

2.6 Continuity

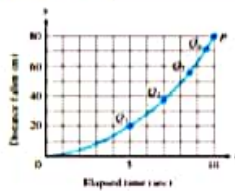


FIGURE 2.49 Connecting plotted points by an unbroken curve from experimental data Q_1, Q_2, Q_3, \dots for a falling object

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.49). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. The limit of a continuous function as x approaches c can be found simply by calculating the value of the function at c (We found this to be true for polynomials in Section 2.2.)

Any function $y = f(x)$ whose graph can be sketched over its domain in one continuous motion without lifting the pencil is an example of a *continuous function*. In this section we investigate more precisely what it means for a function to be continuous. We also study the properties of continuous functions, and see that many of the function types presented in Section 1.4 are continuous.

Continuity at a Point

To understand continuity, we need to consider a function like the one in Figure 2.50 whose limits we investigated in Example 2, Section 2.4.

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.50 is continuous and the points at which f is discontinuous.

Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

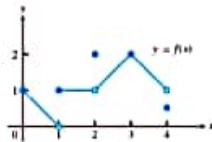


FIGURE 2.50 The function is continuous on $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$ (Example 1)

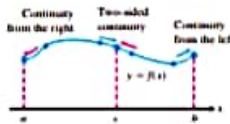


FIGURE 2.51 Continuity at points a , b , and c .

Points at which f is continuous:

- At $x = 0$, $\lim_{x \rightarrow 0} f(x) = f(0)$
- At $x = 3$, $\lim_{x \rightarrow 3} f(x) = f(3)$
- At $0 < c < 4$, $c \neq 1, 2$, $\lim_{x \rightarrow c} f(x) = f(c)$

Points at which f is discontinuous:

- At $x = 1$, $\lim_{x \rightarrow 1} f(x)$ does not exist
- At $x = 2$, $\lim_{x \rightarrow 2} f(x) = 1$, but $1 \neq f(2)$
- At $x = 4$, $\lim_{x \rightarrow 4} f(x) = 1$, but $1 \neq f(4)$
- At $c < 0$, $c > 4$, these points are not in the domain of f

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 2.51).

DEFINITION Continuous at a Point

Interior point A function $y = f(x)$ is *continuous at an interior point* c of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Endpoint A function $y = f(x)$ is *continuous at a left endpoint* a or is *continuous at a right endpoint* b of its domain if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \quad \text{respectively}$$

If a function f is not continuous at a point c , we say that f is *discontinuous at* c and c is a *point of discontinuity of* f . Note that c need not be in the domain of f .

A function f is *right-continuous (continuous from the right)* at a point $x = c$ in its domain if $\lim_{x \rightarrow c^+} f(x) = f(c)$. It is *left-continuous (continuous from the left)* at c if $\lim_{x \rightarrow c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b . A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.51).

EXAMPLE 2 A Function Continuous Throughout Its Domain

The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Figure 2.52), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous.

EXAMPLE 3 The Unit Step Function Has a Jump Discontinuity

The unit step function $U(x)$, graphed in Figure 2.53, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a *jump discontinuity* at $x = 0$.

We summarize continuity at a point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \rightarrow c} f(x)$ exists (f has a limit as $x \rightarrow c$)
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

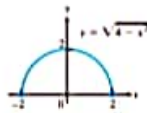


FIGURE 2.52 A function that is continuous at every domain point (Example 2)

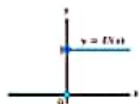


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3)

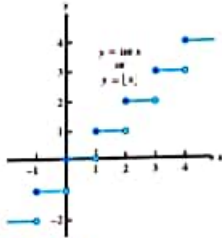


FIGURE 2.54 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

EXAMPLE 4 The Greatest Integer Function

The function $y = [x]$ or $y = \text{int } x$, introduced in Chapter 1, is graphed in Figure 2.54. It is discontinuous at every integer because the limit does not exist at any integer n .

$$\lim_{x \rightarrow n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x \rightarrow n^+} \text{int } x = n$$

so the left-hand and right-hand limits are not equal as $x \rightarrow n$. Since $\text{int } n = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \rightarrow 1.5} \text{int } x = 1 = \text{int } 1.5$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \rightarrow c} \text{int } x = n - 1 = \text{int } c.$$

Figure 2.55 is a catalog of discontinuity types. The function in Figure 2.55b would be continuous if $f(0)$ were 1 . The function in Figure 2.55c would be continuous if $f(0)$ were 2 . Each function has a removable discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.55d through f are more serious. $\lim_{x \rightarrow 0} f(x)$ does not exist, and there is no way to improve the situation by changing f at 0 . The step function in Figure 2.55d has a **jump discontinuity**. The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.55e has an **infinite discontinuity**. The function in Figure 2.55f has an **oscillating discontinuity**. It oscillates too much to have a limit as $x \rightarrow 0$.

146 / 1564

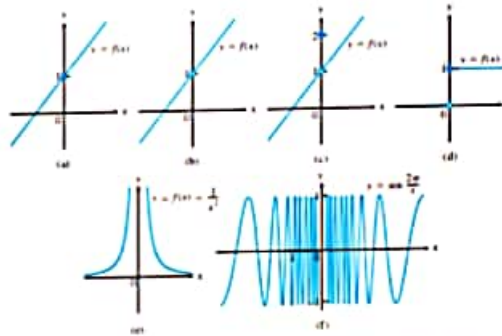


FIGURE 2.55 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

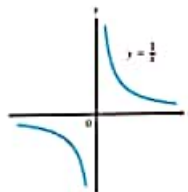


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 2.52 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$ (Figure 2.56), but it is continuous over its domain $(-\infty, 0) \cup (0, \infty)$.

EXAMPLE 5 Identifying Continuous Functions

- (a) The function $y = 1/x$ (Figure 2.56) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there.
- (b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3.

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. Sums $f + g$
2. Differences $f - g$
3. Products $f \cdot g$
4. Constant multiples $k \cdot f$, for any number k
5. Quotients f/g provided $g(c) \neq 0$
6. Powers f^r , provided it is defined on an open interval containing c , where r and s are integers

Most of the results in Theorem 9 are easily proved from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned} \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x), && \text{Sum Rule, Theorem 1} \\ &= f(c) + g(c), && \text{Continuity of } f, g \text{ at } c \\ &= (f + g)(c). \end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6 Polynomial and Rational Functions Are Continuous

- (a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is continuous because $\lim_{x \rightarrow c} P(x) = P(c)$ by Theorem 2, Section 2.2.

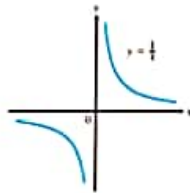


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 2.52 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$ (Figure 2.56), but it is continuous over its domain $(-\infty, 0) \cup (0, \infty)$.

EXAMPLE 5 Identifying Continuous Functions

- (a) The function $y = 1/x$ (Figure 2.56) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there.
 - (b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3.
- Algebraic combinations of continuous functions are defined.

147 / 1564

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. **Sum:** $f + g$
2. **Difference:** $f - g$
3. **Product:** $f \cdot g$
4. **Constant multiples:** $k \cdot f$, for any number k
5. **Quotient:** f/g , provided $g(c) \neq 0$
6. **Powers:** f^r , provided it is defined on an open interval containing c , where r and s are integers

Most of the results in Theorem 9 are easily proved from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned} \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x), && \text{Sum Rule, Theorem 1} \\ &= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\ &= (f + g)(c). \end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6 Polynomial and Rational Functions Are Continuous

- (a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is continuous because $\lim_{x \rightarrow c} P(x) = P(c)$ by Theorem 2, Section 2.2.

- (b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(x) \neq 0$) by the Quotient Rule in Theorem 9.

EXAMPLE 7 Continuity of the Absolute Value Function

The function $f(x) = |x|$ is continuous at every value of x . If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \rightarrow 0} |x| = 0 = |0|$.

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 6 of Section 2.2. Both functions are, in fact, continuous everywhere (see Exercise 62). It follows from Theorem 9 that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\dots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \dots$.

Composites

All composites of continuous functions are continuous. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.57). In this case, the limit as $x \rightarrow c$ is $g(f(c))$.

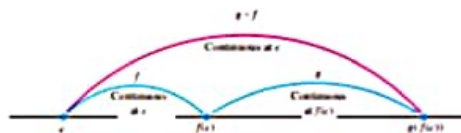


FIGURE 2.57 Composites of continuous functions are continuous.

THEOREM 10 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c .

Intuitively, Theorem 10 is reasonable because if x is close to c , then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied. For an outline of the proof of Theorem 10, see Exercise 6 in Appendix 2.



EXAMPLE 8 Applying Theorems 9 and 10

Show that the following functions are continuous everywhere on their respective domains.

- (a) $y = \sqrt{x^2 - 2x - 5}$
- (b) $y = \frac{x-3}{1+x^2}$
- (c) $y = \left| \frac{x-2}{x^2-2} \right|$
- (d) $y = \frac{x \sin x}{x^2 + 2}$

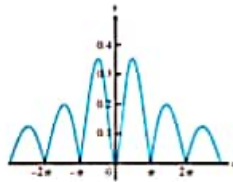


FIGURE 2.58 The graph suggests that $y = |(x \sin x)(x^2 + 2)|$ is continuous (Example 8d).

Solution

- (a) The square root function is continuous on $[0, \infty)$ because it is a rational power of the continuous identity function $f(x) = x$ (Part 6, Theorem 9). The given function is then the composite of the polynomial $f(x) = x^2 - 2x + 5$ with the square root function $g(t) = \sqrt{t}$.
- (b) The numerator is a rational power of the identity function, the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.
- (c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).
- (d) Because the sine function is everywhere-continuous (Exercise 62), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.58).

Continuous Extension to a Point

The function $y = (\sin x)/x$ is continuous at every point $x \neq 0$. But $y = (\sin x)/x$ is different from $y = 1/x$. It is therefore possible to extend point $x = 0$ in such a way that the extended function

149 / 1564

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

The function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = F(0)$$

(Figure 2.59).

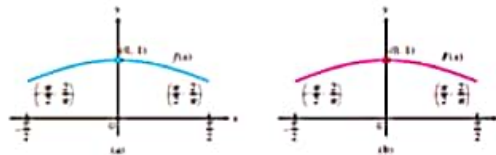


FIGURE 2.59 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \rightarrow 0} f(x)$.

More generally, a function (such as a rational function) may have a limit even at a point where it is not defined. If $f(x)$ is not defined, but $\lim_{x \rightarrow c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension** of f to $x = c$. For rational functions f , continuous extensions are usually found by canceling common factors.



EXAMPLE 9 A Continuous Extension

Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} f(x) = \frac{5}{4}$$

The graph of f is shown in Figure 2.60. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f with its point of discontinuity at $x = 2$ removed.

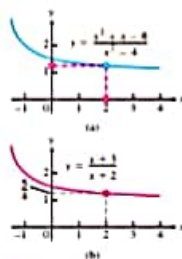


FIGURE 2.60 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 9).

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the **Intermediate Value Property**. A function is said to have the **Intermediate Value Property** if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.