

## 5.5

## Indefinite Integrals and the Substitution Rule

A definite integral is a number defined by taking the limit of Riemann sums associated with partitions of a finite closed interval whose norms go to zero. The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed easily if we can find an antiderivative of the function. Antiderivatives generally turn out to be more difficult to find than derivatives. However, it is well worth the effort to learn techniques for computing them.

Recall from Section 4.8 that the set of all antiderivatives of the function  $f$  is called the **Indefinite Integral of  $f$**  with respect to  $x$ , and is symbolized by

$$\int f(x) dx.$$

The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation. When finding the indefinite integral of a function  $f$ , remember that it always includes an arbitrary constant  $C$ .

We must distinguish carefully between definite and indefinite integrals. A definite integral  $\int_a^b f(x) dx$  is a **number**. An indefinite integral  $\int f(x) dx$  is a **function** plus an arbitrary constant  $C$ .

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives. The first integration techniques we develop are obtained by inverting rules for finding derivatives, such as the Power Rule and the Chain Rule.

## The Power Rule in Integral Form

If  $u$  is a differentiable function of  $x$  and  $n$  is a rational number different from  $-1$ , the Chain Rule tells us that

$$\frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

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From another point of view, this same equation says that  $u^{n+1}/(n+1)$  is one of the antiderivatives of the function  $u^n (du/dx)$ . Therefore,

$$\int \left( u^n \frac{du}{dx} \right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n du,$$

obtained by treating the  $dx$ 's as differentials that cancel. We are thus led to the following rule.

If  $u$  is any differentiable function, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1, n \text{ rational}). \quad (1)$$

Equation (1) actually holds for any real exponent  $n \neq -1$ , as we see in Chapter 7.

In deriving Equation (1), we assumed  $u$  to be a differentiable function of the variable  $x$ , but the name of the variable does not matter and does not appear in the final formula. We could have represented the variable with  $\theta$ ,  $t$ ,  $y$ , or any other letter. Equation (1) says that whenever we can cast an integral in the form

$$\int u^n du, \quad (n \neq -1),$$

with  $u$  a differentiable function and  $du$  its differential, we can evaluate the integral as  $[u^{n+1}/(n+1)] + C$ .

## EXAMPLE 1 Using the Power Rule

$$\begin{aligned} \int \sqrt{1+y^2} \cdot 2y dy &= \int \sqrt{u} \cdot \left( \frac{du}{dy} \right) dy && \text{Let } u = 1+y^2, \\ &= \int u^{1/2} du && \frac{du}{dy} = 2y \\ &= \frac{u^{1/2+1}}{(1/2)+1} + C && \text{Integrate using Eq. (1)} \\ &= \frac{2}{3} u^{3/2} + C && \text{Simpler form} \\ &= \frac{2}{3} (1+y^2)^{3/2} + C && \text{Replace } u \text{ by } 1+y^2. \end{aligned}$$

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**EXAMPLE 2** Adjusting the Integrand by a Constant

$$\begin{aligned}
 \int \sqrt{4t-1} dt &= \int \frac{1}{4} \cdot \sqrt{4t-1} \cdot 4 dt \\
 &= \frac{1}{4} \int \sqrt{u} \cdot \left( \frac{du}{dt} \right) dt \\
 &= \frac{1}{4} \int u^{1/2} du \\
 &= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C \\
 &= \frac{1}{6} u^{3/2} + C \\
 &= \frac{1}{6} (4t-1)^{3/2} + C
 \end{aligned}$$

Replace  $u$  by  $4t-1$  ■

**Substitution: Running the Chain Rule Backwards**

The substitutions in Examples 1 and 2 are instances of the following general rule.

**THEOREM 5 The Substitution Rule**

If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

**Proof** The rule is true because, by the Chain Rule,  $F(g(x))$  is an antiderivative of  $f(g(x)) \cdot g'(x)$  whenever  $F$  is an antiderivative of  $f$ :

$$\begin{aligned}
 \frac{d}{dx} F(g(x)) &= F'(g(x)) \cdot g' \\
 &= f(g(x)) \cdot g' \quad 433 / 1564
 \end{aligned}$$

If we make the substitution  $u = g(x)$  then

$$\begin{aligned}
 \int f(g(x))g'(x) dx &= \int \frac{d}{dx} F(g(x)) dx \\
 &= F(g(x)) + C \quad \text{Fundamental Theorem} \\
 &= F(u) + C \quad u = g(x) \\
 &= \int F(u) du \quad \text{Fundamental Theorem} \\
 &= f(u) du \quad F' = f
 \end{aligned}$$

■

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The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x) dx,$$

when  $f$  and  $g'$  are continuous functions:

1. Substitute  $u = g(x)$  and  $du = g'(x) dx$  to obtain the integral

$$\int f(u) du.$$

2. Integrate with respect to  $u$ .
3. Replace  $u$  by  $g(x)$  in the result.

**EXAMPLE 3** Using Substitution

$$\begin{aligned}
 \int \cos(7\theta + 5) d\theta &= \int \cos u \cdot \frac{1}{7} du \\
 &= \frac{1}{7} \int \cos u du \\
 &= \frac{1}{7} \sin u + C \\
 &= \frac{1}{7} \sin(7\theta + 5) + C
 \end{aligned}$$

Replace  $u$  by  $7\theta + 5$  ■

We can verify this solution by differentiating and checking that we obtain the original function  $\cos(7\theta + 5)$ .

**EXAMPLE 4** Using Substitution

$$\begin{aligned}
 \int x^2 \sin(x^3) dx &= \int \sin(u) \cdot x^2 dx \\
 &= \int \sin u \cdot \frac{1}{3} du \\
 &= \frac{1}{3} \int \sin u du \\
 &= \frac{1}{3} (-\cos u) + C \\
 &= -\frac{1}{3} \cos(x^3) + C
 \end{aligned}$$

Replace  $u$  by  $x^3$  ■

## Evaluating Integrals

Evaluate the indefinite integrals in Exercises 1–12 by using the given substitutions to reduce the integrals to standard form.



1.  $\int \sin 3x \, dx, \quad u = 3x \quad 2. \int x \tan(2x^2) \, dx, \quad u = 2x^2$

3.  $\int \sec 2t \tan 2t \, dt, \quad u = 2t$

4.  $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} \, dt, \quad u = 1 - \cos \frac{t}{2}$

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3.  $\int 28(7x - 2)^4 \, dx, \quad u = 7x - 2$
4.  $\int x^3(x^4 - 1)^2 \, dx, \quad u = x^4 - 1$
7.  $\int \frac{u_2^{-2} \, du}{\sqrt{1 - u^2}}, \quad u = 1 - x^2$
8.  $\int 12(y^4 + 4y^2 + 1)^2(y^3 + 2y) \, dy, \quad u = y^4 + 4y^2 + 1$
9.  $\int \sqrt{x} \sin^2(x^{1/2} - 1) \, dx, \quad u = x^{1/2} - 1$
10.  $\int \frac{1}{x} \cos^2\left(\frac{1}{x}\right) \, dx, \quad u = -\frac{1}{x}$
11.  $\int \csc^2 2\theta \cot 2\theta \, d\theta$ 
  - a. Using  $u = \cot 2\theta$
  - b. Using  $u = \csc 2\theta$
12.  $\int \frac{dt}{\sqrt{5t + 8}}$ 
  - a. Using  $u = 5t + 8$
  - b. Using  $u = \sqrt{5t + 8}$

Evaluate the integrals in Exercises 13–40.



13.  $\int \sqrt{3 - 2x} \, dx$
14.  $\int (2x + 1)^4 \, dx$
15.  $\int \frac{1}{\sqrt{3x + 4}} \, dx$
16.  $\int \frac{3 \, dx}{(2 - x)^2}$
17.  $\int \theta \sqrt{1 - \theta^2} \, d\theta$
18.  $\int \theta \theta \sqrt{\theta^2 - 1} \, d\theta$
19.  $\int 3x \sqrt{7 - 3x^2} \, dx$
20.  $\int \frac{4x \, dx}{\sqrt{2x^2 + 1}}$
21.  $\int \frac{1}{\sqrt{1 + \sqrt{x}}} \, dx$
22.  $\int \frac{(1 + \sqrt{v})^3}{\sqrt{v}} \, dv$
23.  $\int \cos(3x + 4) \, dx$
24.  $\int \sin(8x - 5) \, dx$
25.  $\int \sec^2(3x + 2) \, dx$
26.  $\int \tan^2 x \sec^2 x \, dx$
27.  $\int \sin^2 \frac{x}{3} \cos \frac{x}{3} \, dx$
28.  $\int \tan^2 \frac{x}{2} \sec^2 \frac{x}{2} \, dx$
29.  $\int r^2 \left(\frac{r'}{10} - 1\right)^3 \, dr$
30.  $\int r^4 \left(7 - \frac{r'}{10}\right)^3 \, dr$
31.  $\int x^{1/2} \sin(x^{1/2} + 1) \, dx$
32.  $\int x^{1/3} \sin(x^{4/3} - 8) \, dx$
33.  $\int \sec\left(v + \frac{\pi}{2}\right) \tan\left(v + \frac{\pi}{2}\right) \, dv$
34.  $\int \csc\left(\frac{v - \pi}{2}\right) \cot\left(\frac{v - \pi}{2}\right) \, dv$
35.  $\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)} \, dt$
36.  $\int \frac{6 \cos t}{(2 + \sin t)^3} \, dt$

37.  $\int \sqrt{\cot x} \csc^2 x \, dx$
38.  $\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz$
39.  $\int \frac{1}{t^2} \cos\left(\frac{1}{t} - 1\right) \, dt$
40.  $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3) \, dt$
41.  $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} \, d\theta$
42.  $\int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} \, d\theta$
43.  $\int (x^3 + 2x^2 - 5x + 5)(3x^2 + 4x - 5) \, dx$
44.  $\int (\theta^3 - 2\theta^2 + 8\theta - 2)(\theta^3 - \theta + 2) \, d\theta$
45.  $\int t^2(1 + t^4)^3 \, dt$
46.  $\int \sqrt{\frac{t-1}{t^2}} \, dt$
47.  $\int x^3 \sqrt{x^2 + 1} \, dx$
48.  $\int 3x^4 \sqrt{x^2 + 1} \, dx$



## Simplifying Integrals Step by Step

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 49 and 50.

49.  $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^2 x)^2} \, dx$ 
  - a.  $u = \tan x$ , followed by  $v = u^2$ , then by  $w = 2 + v$
  - b.  $u = \tan^2 x$ , followed by  $v = 2 + u$
  - c.  $u = 2 + \tan^2 x$
50.  $\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx$ 
  - a.  $u = x - 1$ , followed by  $v = \sin u$ , then by  $w = 1 + v^2$
  - b.  $u = \sin(x - 1)$ , followed by  $v = 1 + u^2$
  - c.  $u = 1 + \sin^2(x - 1)$

Evaluate the integrals in Exercises 51 and 52.

51.  $\int \frac{(2r - 1) \cos \sqrt{3(2r - 1)^2 + 6}}{\sqrt{3(2r - 1)^2 + 6}} \, dr$
52.  $\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^2 \sqrt{\theta}} \, d\theta$



## Initial Value Problems

Solve the initial value problems in Exercises 53–58.

53.  $\frac{dx}{dt} = 12(x^2 - 1)^3, \quad x(1) = 3$
54.  $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, \quad y(0) = 0$
55.  $\frac{dx}{dt} = 8 \sin^2(t + \frac{\pi}{12}), \quad x(0) = 8$
56.  $\frac{dr}{dt} = 3 \cos^2\left(\frac{\pi}{4} - \theta\right), \quad r(0) = \frac{\pi}{8}$

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57.  $\frac{d^2x}{dt^2} = -4 \sin\left(2t - \frac{\pi}{2}\right), \quad x'(0) = 100, \quad x(0) = 0$
58.  $\frac{d^2y}{dt^2} = 4 \sec^2 2t \tan 2t, \quad y'(0) = 4, \quad y(0) = -1$
59. The velocity of a particle moving back and forth on a line is  $v = dx/dt = 6 \sin 2t$  m/sec for all  $t$ . If  $x = 0$  when  $t = 0$ , find the value of  $x$  when  $t = \pi/2$  sec.
60. The acceleration of a particle moving back and forth on a line is  $a = d^2x/dt^2 = -n^2 \cos \pi t$  m/sec<sup>2</sup> for all  $t$ . If  $x = 0$  and  $v = 8$  m/sec when  $t = 0$ , find  $x$  when  $t = 1$  sec.

## Theory and Examples

61. It looks as if we can integrate  $2 \sin x \cos x$  with respect to  $x$  in three different ways.

- a.  $\int 2 \sin x \cos x \, dx = \int 2u \, du, \quad u = \sin x$   
 $= u^2 + C_1 = \sin^2 x + C_1$
- b.  $\int 2 \sin x \cos x \, dx = \int -2u \, du, \quad u = \cos x$   
 $= -u^2 + C_2 = -\cos^2 x + C_2$
- c.  $\int 2 \sin x \cos x \, dx = \int \sin 2x \, dx, \quad 2 \sin x \cos x = \sin 2x$   
 $= -\frac{\cos 2x}{2} + C_3$

Can all three integrations be correct? Give reasons for your answer.

62. The substitution  $u = \tan x$  gives

$$\int \sec^2 x \tan x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C.$$

The substitution  $u = \sec x$  gives

$$\int \sec^2 x \tan x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C.$$

Can both integrations be correct? Give reasons for your answer.

63. (Continuation of Example 9.)

- a. Show by evaluating the integral in the expression

$$\frac{1}{(1/60) - 0} \int_0^{1/60} V_{\text{max}} \sin 120 \pi t \, dt$$

that the average value of  $V = V_{\text{max}} \sin 120 \pi t$  over a full cycle is zero.

- b. The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

- c. Show that

$$\int_0^{1/60} (V_{\text{max}})^2 \sin^2 120 \pi t \, dt = \frac{(V_{\text{max}})^2}{120}$$