5.3 The Definite Integral

In Section 5.2 we investigated the limit of a finite sum for a function defined over a closed interval $[a, b]$ using n subintervals of equal width (or length), $(b - a)/n$. In this section we consider the limit of more general Riemann sums as the norm of the partitions of $[a, b]$ approaches zero. For general Riemann sums the subintervals of the partitions need not have equal widths. The limiting process then leads to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

Limits of Riemann Sums

The definition of the definite integral is based on the idea that for certain functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann

sums approach a limiting value I. What we mean by this converging idea is that a Riemann sum will be close to the number I provided that the norm of its partition is sufficiently small (so that all of its subintervals have thin enough widths). We introduce the symbol ϵ as a small positive number that specifies how close to I the Riemann sum must be, and the symbol δ as a second small positive number that specifies how small the norm of a partition must be in order for that to happen. Here is a precise formulation.

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$

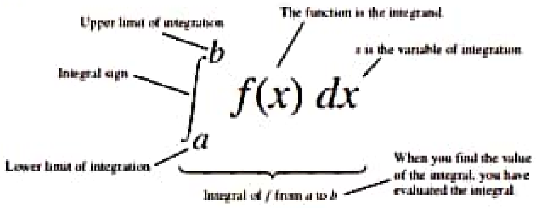
Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums. He envisioned the finite sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ becoming an infinite sum of function values $f(x)$ multiplied by "infinitesimal" subinterval widths dx. The sum symbol $\sum$ is replaced in the limit by the integral symbol $\int$, whose origin is in the letter "S." The function values $f(c_k)$ are replaced by a continuous selection of function values $f(x)$. The subinterval widths Δx_k become the differential dx. It is as if we are summing all products of the form $f(x) \cdot dx$ as x goes from a to b. While this notation captures the process of constructing an integral, it is Riemann's definition that gives a precise meaning to the definite integral.

Notation and Existence of the Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



When the definition is satisfied, we say the Riemann sums of f on $[a, b]$ **converge** to the definite integral $I = \int_a^b f(x)\,dx$ and that f is **integrable** over $[a, b]$. We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit I, no matter what choices are made. When the limit exists we write it as the definite integral

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = I = \int_a^b f(x)\,dx.$$

You Try It

When each partition has n equal subintervals, each of width $\Delta x = (b - a)/n$, we will also write

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When each partition has n equal subintervals, each of width $\Delta x = (b - a)/n$, we will also write

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = I = \int_a^b f(x)\,dx.$$

The limit is always taken as the norm of the partitions approaches zero and the number of subintervals goes to infinity.

The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \qquad \text{or} \qquad \int_a^b f(u)\,du \qquad \text{instead of} \qquad \int_a^b f(x)\,dx.$$

No matter how we write the integral, it is still the same number, defined as a limit of Riemann sums. Since it does not matter what letter we use, the variable of integration is called a **dummy variable**.

Since there are so many choices to be made in taking a limit of Riemann sums, it might seem difficult to show that such a limit exists. It turns out, however, that no matter what choices are made, the Riemann sums associated with a *continuous* function converge to the same limit.

THEOREM 1 The Existence of Definite Integrals

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

By the Extreme Value Theorem (Theorem 1, Section 4.1), when f is continuous we can choose c_k so that $f(c_k)$ gives the maximum value of f on $[x_{k-1}, x_k]$, giving an **upper sum**. We can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$, giving a **lower sum**. We can pick c_k to be the midpoint of $[x_{k-1}, x_k]$, the rightmost point x_k, or a random point. We can take the partitions of equal or varying widths. In each case we get the same limit for $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ as $\|P\| \to 0$. The idea behind Theorem 1 is that a Riemann sum associated with a partition is no more than the upper sum of that partition and no less than the lower sum. The upper and lower sums converge to the same value when $\|P\| \to 0$. All other Riemann sums lie between the upper and lower sums and have the same limit. A proof of Theorem 1 involves a careful analysis of functions, partitions, and limits along this line of thinking and is left to a more advanced text. An indication of this proof is given in Exercises 80 and 81.

Theorem 1 says nothing about how to *calculate* definite integrals. A method of calculation will be developed in Section 5.4, through a connection to the process of taking antiderivatives.

Integrable and Nonintegrable Functions

Theorem 1 tells us that functions continuous over the interval $[a, b]$ are integrable there. Functions that are not continuous may or may not be integrable. Discontinuous functions that are integrable include those that are increasing on $[a, b]$ (Exercise 77), and the *piecewise-continuous functions* defined in the Additional Exercises at the end of this chapter. (The latter are continuous except at a finite number of points in $[a, b]$.) For integrability to fail, a function needs to be sufficiently discontinuous so that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectangles. Here is an example of a function that is not integrable.

Video

EXAMPLE 1 A Nonintegrable Function on $[0, 1]$

The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there is both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the x-axis to be approximated by rectangles, no matter how thin they are. We show, in fact, that upper sum approximations and lower sum approximations converge to different limiting values.

If we pick a partition P of $[0, 1]$ and choose c_k to be the maximum value for f on $[x_{k-1}, x_k]$ then the corresponding Riemann sum is

$$U = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (1)\,\Delta x_k = 1,$$

since each subinterval $[x_{k-1}, x_k]$ contains a rational number where $f(c_k) = 1$. Note that the lengths of the intervals in the partition sum to 1, $\sum_{k=1}^{n} \Delta x_k = 1$. So each such Riemann sum equals 1, and a limit of Riemann sums using these choices equals 1.

On the other hand, if we pick c_k to be the minimum value for f on $[x_{k-1}, x_k]$, then the Riemann sum is

$$L = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (0)\,\Delta x_k = 0,$$

since each subinterval $[x_{k-1}, x_k]$ contains an irrational number c_k where $f(c_k) = 0$. The limit of Riemann sums using these choices equals zero. Since the limit depends on the choices of c_k, the function f is not integrable. ■

Properties of Definite Integrals

In defining $\int_a^b f(x)\,dx$ as a limit of sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we instead move right to left, starting with $x_0 = b$ and ending at $x_n = a$. Each Δx_k in the Riemann sum would change its sign, with $x_k - x_{k-1}$ now negative instead of positive. With the same choices of c_k in each subinterval, the sign of any Riemann sum would change, as would the sign of the limit, the integral

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$\int_b^a f(x)\,dx$. Since we have not previously given a meaning to integrating backward, we are led to define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Another extension of the integral is to an interval of zero width, when $a = b$. Since $f(c_k)\,\Delta x_k$ is zero when the interval width $\Delta x_k = 0$, we define

$$\int_a^a f(x)\,dx = 0.$$

Theorem 2 states seven properties of integrals, given as rules that they satisfy, including the two above. These rules become very useful in the process of computing integrals. We will refer to them repeatedly to simplify our calculations.

Rules 2 through 7 have geometric interpretations, shown in Figure 5.11. The graphs in these figures are of positive functions, but the rules apply to general integrable functions.

THEOREM 2
When f and g are integrable, the definite integral satisfies Rules 1 to 7 in Table 5.3.

TABLE 5.3 Rules satisfied by definite integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2. *Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
	$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6. *Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$	
7. *Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
	$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

In short, all Riemann sums for f on $[a, b]$ satisfy the inequality

$$\min f \cdot (b - a) \le \sum_{k=1}^{n} f(c_k)\,\Delta x_k \le \max f \cdot (b - a).$$

Hence their limit, the integral, does too. ■

EXAMPLE 2 Using the Rules for Definite Integrals

Suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \qquad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\displaystyle\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ Rule 1

2. $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$ Rules 3 and 4
$\qquad\qquad = 2(5) + 3(7) = 31$

3. $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$ Rule 5 ■

EXAMPLE 3 Finding Bounds for an Integral

Show that the value of $\int_0^1 \sqrt{1 + \cos x}\,dx$ is less than $3/2$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and that $\max f \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x}\,dx \le \sqrt{2} \cdot (1 - 0) = \sqrt{2}.$$

Since $\int_0^1 \sqrt{1 + \cos x}\,dx$ is bounded from above by $\sqrt{2}$ (which is 1.414 ...), the integral is less than $3/2$. ■

Area Under the Graph of a Nonnegative Function

We now make precise the notion of the area of a region with curved boundary, capturing the idea of approximating a region by increasingly many rectangles. The area under the graph of a nonnegative continuous function is defined to be a definite integral.

DEFINITION **Area Under a Curve as a Definite Integral**

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Using Properties and Known Values to Find Other Integrals

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4,\quad \int_1^5 f(x)\,dx = 6,\quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find

a. $\int_2^2 g(x)\,dx$ b. $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ d. $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ f. $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1,\quad \int_7^9 f(x)\,dx = 5,\quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find

a. $\int_1^9 -2f(x)\,dx$ b. $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ d. $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ f. $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ b. $\int_1^2 \sqrt{3}f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ d. $\int_1^2 [-f(x)]\,dx$

12. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$ b. $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$ d. $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ b. $\int_4^3 f(t)\,dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\int_1^3 h(r)\,dr$ b. $-\int_3^1 h(u)\,du$

Using Area to Evaluate Definite Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right)dx$ 16. $\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\int_{-3}^3 \sqrt{9 - x^2}\,dx$ 18. $\int_{-4}^0 \sqrt{16 - x^2}\,dx$

19. $\int_{-2}^1 |x|\,dx$ 20. $\int_{-1}^1 (1 - |x|)\,dx$

21. $\int_{-1}^1 (2 - |x|)\,dx$ 22. $\int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right)dx$

Use areas to evaluate the integrals in Exercises 23–26.

23. $\int_0^b \frac{x}{2}\,dx,\quad b > 0$ 24. $\int_0^b 4x\,dx,\quad b > 0$

25. $\int_a^b 2s\,ds,\quad 0 < a < b$ 26. $\int_a^b 3t\,dt,\quad 0 < a < b$

Evaluations

Use the results of Equations (1) and (3) to evaluate the integrals in Exercises 27–38.

27. $\int_1^{\sqrt{2}} x\,dx$ 28. $\int_{0.5}^{2.5} x\,dx$ 29. $\int_\pi^{2\pi} \theta\,d\theta$

30. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$ 31. $\int_0^{\sqrt[3]{7}} x^2\,dx$ 32. $\int_0^{0.3} s^2\,ds$

33. $\int_0^{1/2} t^2\,dt$ 34. $\int_0^{\pi/2} \theta^2\,d\theta$ 35. $\int_a^{2a} x\,dx$

36. $\int_a^{\sqrt{3}a} x\,dx$ 37. $\int_0^{\sqrt[3]{b}} x^2\,dx$ 38. $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.3 and Equations (1)–(3) to evaluate the integrals in Exercises 39–50.

39. $\int_3^1 7\,dx$ 40. $\int_0^{-2} \sqrt{2}\,dx$

41. $\int_0^2 5x\,dx$

43. $\int_0^2 (2t - 3)\,dt$

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45. $\int_2^1 \left(1 + \frac{z}{2}\right)dz$ 46. [illegible]

47. $\int_1^2 3u^2\,du$ 48. $\int_{1/2}^1 24u^2\,du$

49. $\int_0^2 (3x^2 + x - 5)\,dx$ 50. $\int_1^0 (3x^2 + x - 5)\,dx$

Finding Area

In Exercises 51–54 use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

51. $y = 3x^2$ 52. $y = \pi x^2$

53. $y = 2x$ 54. $y = \frac{x}{2} + 1$

Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$ 57. $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on a. $[-1, 1]$, b. $[1, 3]$, and c. $[-1, 3]$

62. $h(x) = -|x|$ on a. $[-1, 0]$, b. $[0, 1]$, and c. $[-1, 1]$

Theory and Examples

63. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\,dx?$$

(*Hint:* Where is the integrand positive?)

64. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx?$$

65. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

66. (*Continuation of Exercise 65*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\,dx \quad\text{and}\quad \int_{0.5}^1 \frac{1}{1 + x^2}\,dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

67. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

68. Show that the value of $\int_0^1 \sqrt{x + 8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

69. **Integrals of nonnegative functions** Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \text{ on } [a, b] \quad\Rightarrow\quad \int_a^b f(x)\,dx \ge 0.$$

70. **Integrals of nonpositive functions** Show that if f is integrable then

$$f(x) \le 0 \text{ on } [a, b] \quad\Rightarrow\quad \int_a^b f(x)\,dx \le 0.$$

72. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

73. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $\text{av}(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b \text{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

74. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$

b. $\text{av}(kf) = k\,\text{av}(f)$ (any number k)

c. $\text{av}(f) \le \text{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

75. Use limits of Riemann sums as in Example 4a to establish Equation (2).

76. Use limits of Riemann sums as in Example 4a to establish Equation (3).

77. **Upper and lower sums for increasing functions**

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\,\Delta x_{max},$$

where Δx_{max} is the norm of P, and hence that $\lim_{||P|| \to 0} (U - L) = 0$.