seemed to be insufficient space in the text.

A short bibliography is included at the end of the book. All references are given in a form such as Fasenmyer [2], meaning item number two under the listing of references to the work of Sister M. Celine Fasenmyer, or Brafman [1;944], meaning page 944 of item number one under the listing of references to the work of Fred Brafman. In general, specific reference to material a century or more old is omitted. The work of the giants in the field, Euler, Gauss, Legendre, etc., is easily located either in standard treatises or in the collected works of the pertinent mathematician.

**2. Definition of an infinite product.** The elementary theory of infinite products closely parallels that of infinite series. Given a sequence $a_k$ defined for all positive integral $k$, consider the finite product

$$(1) \qquad P_n = \prod_{k=1}^{n}(1 + a_k) = (1 + a_1)(1 + a_2)\cdots(1 + a_n).$$

If $\lim_{n\to\infty} P_n$ exists and is equal to $P \neq 0$, we say that the infinite product

$$\prod_{n=1}^{\infty} (1 + a_n) \tag{2}$$

*converges* to the value $P$. If at least one of the factors of the product (2) is zero, if only a finite number of the factors of (2) are zero, and if the infinite product with the zero factors deleted converges to a value $P \neq 0$, we say that the infinite product *converges to zero.*

If the infinite product is not convergent, it is said to be *divergent.* If that divergence is due not to the failure of $\lim_{n\to\infty} P_n$ to exist but to the fact that the limit is zero, the product is said to *diverge to zero.* We make no attempt to treat products with an infinity of zero factors.

The peculiar role which zero plays in multiplication is the reason for the slight difference between the definition of convergence of an infinite product and the analogous definition of convergence of an infinite series.

**3. A necessary condition for convergence.** The general term of a convergent infinite series must approach zero as the index of summation approaches infinity. A similar result will now be obtained for infinite products.

THEOREM 1. *If* $\prod_{n=1}^{\infty} (1 + a_n)$ *converges,*

$$\lim_{n\to\infty} a_n = 0.$$

*Proof:* If the product converges to $P \neq 0$,

$$1 = \frac{P}{P} = \frac{\lim_{n\to\infty} \prod_{k=1}^{n} (1 + a_k)}{\lim_{n\to\infty} \prod_{k=1}^{n-1} (1 + a_k)} = \lim_{n\to\infty} (1 + a_n).$$

Hence $\lim_{n\to\infty} a_n = 0$, as desired. If the product converges to zero, remove the zero factors and repeat the argument.

**4. The associated series of logarithms.** Any product without zero factors has associated with it the series of principal values of the logarithms of the separate factors in the following sense.

THEOREM 2. *If no* $a_n = -1$, $\prod_{n=1}^{\infty}(1 + a_n)$ *and* $\sum_{n=1}^{\infty} \operatorname{Log}(1 + a_n)$ *converge or diverge together.*

*Proof:* Let the partial product and partial sum be indicated as follows:

$$P_n = \prod_{k=1}^{n}(1 + a_k), \qquad S_n = \sum_{k=1}^{n} \operatorname{Log}(1 + a_k).$$

Then* $\exp S_n = P_n$. We know from the theory of complex variables that $\lim_{n\to\infty} \exp S_n = \exp \lim_{n\to\infty} S_n$. Therefore $P_n$ approaches a limit if and only if $S_n$ approaches a limit, and $P_n$ cannot approach zero because the exponential function cannot take on the value zero.

**5. Absolute convergence.** Assume that the product $\prod_{n=1}^{\infty}(1 + a_n)$ has had its zero factors, if any, deleted. We define *absolute convergence* of the product by utilizing the associated series of logarithms.

The product $\prod_{n=1}^{\infty}(1 + a_n)$, with zero factors deleted, is said to be absolutely convergent if and only if the series $\sum_{n=1}^{\infty} \operatorname{Log}(1 + a_n)$ is absolutely convergent.

THEOREM 3. *The product* $\prod_{n=1}^{\infty}(1 + a_n)$, *with zero factors deleted, is absolutely convergent if and only if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

*Proof:* First throw out any $a_n$'s which are zero; they contribute only unit factors in the product and zero terms in the sum and thus have no bearing on convergence.

We know that if either the series or the product in the theorem converges, $\lim_{n\to\infty} a_n = 0$. Let us then consider $n$ large enough, $n > n_0$, so that $|a_n| < \frac{1}{2}$ for all $n > n_0$. We may now write

$$\frac{\operatorname{Log}(1 + a_n)}{a_n} = \sum_{k=0}^{\infty} \frac{(-1)^k a_n{}^k}{k + 1}, \tag{1}$$

from which it follows that

$$\left| \frac{\operatorname{Log}(1 + a_n)}{a_n} - 1 \right| \leqq \sum_{k=1}^{\infty} \frac{|a_n|^k}{k+1} < \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}.$$

*We make frequent use of the common notation $\exp u = e^u$.

Thus we have

$$\frac{1}{2} < \left| \frac{\text{Log}\,(1 + a_n)}{a_n} \right| < \frac{3}{2},$$

from which

$$\left| \frac{\text{Log}\,(1 + a_n)}{a_n} \right| < \frac{3}{2} \qquad \text{and} \qquad \left| \frac{a_n}{\text{Log}\,(1 + a_n)} \right| < 2.$$

By the comparison test it follows that the absolute convergence of either of $\sum_{n=1}^{\infty} \text{Log}\,(1 + a_n)$ or $\sum_{n=1}^{\infty} a_n$ implies the absolute convergence of the other. We then use the definition of absolute convergence of the product to complete the proof of Theorem 3.

Because of Theorem 2 it follows at once that an infinite product which is absolutely convergent is also convergent.

Example (*a*): Show that the following product converges and find its value:

$$\prod_{n=1}^{\infty} \left[ 1 + \frac{1}{(n+1)(n+3)} \right].$$

The series of positive numbers

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$$

is known to be convergent. It can easily be tested by the polynomial test or by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Hence our product is absolutely convergent by Theorem 3.

The partial products are often useful in evaluating an infinite product. When the following method is employed, there is no need for the separate testing for convergence made in the preceding paragraph. Consider the partial products

$$P_n = \prod_{k=1}^{n} \left[ 1 + \frac{1}{(k+1)(k+3)} \right] = \prod_{k=1}^{n} \frac{(k+2)^2}{(k+1)(k+3)}$$

$$= \frac{[3 \cdot 4 \cdot 5 \cdots (n+2)]^2}{[2 \cdot 3 \cdot 4 \cdots (n+1)][4 \cdot 5 \cdot 6 \cdots (n+3)]} = \frac{n+2}{2} \cdot \frac{3}{n+3}.$$

At once $\lim_{n \to \infty} P_n = \frac{3}{2}$, from which we conclude both that the infinite product converges and that its value is $\frac{3}{2}$.

Example (*b*): Show that if $z$ is not a negative integer,

$$\lim_{n\to\infty} \frac{(n-1)!\ n^z}{(z+1)(z+2)(z+3)\cdots(z+n-1)}$$

exists.

We shall form an infinite product for which the expression

$$P_n = \frac{(n-1)!\ n^z}{(z+1)(z+2)(z+3)\cdots(z+n-1)}$$

is a partial product, prove that the infinite product converges, and thus conclude that $\lim_{n\to\infty} P_n$ exists.

Write

$$\begin{aligned} P_{n+1} &= \frac{n!\ (n+1)^z}{(z+1)(z+2)\cdots(z+n)} \\ &= \frac{n!}{(z+1)(z+2)\cdots(z+n)} \cdot \frac{2^z}{1^z} \cdot \frac{3^z}{2^z} \cdot \frac{4^z}{3^z} \cdots \frac{(n+1)^z}{n^z} \\ &= \prod_{k=1}^{n} \left[\frac{k}{z+k} \cdot \frac{(k+1)^z}{k^z}\right] = \prod_{k=1}^{n} \left[\left(1+\frac{z}{k}\right)^{-1}\left(1+\frac{1}{k}\right)^z\right]. \end{aligned}$$

Consider now the product*

$$\prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^z\right]. \tag{2}$$

Since

$$\lim_{n\to\infty} n^2\left[\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^z - 1\right]$$

$$= \lim_{\beta\to 0} \frac{(1+z\beta)^{-1}(1+\beta)^z - 1}{\beta^2} = \lim_{\beta\to 0} \frac{(1+\beta)^z - 1 - z\beta}{\beta^2}$$

$$= \lim_{\beta\to 0} \frac{z[(1+\beta)^{z-1} - 1]}{2\beta} = \lim_{\beta\to 0} \frac{z(z-1)(1+\beta)^{z-2}}{2} = \tfrac{1}{2}z(z-1),$$

we conclude with the aid of the comparison test and the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ that the product (2) converges. Therefore $\lim_{n\to\infty} P_n$ exists.

**6. Uniform convergence.** Let the factors in the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ be dependent upon a complex variable $z$. Let $R$

*We shall find in Chapter 2 that this product has the value $z\Gamma(z)$.

be a closed region in the $z$-plane. If the product converges in such a way that, given any $\epsilon > 0$, there exists an $n_0$ independent of $z$ for all $z$ in $R$ such that

$$\left| \prod_{k=1}^{n_0+p} [1 + a_k(z)] - \prod_{k=1}^{n_0} [1 + a_k(z)] \right| < \epsilon$$

for all positive integral $p$, we say that the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ is *uniformly convergent* in the region $R$.

Again the convergence properties parallel those of infinite series. We need a Weierstrass $M$-test.

THEOREM 4. *If there exist positive constants $M_n$ such that $\sum_{n=1}^{\infty} M_n$ is convergent and $|a_n(z)| < M_n$ for all $z$ in the closed region $R$, the product $\prod_{n=1}^{\infty} [1 + a_n(z)]$ is uniformly convergent in $R$.*

*Proof:* Since $\sum_{n=1}^{\infty} M_n$ is convergent and $M_n > 0$, $\prod_{n=1}^{\infty} (1 + M_n)$ is convergent and $\lim_{n \to \infty} \prod_{k=1}^{n} (1 + M_k)$ exists. Therefore, given any $\epsilon > 0$, there exists an $n_0$ such that

$$\prod_{k=1}^{n_0+p} (1 + M_k) - \prod_{k=1}^{n_0} (1 + M_k) < \epsilon$$

for all positive integers $p$. For all $z$ in $R$, each $a_k(z)$ is such that $|a_k(z)| < M_k$. Hence

$$\left| \prod_{k=1}^{n_0+p} [1 + a_k(z)] - \prod_{k=1}^{n_0} [1 + a_k(z)] \right|$$

$$= \left| \prod_{k=1}^{n_0} [1 + a_k(z)] \right| \cdot \left| \prod_{k=n_0+1}^{n_0+p} [1 + a_k(z)] - 1 \right|$$

$$< \prod_{k=1}^{n_0} (1 + M_k) \left[ \prod_{k=n_0+1}^{n_0+p} (1 + M_k) - 1 \right]$$

$$< \prod_{k=1}^{n_0+p} (1 + M_k) - \prod_{k=1}^{n_0} (1 + M_k) < \epsilon,$$

which was to be proved.

## EXERCISES

1. Show that the following product converges, and find its value:

$$\prod_{n=1}^{\infty}\left[1+\frac{6}{(n+1)(2n+9)}\right]. \qquad \textit{Ans. } \frac{21}{8}.$$

2. Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$

3. Show that $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ diverges to zero.

4. Investigate the product $\prod_{n=0}^{\infty}(1+z^{2^n})$ in $|z|<1$.

*Ans.* Abs. conv. to $\frac{1}{1-z}$.

5. Show that $\prod_{n=1}^{\infty}\exp\left(\frac{1}{n}\right)$ diverges.

6. Show that $\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}\right)$ diverges to zero.

7. Test $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$. *Ans.* Abs. conv. for all finite $z$.

8. Show that $\prod_{n=1}^{\infty}\left[1+\frac{(-1)^{n+1}}{n}\right]$ converges to unity.

9. Test for convergence: $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^p}\right)$ for real $p\neq 0$.

*Ans.* Conv. for $p>1$· div. for $p\leqq 1$.

10. Show that $\prod_{n=1}^{\infty}\frac{\sin(z/n)}{z/n}$ is absolutely convergent for all finite $z$ with the usual convention at $z=0$. *Hint:* Show first that

$$\lim_{n\to\infty} n^2\left[\frac{\sin(z/n)}{z/n}-1\right]=-\frac{z^2}{6}.$$

11. Show that if $c$ is not a negative integer,

$$\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{c+n}\right)\exp\left(\frac{z}{n}\right)\right]$$

is absolutely convergent for all finite $z$. *Hint*: Show first that

$$\lim_{n\to\infty} n^2\left[\left(1-\frac{z}{c+n}\right)\exp\left(\frac{z}{n}\right)-1\right]=z\left(c-\frac{1}{2}z\right).$$