

8.2

Integration by Parts

Since

$$\int x \, dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2 \, dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx.$$

In other words, the integral of a product is generally *not* the product of the individual integrals.

$$\int f(x)g(x) \, dx \text{ is not equal to } \int f(x) \, dx \cdot \int g(x) \, dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x) \, dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int x e^x \, dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x \, dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x , the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)] \, dx = \int [f'(x)g(x) + f(x)g'(x)] \, dx$$

656 / 1564

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley

562 Chapter 8: Techniques of Integration

or

$$\int \frac{d}{dx}[f(x)g(x)] \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x) \, dx = \int \frac{d}{dx}[f(x)g(x)] \, dx - \int f'(x)g(x) \, dx$$

leading to the **integration by parts formula**

$$\boxed{\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx} \quad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x) \, dx$ and $dv = g'(x) \, dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\boxed{\int u \, dv = uv - \int v \, du} \quad (2)$$

This formula expresses one integral, $\int u \, dv$, in terms of a second integral, $\int v \, du$. With a proper choice of u and v , the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv . The next examples illustrate the technique.

EXAMPLE 1 Using Integration by Parts

Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \quad du = \cos x \, dx,$$

$$dv = dx, \quad v = \sin x. \quad \text{Since the antiderivative of } \cos x \text{ is } \sin x.$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C. \quad \blacksquare$$

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x \cos x \, dx = \int u \, dv$$

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x \, dx$ to get v .

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x) \, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v \, du = \int (x \cos x - x^2 \sin x) \, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x \, dx, \\ du &= -\sin x \, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v \, du = -\int \frac{x^2}{2} \sin x \, dx.$$

This, too, is worse.

The goal of integration by parts is to go from an integral $\int v \, du$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx , as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$$\begin{array}{lll} u = \ln x & \text{Simplifies when differentiated} & dv = dx \quad \text{Easy to integrate} \\ du = \frac{1}{x} \, dx & & v = x. \quad \text{Simplest antiderivative} \end{array}$$

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C.$$

Sometimes we have to use integration by parts more.

658 / 1564

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison Wesley

564 Chapter 8: Techniques of Integration

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

The technique of Example 4 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We say more about this later in this section when we discuss *tabular integration*.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x \, dx, \quad v = -\cos x, \quad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.



1. $\int x \sin \frac{1}{2} dx$
 2. $\int \theta \cos \pi \theta d\theta$
 3. $\int t^2 \cos dt$
 4. $\int z^2 \sin z dz$
 5. $\int_0^2 x \ln x dx$
 6. $\int_1^e z^3 \ln z dz$
 7. $\int \tan^{-1} y dy$
 8. $\int \sin^{-1} y dy$
 9. $\int x \sec^2 x dx$
 10. $\int 4x \sec^2 2x dx$
 11. $\int z^2 e^z dz$
 12. $\int p^4 e^{-p} dp$

13. $\int (z^2 - 5z) e^z dz$
 14. $\int (x^2 + x + 1) e^x dx$
 15. $\int z^3 e^z dz$
 16. $\int t^2 e^{-t} dt$
 17. $\int_0^{\pi/2} \theta^2 \sin 2\theta d\theta$
 18. $\int_0^{\pi/2} z^3 \cos 2z dz$
 19. $\int_{2\sqrt{2}}^2 t \sec^{-1} t dt$
 20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx$
 21. $\int z^2 \sin \theta d\theta$
 22. $\int e^{-t} \cos y dy$
 23. $\int z^{2z} \cos 3z dz$
 24. $\int e^{-2z} \sin 2z dz$

Copyright © 2005 Pearson Education, Inc., publishing as Pearson Addison-Wesley

663-664 / 1564

8.2 Integration by Parts 569

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

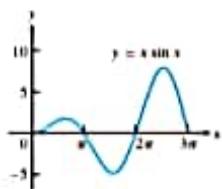


25. $\int e^{\sqrt{1+t^2}} dt$
 26. $\int_0^1 s \sqrt{1-s^2} ds$
 27. $\int_0^{\pi/4} z \tan^2 z dz$
 28. $\int \ln(z+z^2) dz$
 29. $\int \sin(\ln z) dz$
 30. $\int z \ln(2z) dz$

Theory and Examples

31. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the x -axis (see the accompanying figure) for
 a. $0 \leq x \leq \pi$ b. $\pi \leq x \leq 2\pi$ c. $2\pi \leq x \leq 3\pi$.

- d. What pattern do you see here? What is the area between the curve and the x -axis for $n\pi \leq x \leq (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.

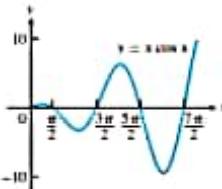


32. **Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the x -axis (see the accompanying figure) for
 a. $\pi/2 \leq x \leq 3\pi/2$ b. $3\pi/2 \leq x \leq 5\pi/2$
 c. $5\pi/2 \leq x \leq 7\pi/2$

- d. What pattern do you see? What is the area between the curve and the x -axis for

$$\left(\frac{2n-1}{2}\right)\pi \leq x \leq \left(\frac{2n+1}{2}\right)\pi,$$

- n an arbitrary positive integer? Give reasons for your answer.



33. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $y = 1$

- a. about the y -axis b. about the line $x = 1$

35. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $r = \cos \theta$, $0 \leq \theta \leq \pi/2$, about

- a. the y -axis b. the line $x = \pi/2$.

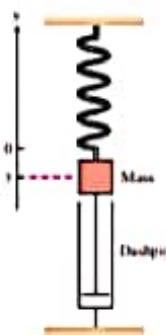
36. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x -axis and the curve $y = x \sin x$, $0 \leq x \leq \pi$, about

- a. the y -axis b. the line $x = \pi$.

(See Exercise 31 for a graph.)

37. **Average value** A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \geq 0$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.

38. **Average value** In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t} (\sin t - \cos t), \quad t \geq 0$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.