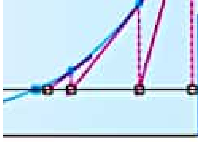


Chapter 4

APPLICATIONS OF DERIVATIVES

OVERVIEW This chapter studies some of the important applications of derivatives. We learn how derivatives are used to find extreme values of functions, to determine and analyze the shapes of graphs, to calculate limits of fractions whose numerators and denominators both approach zero or infinity, and to find numerically where a function equals zero. We also consider the process of recovering a function from its derivative. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in Chapter 5.

4.1 Extreme Values of Functions

This section shows how to locate and identify extreme values of a continuous function from its derivative. Once we can do this, we can solve a variety of *optimization problems* in which we find the optimal (best) way to do something in a given situation.

DEFINITIONS Absolute Maximum, Absolute M

Let f be a function with domain D. Then f has an ab
D at a point c if

$$f(x) \le f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \quad \text{for all } x \text{ in } D.$$

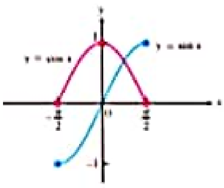


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema, to distinguish them from *local extrema* defined below.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions with the same defining rule can have different extrema, depending on the domain.

EXAMPLE 1 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

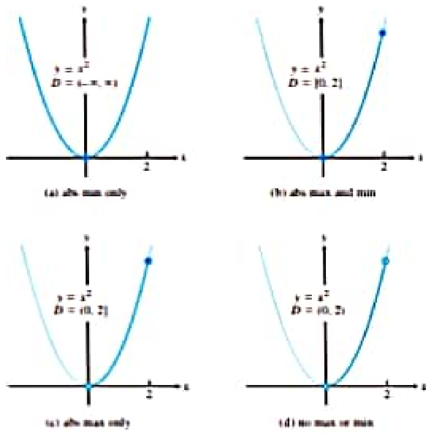


FIGURE 4.2 Graphs for Example 1.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

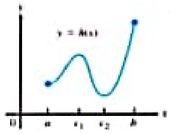
HISTORICAL BIOGRAPHY
Daniel Bernoulli
(1700–1789)

The following theorem asserts that a function which is continuous at every point of a closed interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function.

EXERCISES 4.1

Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

1.


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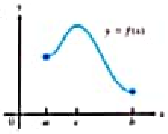

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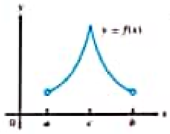

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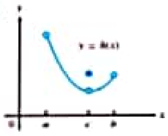

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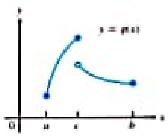

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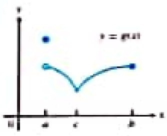
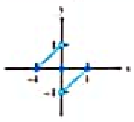


In Exercises 7–10, find the extreme values and where they occur.

7.


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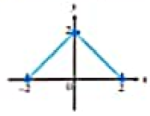

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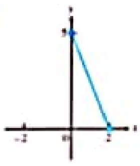

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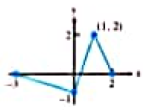


In Exercises 11–14, match the table with a graph.

11.

x	$f'(x)$
a	0
b	0
c	5

12.

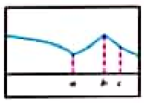
x	$f'(x)$
a	0
b	0
c	−5

13.

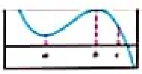
x	$f'(x)$
a	does not exist
b	0
c	−2

14.

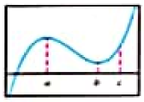
x	$f'(x)$
a	does not exist



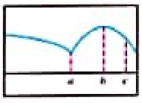
(a)



(b)



(c)



(d)

Absolute Extrema on Finite Closed Intervals

In Exercises 15–30, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

15. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$
16. $f(x) = -x - 4, \quad -4 \le x \le 1$
17. $f(x) = x^2 - 1, \quad -1 \le x \le 2$
18. $f(x) = 4 - x^2, \quad -3 \le x \le 1$
19. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$
20. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$
21. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$
22. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$
23. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$
24. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$
25. $f(\theta) = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$
26. $f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$
27. $g(x) = \csc x, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$
28. $g(x) = \sec x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$
29. $f(t) = 2 - |t|, \quad -1 \le t \le 3$
30. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 31–34, find the function's absolute maximum and minimum values and say where they are assumed.

31. $f(x) = x^{4/3}, \quad -1 \le x \le 8$
32. $f(x) = x^{5/3}, \quad -1 \le x \le 8$
33. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$
34. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Finding Extreme Values

In Exercises 35–44, find the extreme values of the function and where they occur.

35. $y = 2x^2 - 8x + 9$
36. $y = x^3 - 2x + 4$
37. $y = x^3 + x^2 - 8x + 5$
38. $y = x^3 - 3x^2 + 3x - 2$
39. $y = \sqrt{x^2 - 1}$
40. $y = \frac{1}{\sqrt{1 - x^2}}$
41. $y = \frac{1}{\sqrt[3]{1 - x^2}}$
42. $y = \sqrt{3 + 2x - x^2}$
43. $y = \frac{x}{x^2 + 1}$
44. $y = \frac{x + 1}{x^2 + 2x + 2}$

Local Extrema and Critical Points

In Exercises 45–52, find the derivative at each critical point and determine the local extreme values.

45. $y = x^{2/3}(x + 2)$
46. $y = x^{2/3}(x^2 - 4)$
47. $y = x\sqrt{4 - x^2}$
48. $y = x^2\sqrt{3 - x}$
49. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$
50. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$
51. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$
52. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

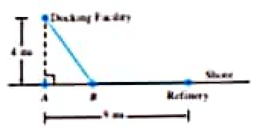
In Exercises 53 and 54, give reasons for your answers.

53. Let $f(x) = (x - 2)^{2/3}$.
   a. Does $f'(2)$ exist?
   b. Show that the only local extreme value of f occurs at $x = 2$.
   c. Does the result in part (b) contradict the Extreme Value Theorem?
   d. Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.
54. Let $f(x) = |x^3 - 9x|$.
   a. Does $f'(0)$ exist?
   b. Does $f'(3)$ exist?
   c. Does $f'(-3)$ exist?
   d. Determine all extrema of f.

Optimization Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph the function over the domain that is appropriate to the problem you are solving. The graph will provide insight before you begin to calculate and will furnish a visual context for understanding your answer.

55. **Constructing a pipeline** Supertankers off-load oil at a docking facility 4 mi offshore. The nearest refinery is 9 mi east of the shore point nearest the docking facility. A pipeline must be constructed connecting the docking facility with the refinery. The pipeline costs \$300,000 per mile if constructed underwater and \$200,000 per mile if overland.



   a. Locate Point B to minimize the cost of the construction.