Chapter 5
ANGULAR MOMENTUM AND ROTATIONS

In classical mechanics the total angular momentum $\vec{L}$ of an isolated system about any fixed point is conserved. The existence of a conserved vector $\vec{L}$ associated with such a system is itself a consequence of the fact that the associated Hamiltonian (or Lagrangian) is invariant under rotations, i.e., if the coordinates and momenta of the entire system are rotated "rigidly" about some point, the energy of the system is unchanged and, more importantly, is the same function of the dynamical variables as it was before the rotation. Such a circumstance would not apply, e.g., to a system lying in an externally imposed gravitational field pointing in some specific direction. Thus, the invariance of an isolated system under rotations ultimately arises from the fact that, in the absence of external fields of this sort, space is isotropic; it behaves the same way in all directions.

Not surprisingly, therefore, in quantum mechanics the individual Cartesian components L_i of the total angular momentum operator $\vec{L}$ of an isolated system are also constants of the motion. The different components of $\vec{L}$ are not, however, compatible quantum observables. Indeed, as we will see the operators representing the components of angular momentum along different directions do not generally commute with one another. Thus, the vector operator $\vec{L}$ is not, strictly speaking, an observable, since it does not have a complete basis of eigenstates (which would have to be simultaneous eigenstates of all of its non-commuting components). This lack of commutivity often seems, at first encounter, as somewhat of a nuisance but, in fact, it intimately reflects the underlying structure of the three dimensional space in which we are immersed, and has its source in the fact that rotations in three dimensions about different axes do not commute with one another. Indeed, it is this lack of commutivity that imparts to angular momentum observables their rich characteristic structure and makes them quite useful, e.g., in classifying the bound states of atomic, molecular, and nuclear systems containing one or more particles, and in decomposing the scattering states of such systems into components associated with different angular momenta. Just as importantly, the existence of internal "spin" degrees of freedom, i.e., intrinsic angular momenta associated with the internal structure of fundamental particles, provides additional motivation for the study of angular momentum and to the general properties exhibited by dynamical quantum systems under rotations.

5.1 Orbital Angular Momentum of One or More Particles

The classical orbital angular momentum of a single particle about a given origin is given by the cross product

$$\vec{\ell} = \vec{r} \times \vec{p} \tag{5.1}$$

of its position and momentum vectors. The total angular momentum of a system of such structureless point particles is then the vector sum

$$\vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \tag{5.2}$$

of the individual angular momenta of the particles making up the collection. In quantum mechanics, of course, dynamical variables are replaced by Hermitian operators, and so we are led to consider the vector operator

$$\vec{\ell} = \vec{R} \times \vec{P} \tag{5.3}$$

or its dimensionless counterpart

$$\vec{l} = \vec{R} \times \vec{K}, = \frac{\vec{\ell}}{\hbar}, \tag{5.4}$$

either of which we will refer to as an angular momentum (i.e., we will, for the rest of this chapter, effectively be working in a set of units for which $\hbar = 1$). Now, a general vector operator $\vec{B}$ can always be defined in terms of its operator components $\{B_x, B_y, B_z\}$ along any three orthogonal axes. The component of $\vec{B}$ along any other direction, defined, e.g., by the unit vector $\hat{u}$, is then the operator $\vec{B} \cdot \hat{u} = B_x u_x + B_y u_y + B_z u_z$. So it is with the operator $\vec{l}$, whose components are, by definition, the operators

$$l_x = Y K_z - Z K_y \qquad l_y = Z K_x - X K_z \qquad l_z = X K_y - Y K_x. \tag{5.5}$$

The components of the cross product can also be written in a more compact form

$$l_i = \sum_{j,k} \varepsilon_{ijk} X_j K_k \tag{5.6}$$

in terms of the Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}. \tag{5.7}$$

Although the normal product of two Hermitian operators is itself Hermitian if and only if they commute, this familiar rule does not extend to the cross product of two vector operators. Indeed, even though $\vec{R}$ and $\vec{K}$ do *not* commute, their cross product $\vec{l}$ is readily shown to be Hermitian. From (5.6),

$$l_i^+ = \sum_{j,k} \varepsilon_{ijk} K_k^+ X_j^+ = \sum_{j,k} \varepsilon_{ijk} K_k X_j = \sum_{j,k} \varepsilon_{ijk} X_j K_k = l_i, \tag{5.8}$$

where we have used the fact the components of $\vec{R}$ and $\vec{K}$ are Hermitian and that, since $\varepsilon_{ijk} = 0$ if $k = j$, only commuting components of $\vec{R}$ and $\vec{K}$ appear in each term of the cross product. It is also useful to define the scalar operator

$$l^2 = \vec{l} \cdot \vec{l} = l_x^2 + l_y^2 + l_z^2 \tag{5.9}$$

which, being the sum of the squares of Hermitian operators, is itself both Hermitian and positive.

So the components of $\vec{l}$, like those of the vector operators $\vec{R}$ and $\vec{P}$, are Hermitian. We will assume that they are also observables. Unlike the components of $\vec{R}$ and $\vec{P}$, however, the components of $\vec{l}$ along different directions do not commute *with each other*. This is readily established; e.g.,

$$\begin{aligned} [l_x, l_y] &= [Y K_z - Z K_y, Z K_x - K_z X] \\ &= Y K_x [K_z, Z] + K_y X [Z, K_z] \\ &= i (X K_y - Y K_x) = i l_z. \end{aligned}$$

The other two commutators are obtained in a similar fashion, or by a cyclic permutation of x, y, and z, giving

$$[l_x, l_y] = il_z \qquad [l_y, l_z] = il_x \qquad [l_z, l_x] = il_y, \tag{5.10}$$

which can be written more compactly using the Levi-Civita symbol in either of two ways,

$$[l_i, l_j] = i\sum_k \varepsilon_{ijk} l_k, \tag{5.11}$$

or

$$\sum_{i,j} \varepsilon_{ijk} l_i l_j = il_k,$$

the latter of which is, component-by-component, equivalent to the vector relation

$$\vec{l} \times \vec{l} = i\vec{l}. \tag{5.12}$$

These can also be used to derive the following generalization

$$\left[\vec{l} \cdot \hat{a}, \vec{l} \cdot \hat{b}\right] = i\vec{l} \cdot \left(\hat{a} \times \hat{b}\right) \tag{5.13}$$

involving the components of $\vec{l}$ along arbitrary directions $\hat{a}$ and $\hat{b}$.

It is also straightforward to compute the commutation relations between the components of $\vec{l}$ and l^2, i.e.,

$$\begin{aligned}
\left[l_j, l^2\right] &= \sum_i \left[l_j, l_i^2\right] = \sum_i l_i \left[l_j, l_i\right] + \sum_i \left[l_j, l_i\right] l_i \\
&= i\sum_{i,k} (\varepsilon_{ijk} l_i l_k + \varepsilon_{ijk} l_k l_i) = i\sum_{i,k} (\varepsilon_{ijk} l_i l_k + \varepsilon_{kji} l_i l_k) \\
&= i\sum_{i,k} \varepsilon_{ijk} (l_i l_k - l_i l_k) = 0
\end{aligned} \tag{5.14}$$

where in the second line we have switched summation indices in the second sum and then used the fact that $\varepsilon_{kji} = -\varepsilon_{ijk}$. Thus each component of $\vec{l}$ commutes with l^2. We write

$$\left[\vec{l}, l^2\right] = 0 \qquad [l_i, l_j] = i\sum_k \varepsilon_{ijk} l_k. \tag{5.15}$$

The same commutation relations are also easily shown to apply to the operator representing the total orbital angular momentum $\vec{L}$ of a system of particles. For such a system, the state space of which is the direct product of the state spaces for each particle, the operators for one particle automatically commute with those of any other, so that

$$\begin{aligned}
[L_i, L_j] &= \sum_{\alpha,\beta} [l_{i,\alpha}, l_{j,\beta}] = i\sum_k \varepsilon_{ijk} \sum_{\alpha,\beta} \delta_{\alpha,\beta}\, l_{k,\alpha} = i\sum_k \varepsilon_{ijk} \sum_\alpha l_{k,\alpha} \\
&= i\sum_k \varepsilon_{ijk} L_k
\end{aligned} \tag{5.16}$$

Similarly, from these commutation relations for the components of $\vec{L}$, it can be shown that $\left[L_i, L^2\right] = 0$ using the same proof as above for $\vec{l}$. Thus, for each particle, and for the total orbital angular momentum itself, we have the same characteristic commutation relations

$$[L_i, L_j] = i\sum_k \varepsilon_{ijk} L_k \qquad \left[\vec{L}, L^2\right] = 0. \tag{5.17}$$

As we will see, these commutation relations determine to a very large extent the allowed spectrum and structure of the eigenstates of angular momentum. It is convenient to adopt the viewpoint, therefore, that *any* vector operator obeying these characteristic commutation relations represents an angular momentum of some sort. We thus generally say that an arbitrary vector operator $\vec{J}$ is an angular momentum if its Cartesian components are observables obeying the following characteristic commutation relations

$$[J_i, J_j] = i\sum_k \varepsilon_{ijk} J_k \qquad \left[\vec{J}, J^2\right] = 0. \tag{5.18}$$

It is actually possible to go considerably further than this. It can be shown, under very general circumstances, that for every quantum system there *must* exist a vector operator $\vec{J}$ obeying the commutation relations (5.18), the components of which characterize the way that the quantum system transforms under rotations. This vector operator $\vec{J}$ can usually, in such circumstances, be taken as a *definition* of the total angular momentum of the associated system. Our immediate goals, therefore, are twofold. First we will explore this underlying relationship that exists between rotations and the angular momentum of a physical system. Then, afterwards, we will return to the commutation relations (5.18), and use them to determine the allowed spectrum and the structure of the eigenstates of arbitrary angular momentum observables.

5.2 Rotation of Physical Systems

A rotation $\mathbf{R}$ of a physical system is a distance preserving mapping of R^3 onto itself that leaves a single point O, and the handedness of coordinate systems invariant. This definition excludes, e.g., reflections and other "improper" transformations, which always invert coordinate systems. There are two different, but essentially equivalent ways of mathematically describing rotations. An **active** rotation of a physical system is one in which all position and velocity vectors of particles in the system are rotated about the fixed point O, while the coordinate system used to describe the system is left unchanged. A **passive** rotation, by contrast, is one in which the coordinate axes are rotated, but the physical vectors of the system are left alone. In either case the result, generally, is a change in the Cartesian components of any vector in the system with respect to the coordinate axes used to represent them. It is important to note, however, that a clockwise active rotation of a physical system about a given axis is equivalent in terms of the change it produces on the coordinates of a vector to a counterclockwise passive rotation about the same axis.

There are also two different methods commonly adopted for indicating specific rotations, each requiring three independent parameters. One method specifies particular rotations through the use of the so-called Euler angles introduced in the study of rigid bodies. Thus, e.g., $\mathbf{R}(\alpha, \beta, \gamma)$ would indicate the rotation equivalent to the three separate rotations defined by the Euler angles (α, β, γ).

Alternatively, we can indicate a rotation by choosing a specific rotation axis, described by a unit vector $\hat{u}$ (defined, e.g., through its polar angles θ and ϕ), and a rotation angle α. Thus, a rotation about $\hat{u}$ through an angle α (positive or negative, according to the right-hand-rule applied to $\hat{u}$) would be written $\mathbf{R}_{\hat{u}}(\alpha)$. We will, in what follows, make more use of this latter approach than we will of the Euler angles.

Independent of their means of specification, the rotations about a specified point O in three dimensions form a **group**, referred to as the three-dimensional rotation group. Recall that a set G of elements $R_1, R_2, \cdots$, that is closed under an associative binary operation,

$$R_i R_j = R_k \in G \qquad \text{for all } R_i, R_j \in G, \tag{5.19}$$

is said to form a group if (i) there exists in G an identity element $\mathbf{1}$ such that $R\mathbf{1} = \mathbf{1}R = R$ for all R in G and (ii) there is in G, for each R, an inverse element R^{-1}, such that $RR^{-1}=R^{-1}R = \mathbf{1}$.

For the rotation group $\{\mathbf{R}_{\hat{u}}(\alpha)\}$ the product of any two rotations is just the rotation obtained by performing each rotation in sequence, i.e., $\mathbf{R}_{\hat{u}}(\alpha)\mathbf{R}_{\hat{u}'}(\alpha')$ corresponds to a rotation of the physical system through an angle α' about $\hat{u}'$, followed by a rotation through α about $\hat{u}$. The identity rotation corresponds to the limiting case of a rotation of $\alpha = 0$ about any axis (i.e., the identity mapping). The inverse of $\mathbf{R}_{\hat{u}}(\alpha)$ is the rotation

$$\mathbf{R}_{\hat{u}}^{-1}(\alpha) = \mathbf{R}_{\hat{u}}(-\alpha) = \mathbf{R}_{-\hat{u}}(\alpha), \tag{5.20}$$

that rotates the system in the opposite direction about the same axis.

It is readily verified that, in three dimensions, the product of two rotations generally depends upon the order in which they are taken. That is, in most cases,

$$\mathbf{R}_{\hat{u}}(\alpha)\mathbf{R}_{\hat{u}'}(\alpha') \neq \mathbf{R}_{\hat{u}'}(\alpha')\mathbf{R}_{\hat{u}}(\alpha). \tag{5.21}$$

The rotation group, therefore, is said to be a noncommutative or non-Abelian group.

There are, however, certain subsets of the rotation group that form commutative **subgroups** (subsets of the original group that are themselves closed under the same binary operation). For example, the set of rotations $\{\mathbf{R}_{\hat{u}}(\alpha) \mid \text{fixed } \hat{u}\}$ about any single *fixed* axis forms an Abelian subgroup of the 3D rotation group, since the product of two rotations in the plane perpendicular to $\hat{u}$ corresponds to a single rotation in that plane through an angle equal to the (commutative) sum of the individual rotation angles,

$$\mathbf{R}_{\hat{u}}(\alpha)\mathbf{R}_{\hat{u}}(\beta) = \mathbf{R}_{\hat{u}}(\alpha+\beta) = \mathbf{R}_{\hat{u}}(\beta)\mathbf{R}_{\hat{u}}(\alpha). \tag{5.22}$$

The subgroups of this type are all isomorphic to one another. Each one forms a realization of what is referred to for obvious reasons as the two dimensional rotation group.

Another commutative subgroup comprises the set of **infinitesimal rotations**. A rotation $\mathbf{R}_{\hat{u}}(\delta\alpha)$ is said to be infinitesimal if the associated rotation angle $\delta\alpha$ is an infinitesimal (it being understood that quantities of order $\delta^2\alpha$ are always to be neglected with respect to quantities of order $\delta\alpha$). The effect of an infinitesimal rotation on a physical quantity of the system is to change it, at most, by an infinitesimal amount. The general properties of such rotations are perhaps most easily demonstrated by considering their effect on normal vectors of R^3.

The effect of an arbitrary rotation $\mathbf{R}$ on a vector $\vec{v}$ of R^3 is to transform it into a new vector

$$\vec{v}' = \mathbf{R}\left[\vec{v}\right]. \tag{5.23}$$

Because rotations preserves the relative orientations and lengths of all vectors in the system, it also preserves the basic linear relationships of the vector space itself, i.e.,

$$\mathbf{R}\left[\vec{v}_1 + \vec{v}_2\right] = \mathbf{R}\left[\vec{v}_1\right] + \mathbf{R}\left[\vec{v}_2\right]. \tag{5.24}$$

Thus, the effect of any rotation $\mathbf{R}$ on vectors in the R^3 can be described through the action of an associated linear operator $A_{\mathbf{R}}$, such that

$$\mathbf{R}\left[\vec{v}\right] = \vec{v}' = A_{\mathbf{R}}\vec{v}. \tag{5.25}$$

This linear relationship can be expressed in any Cartesian coordinate system in component form

$$v_i' = \sum_j A_{ij} v_j \tag{5.26}$$

A systematic study of rotations reveals that the 3×3 matrix A representing the linear operator $A_{\mathbf{R}}$ must be real, orthogonal, and unimodular, i.e.

$$A_{ij} = A_{ij}^* \qquad A^T A = A A^T = \mathbf{1} \qquad \det(A) = 1. \tag{5.27}$$

We will denote by $A_{\hat{u}}(\alpha)$ the linear operator (or any matrix representation thereof, depending upon the context) representing the rotation $\mathbf{R}_{\hat{u}}(\alpha)$. The rotations $\mathbf{R}_{\hat{u}}(\alpha)$ and the orthogonal, unimodular matrices $A_{\hat{u}}(\alpha)$ representing their effect on vectors with respect to a given coordinate system are in a one-to-one correspondence. We say, therefore, that the set of matrices $\{A_{\hat{u}}(\alpha)\}$ forms a representation of the 3D rotation group. The group formed by the matrices themselves is referred to as SO3, which indicates the group of "special" orthogonal 3×3 matrices (special in that it excludes those orthogonal matrices that have determinant of -1, i.e., it excludes reflections and other improper transformations). In this group, the matrix representing the identity rotation is, of course, the identity matrix, while rotations about the three Cartesian axes are effected by the matrices

$$A_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad A_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$A_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.28}$$

Now it is intuitively clear that the matrix associated with an *infinitesimal* rotation barely changes any vector that it acts upon and, as a result, differs from the identity matrix by an infinitesimal amount, i.e.,

$$A_{\hat{u}}(\delta\alpha) = \mathbf{1} + \delta\alpha\, M_{\hat{u}} \tag{5.29}$$

where $M_{\hat{u}}$ is describes a linear transformation that depends upon the rotation axis $\hat{u}$ but is independent of the infinitesimal rotation angle $\delta\alpha$. The easily computed inverse

$$A_{\hat{u}}^{-1}(\delta\alpha) = A_{\hat{u}}(-\delta\alpha) = \mathbf{1} - \delta\alpha\, M_{\hat{u}} \tag{5.30}$$

and the orthogonality of rotation matrices

$$A_{\hat{u}}^{-1}(\delta\alpha) = A_{\hat{u}}^T(\delta\alpha) = \mathbf{1} + \delta\alpha M_{\hat{u}}^T \tag{5.31}$$

leads to the requirement that the matrix

$$M_{\hat{u}} = -M_{\hat{u}}^T \tag{5.32}$$

be real and antisymmetric. Thus, under such an infinitesimal rotation, a vector $\vec{v}$ is taken onto the vector

$$\vec{v}' = \vec{v} + \delta\alpha\, M_{\hat{u}}\vec{v}. \tag{5.33}$$

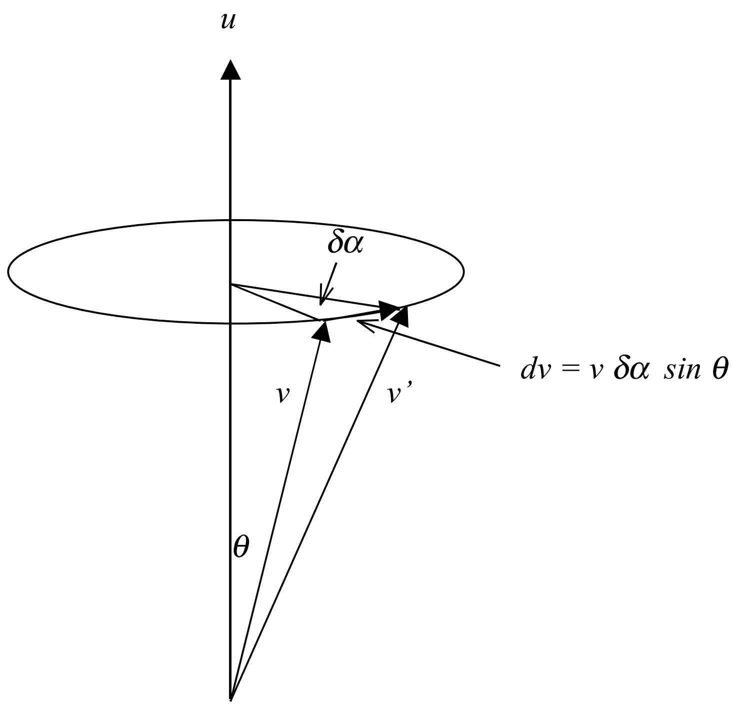

Figure 1 Under an infinitesimal rotation $R_{\hat{u}}(\delta\alpha)$, the change $d\vec{v} = \vec{v}' - \vec{v}$ in a vector $\vec{v}$ is perpendicular to both $\hat{u}$ and $\vec{v}$, and has magnitude $|d\vec{v}| = |\vec{v}|\delta\alpha \sin\theta$.

But an equivalent description of such an infinitesimal transformation on a vector can be determined through simple geometrical arguments. The vector $\vec{v}'$ obtained by rotating the vector $\vec{v}$ about $\hat{u}$ through an infinitesimal angle $\delta\alpha$ is easily verified from Fig. (1) to be given by the expression

$$\vec{v}' = \vec{v} + \delta\alpha\,(\hat{u} \times \vec{v}) \tag{5.34}$$

or, in component form

$$v_i' = v_i + \delta\alpha \sum_{j,k} \varepsilon_{ijk} u_j v_k \tag{5.35}$$

A straightforward comparison of (5.33) and (5.34) reveals that, for these to be consistent, the matrix $M_{\hat{u}}$ must have matrix elements of the form $M_{ik} = \sum_j \varepsilon_{ijk} u_j$, i.e.,

$$M_{\hat{u}} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}, \tag{5.36}$$

where u_x, u_y, and u_z are the components (i.e., direction cosines) of the unit vector $\hat{u}$. Note that we can write (5.36) in the form

$$M_{\hat{u}} = \sum_i u_i M_i = u_x M_x + u_y M_y + u_z M_z \tag{5.37}$$

where the three matrices M_i that characterize rotations about the three different Cartesian axes are given by

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad M_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad M_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.38}$$

Returning to the point that motivated our discussion of infinitesimal rotations, we note that

$$\begin{aligned} A_{\hat{u}}(\delta\alpha)A_{\hat{u}'}(\delta\alpha') &= (\mathbf{1} + \delta\alpha M_{\hat{u}})\,(\mathbf{1} + \delta\alpha' M_{\hat{u}'}) \\ &= \mathbf{1} + \delta\alpha M_{\hat{u}} + \delta\alpha' M_{\hat{u}'} = A_{\hat{u}'}(\delta\alpha)A_{\hat{u}}(\delta\alpha'), \end{aligned} \tag{5.39}$$

which shows that, to lowest order, a product of infinitesimal rotations always commutes. This last expression also reveals that infinitesimal rotations have a particularly simple combination law, i.e., to multiply two or more infinitesimal rotations simply add up the parts corresponding to the deviation of each one from the identity matrix. This rule, and the structure (5.37) of the matrices $M_{\hat{u}}$ implies the following important theorem: an infinitesimal rotation $A_{\hat{u}}(\delta\alpha)$ about an arbitrary axis $\hat{u}$ can *always* be built up as a product of three infinitesimal rotations about any three orthogonal axes, i.e.,

$$\begin{aligned} A_{\hat{u}}(\delta\alpha) &= \mathbf{1} + \delta\alpha\, M_{\hat{u}} \\ &= \mathbf{1} + \delta\alpha\, u_x M_x + \delta\alpha\, u_y M_y + \delta\alpha\, u_z M_z \end{aligned}$$

which implies that

$$A_{\hat{u}}(\delta\alpha) = A_x(u_x\delta\alpha)A_y(u_y\delta\alpha)A_z(u_z\delta\alpha). \tag{5.40}$$

Since this last property only involves products, it must be a *group* property associated with the group SO3 of rotation matrices $\{A_{\hat{u}}(\alpha)\}$, i.e., a property shared by the infinitesimal rotations that they represent, i.e.,

$$\mathbf{R}_{\hat{u}}(\delta\alpha) = \mathbf{R}_x(u_x\delta\alpha)\mathbf{R}_y(u_y\delta\alpha)\mathbf{R}_z(u_z\delta\alpha). \tag{5.41}$$

We will use this group relation associated with infinitesimal rotations in determining their effect on quantum mechanical systems.

5.3 Rotations in Quantum Mechanics

Any quantum system, no matter how complicated, can be characterized by a set of observables and by a state vector $|\psi\rangle$, which is an element of an associated Hilbert space. A rotation performed on a quantum mechanical system will generally result in a transformation of the state vector and to a similar transformation of the observables of the system. To make this a bit more concrete, it is useful to imagine an experiment set up on a rotatable table. The quantum system to be experimentally interrogated is described by some initial suitably-normalized state vector $|\psi\rangle$. The experimental apparatus might be arranged to measure, e.g., the component of the momentum of the system along a particular direction. Imagine, now, that the table containing both the system and the experimental apparatus is rotated about a vertical axis in such a way that the quantum system "moves" rigidly with the table (i.e., so that an observer sitting on the table could distinguish no change in the system). After such a rotation, the system will generally be in a new state $|\psi'\rangle$, normalized in the same way as it was before the rotation. Moreover, the apparatus that has rotated with the table will now be set up to measure the momentum along a *different* direction, as measured by a set of coordinate axes fixed in the laboratory..

Such a transformation clearly describes a mapping of the quantum mechanical state space onto itself in a way that preserves the relationships of vectors in that space, i.e., it describes a *unitary* transformation. Not surprisingly, therefore, the effect of any rotation $\mathbf{R}$ on a quantum system can quite generally be characterized by a unitary operator $U_{\mathbf{R}}$, i.e.,

$$|\psi'\rangle = \mathbf{R}\,[|\psi\rangle] = U_{\mathbf{R}}|\psi\rangle. \tag{5.42}$$

Moreover, the transformation experienced under such a rotation by *observables* of the system must have the property that the mean value and statistical distribution of an observable Q taken with respect to the *original* state $|\psi\rangle$ will be the same as the mean value and distribution of the *rotated* observable $Q' = \mathbf{R}\,[Q]$ taken with respect to the *rotated* state $|\psi'\rangle$, i.e.,

$$\langle\psi|Q|\psi\rangle = \langle\psi'|Q'|\psi'\rangle = \langle\psi|U_{\mathbf{R}}^{+}Q'U_{\mathbf{R}}|\psi\rangle \tag{5.43}$$

From (5.43) we deduce the relationship

$$Q' = \mathbf{R}\,[Q] = U_{\mathbf{R}}QU_{\mathbf{R}}^{+}. \tag{5.44}$$

Thus, the observable Q' is obtained through a unitary transformation of the unrotated observable Q using the same unitary operator that is needed to describe the change in the state vector. Consistent with our previous notation, we will denote by $U_{\hat{u}}(\alpha)$ the unitary transformation describing the effect on a quantum system of a rotation $\mathbf{R}_{\hat{u}}(\alpha)$ about $\hat{u}$ through angle α.

Just as the 3×3 matrices $\{A_{\hat{u}}(\alpha)\}$ form a representation of the rotation group $\{\mathbf{R}_{\hat{u}}(\alpha)\}$, so do the set of unitary operators $\{U_{\hat{u}}(\alpha)\}$ and so also do the set of matrices representing these operators with respect to any given ONB for the state space. Also, as with the case of normal vectors in R^3, an *infinitesimal* rotation on a quantum system will produce an infinitesimal change in the state vector $|\psi\rangle$. Thus, the unitary operator $U_{\hat{u}}(\delta\alpha)$ describing such an infinitesimal rotation will differ from the identity operator by an infinitesimal, i.e.,

$$U_{\hat{u}}(\delta\alpha) = \mathbf{1} + \delta\alpha\,\hat{M}_{\hat{u}} \tag{5.45}$$

where $\hat{M}_u$ is now a linear operator, defined not on R^3 but on the Hilbert space of the quantum system under consideration, that depends on $\hat{u}$ but is independent of $\delta\alpha$. Similar to our previous calculation, the easily computed inverse

$$U_{\hat{u}}^{-1}(\delta\alpha) = U_{\hat{u}}(-\delta\alpha) = \mathbf{1} - \delta\alpha\,\hat{M}_{\hat{u}} \tag{5.46}$$

and the unitarity of these operators ($U^{-1} = U^{+}$) leads to the result that, now, $\hat{M}_{\hat{u}} = -\hat{M}_{\hat{u}}^{+}$ is anti-Hermitian. There exists, therefore, for each quantum system, an Hermitian operator $J_{\hat{u}} = i\hat{M}_{\hat{u}}$, such that

$$U_{\hat{u}}(\delta\alpha) = \mathbf{1} - i\delta\alpha\,J_{\hat{u}}. \tag{5.47}$$

The Hermitian operator $J_{\hat{u}}$ is referred to as the **generator of infinitesimal rotations** about the axis $\hat{u}$. Evidently, there is a different operator $J_{\hat{u}}$ characterizing rotations about each direction in space. Fortunately, as it turns out, all of these different operators $J_{\hat{u}}$ can be expressed as a simple combination of any three operators J_x, J_y, and J_z describing rotations about a given set of coordinate axes. This economy of expression arises from the combination rule (5.41) obeyed by infinitesimal rotations, which implies a corresponding rule

$$U_{\hat{u}}(\delta\alpha) = U_x(u_x\delta\alpha)U_y(u_y\delta\alpha)U_z(u_z\delta\alpha)$$

for the unitary operators that represent them in Hilbert space. Using (5.47), this fundamental relation implies that

$$\begin{aligned} U_{\hat{u}}(\delta\alpha) &= (\mathbf{1} - i\delta\alpha\, u_x J_x)(\mathbf{1} - i\delta\alpha\, u_y J_y)(\mathbf{1} - i\delta\alpha\, u_z J_z) \\ &= \mathbf{1} - i\delta\alpha\,(u_x J_x + u_y J_y + iu_z J_z). \end{aligned} \tag{5.48}$$

Implicit in the form of Eq. (5.48) is the existence of a vector operator $\vec{J}$, with Hermitian components J_x, J_y, and J_z that generate infinitesimal rotations about the corresponding coordinate axes, and in terms of which an arbitrary infinitesimal rotation can be expressed in the form

$$U_{\hat{u}}(\delta\alpha) = \mathbf{1} - i\delta\alpha\, \vec{J} \cdot \hat{u} = \mathbf{1} - i\delta\alpha\, J_u \tag{5.49}$$

where $J_u = \vec{J} \cdot \hat{u}$ now represents the component of the vector operator $\vec{J}$ along $\hat{u}$.

From this form that we have deduced for the unitary operators representing infinitesimal rotations we can now construct the operators representing finite rotations. Since rotations about a fixed axis form a commutative subgroup, we can write

$$U_{\hat{u}}(\alpha + \delta\alpha) = U_{\hat{u}}(\delta\alpha) U_{\hat{u}}(\alpha) = (\mathbf{1} - i\delta\alpha\, J_u) U_{\hat{u}}(\alpha) \tag{5.50}$$

which implies that

$$\frac{dU_{\hat{u}}(\alpha)}{d\alpha} = \lim_{\delta\alpha \to 0} \frac{U_{\hat{u}}(\alpha + \delta\alpha) - U_{\hat{u}}(\alpha)}{\delta\alpha} = -iJ_u U_{\hat{u}}(\alpha). \tag{5.51}$$

The solution to this equation, subject to the obvious boundary condition $U_{\hat{u}}(0) = \mathbf{1}$, is the unitary rotation operator

$$U_{\hat{u}}(\alpha) = \exp\left(-i\alpha J_u\right) = \exp\left(-i\alpha \vec{J} \cdot \vec{u}\right). \tag{5.52}$$

We have shown, therefore, that a description of the behavior of a quantum system under rotations leads automatically to the identification of a vector operator $\vec{J}$, whose components act as generators of infinitesimal rotations and the exponential of which generates the unitary operators necessary to describe more general rotations of arbitrary quantum mechanical systems. It is convenient to adopt the point of view that the vector operator $\vec{J}$ whose existence we have deduced represents, *by definition*, the total angular momentum of the associated system. We will postpone until later a discussion of how angular momentum operators for particular systems are actually identified and constructed. In the meantime, however, to show that this point of view is at least consistent we must demonstrate that the components of $\vec{J}$ satisfy the characteristic commutation relations (5.18) that are, in fact, obeyed by the operators representing the orbital angular momentum of a system of one or more particles.

5.4 Commutation Relations for Scalar and Vector Operators

The analysis of the last section shows that for a general quantum system there exists a vector operator $\vec{J}$, to be identified with the angular momentum of the system, that is essential for describing the effect of rotations on the state vector $|\psi\rangle$ and its observables Q. Indeed, the results of the last section imply that a rotation $\mathbf{R}_{\hat{u}}(\alpha)$ of the physical system will take an arbitrary observable Q onto a generally different observable

$$Q' = U_{\hat{u}}(\alpha) Q U_{\hat{u}}^{+}(\alpha) = e^{-i\alpha J_u} Q e^{i\alpha J_u}. \tag{5.53}$$

For infinitesimal rotations $U_{\hat{u}}(\delta\alpha)$, this transformation law takes the form

$$Q' = (\mathbf{1} - i\delta\alpha\, J_u) Q (\mathbf{1} + i\delta\alpha\, J_u) \tag{5.54}$$

which, to lowest nontrivial order, implies that

$$Q' = Q - i\delta\alpha \left[J_u, Q\right]. \tag{5.55}$$

Now, as in classical mechanics, it is possible to classify certain types of observables of the system according to the manner in which they transform under rotations. Thus, an observable Q is referred to as a **scalar** with respect to rotations if

$$Q' = Q, \tag{5.56}$$

for all $\mathbf{R}$. For this to be true for arbitrary rotations, we must have, from (5.53), that

$$Q' = U_{\mathbf{R}} Q U_{\mathbf{R}}^{+} = Q, \tag{5.57}$$

which implies that $U_{\mathbf{R}} Q = Q U_{\mathbf{R}}$, or

$$[Q, U_{\mathbf{R}}] = 0. \tag{5.58}$$

Thus, for Q to be a scalar it must commute with the complete set of rotation operators for the space. A somewhat simpler expression can be obtained by considering the infinitesimal rotations, where from (5.55) and (5.56) we see that the condition for Q to be a scalar reduces to the requirement that

$$[J_u, Q] = 0, \tag{5.59}$$

for all components J_u, which implies that

$$\left[\vec{J}, Q\right] = 0. \tag{5.60}$$

Thus, by definition, any observable that commutes with the total angular momentum of the system is a scalar with respect to rotations.

A collection of three operators V_x, V_y, and V_z can be viewed as forming the components of a vector operator $\vec{V}$ if the component of $\vec{V}$ along an arbitrary direction $\hat{a}$ is $V_a = \vec{V} \cdot \hat{a} = \sum_i V_i a_i$. By construction, therefore, the operator $\vec{J}$ is a vector operator, since its component along any direction is a linear combination of its three Cartesian components with coefficients that are, indeed, just the associated direction cosines. Now, after undergoing a rotation $\mathbf{R}$, a device initially setup to measure the component V_a of a vector operator $\vec{V}$ along the direction $\hat{a}$ will now measure the component of $\vec{V}$ along the rotated direction

$$\hat{a}' = A_{\mathbf{R}} \hat{a}, \tag{5.61}$$

where $A_{\mathbf{R}}$ is the orthogonal matrix associated with the rotation $\mathbf{R}$. Thus, we can write

$$\mathbf{R}\left[V_a\right] = U_{\mathbf{R}} V_a U_{\mathbf{R}}^{+} = U_{\mathbf{R}} (\vec{V} \cdot \hat{a}) U_{\mathbf{R}}^{+} = \vec{V} \cdot \hat{a}' = V_{a'}. \tag{5.62}$$

Again considering infinitesimal rotations $U_{\hat{u}}(\delta\alpha)$, and applying (5.55), this reduces to the relation

$$\vec{V} \cdot \hat{a}' = \vec{V} \cdot \hat{a} - i\delta\alpha \left[\vec{J} \cdot \hat{u}, \vec{V} \cdot \hat{a}\right]. \tag{5.63}$$

But we also know that, as in (5.34), an infinitesimal rotation $A_{\hat{u}}(\delta\alpha)$ about $\hat{u}$ takes the vector $\hat{a}$ onto the vector

$$\hat{a}' = (\mathbf{1} + \delta\alpha\, M_{\hat{u}})\, \hat{a} = \hat{a} + \delta\alpha\, (\hat{u} \times \hat{a}). \tag{5.64}$$

Consistency of (5.63) and (5.64) requires that

$$\vec{V} \cdot \hat{a}' = \vec{V} \cdot \hat{a} + \delta\alpha\, \vec{V} \cdot (\hat{u} \times \hat{a}) = \vec{V} \cdot \hat{a} - i\delta\alpha \left[\vec{J} \cdot \hat{u}, \vec{V} \cdot \hat{a}\right] \tag{5.65}$$

i.e., that

$$\left[\vec{J}\cdot\hat{u},\vec{V}\cdot\hat{a}\right]=i\vec{V}\cdot(\hat{u}\times\hat{a}). \tag{5.66}$$

Taking $\hat{u}$ and $\hat{a}$ along the ith and jth Cartesian axes, respectively, this latter relation can be written in the form

$$[J_i,V_j]=i\sum_k\varepsilon_{ijk}V_k, \tag{5.67}$$

or more specifically

$$[J_x,V_y]=iV_z \qquad [J_y,V_z]=iV_x \qquad [J_z,V_x]=iV_y \tag{5.68}$$

which shows that the components of *any* vector operator of a quantum system obey commutation relations with the components of the angular momentum that are very similar to those derived earlier for the operators associated with the orbital angular momentum, itself. Indeed, since the operator $\vec{J}$ is a vector operator with respect to rotations, it must also obey these same commutation relations, i.e.,

$$[J_i,J_j]=\sum_k i\varepsilon_{ijk}J_k. \tag{5.69}$$

Thus, our identification of the operator $\vec{J}$ identified above as the total angular momentum of the quantum system is entirely *consistent* with our earlier definition, in which we identified as an angular momentum any vector operator whose components obey the characteristic commutation relations (5.18).

5.5 Relation to Orbital Angular Momentum

To make some of the ideas introduced above a bit more concrete, we show how the generator of rotations $\vec{J}$ relates to the usual definition of angular momentum for, e.g., a single spinless particle. This is most easily done by working in the position representation. For example, let $\psi(\vec{r})=\langle\vec{r}|\psi\rangle$ be the wave function associated with an arbitrary state $|\psi\rangle$ of a single spinless particle. Under a rotation $\mathbf{R}$, the ket $|\psi\rangle$ is taken onto a new ket $|\psi'\rangle=U_R|\psi\rangle$ described by a different wave function $\psi'(\vec{r})=\langle\vec{r}|\psi'\rangle$. The new wave function ψ', obtained from the original by rotation, has the property that the value of the unrotated wavefunction ψ at the point $\vec{r}$ must be the same as the value of the rotated wave function ψ' at the rotated point $\vec{r}'=A_R\vec{r}$. This relationship can be written in several ways, e.g.,

$$\psi(\vec{r})=\psi'(\vec{r}')=\psi'(A_R\vec{r}) \tag{5.70}$$

which can be evaluated at the point $A_R^{-1}\vec{r}$ to obtain

$$\psi'(\vec{r})=\psi(A_R^{-1}\vec{r}). \tag{5.71}$$

Suppose that in (5.71), the rotation $A_R=A_{\hat{u}}(\delta\alpha)$ represents an infinitesimal rotation about the axis $\hat{u}$ through and angle $\delta\alpha$, for which

$$A_R\vec{r}=\vec{r}+\delta\alpha(\hat{u}\times\vec{r}). \tag{5.72}$$

The inverse rotation A_R^{-1} is then given by

$$A_R^{-1}\vec{r}=\vec{r}-\delta\alpha\left(\hat{u}\times\vec{r}\right). \tag{5.73}$$

Thus, under such a rotation, we can write

$$\begin{aligned}\psi'(\vec{r}) &= \psi(A_R^{-1}\vec{r})=\psi\left[\vec{r}-\delta\alpha\left(\hat{u}\times\vec{r}\right)\right]\\ &= \psi\left(\vec{r}\right)-\delta\alpha\left(\hat{u}\times\vec{r}\right)\cdot\vec{\nabla}\psi(\vec{r})\end{aligned} \tag{5.74}$$

where we have expanded $\psi\left[\vec{r}-\delta\alpha\left(\hat{u}\times\vec{r}\right)\right]$ about the point $\vec{r}$, retaining first order infinitesimals. We can use the easily-proven identity

$$(\hat{u}\times\vec{r})\cdot\vec{\nabla}\psi=\hat{u}\cdot(\vec{r}\times\vec{\nabla})\psi$$

which allows us to write

$$\begin{aligned}\psi'(\vec{r}) &= \psi\left(\vec{r}\right)-\delta\alpha\;\hat{u}\cdot(\vec{r}\times\vec{\nabla})\psi(\vec{r})\\ &= \psi\left(\vec{r}\right)-i\delta\alpha\;\hat{u}\cdot\left(\vec{r}\times\frac{\vec{\nabla}}{i}\right)\psi(\vec{r})\end{aligned}\tag{5.75}$$

In Dirac notation this is equivalent to the relation

$$\langle\vec{r}|\psi'\rangle=\langle\vec{r}|U_{\hat{u}}(\delta\alpha)|\psi\rangle=\langle\vec{r}|\left(\mathbf{1}-i\delta\alpha\;\hat{u}\cdot\vec{\ell}\right)|\psi\rangle\tag{5.76}$$

where

$$\vec{\ell}=\vec{R}\times\vec{K}.\tag{5.77}$$

Thus, we identify the vector operator $\vec{J}$ for a single spinless particle with the orbital angular momentum operator $\vec{\ell}$. This allows us to write a general rotation operator for such a particle in the form

$$U_{\hat{u}}(\alpha)=\exp\left(-i\alpha\vec{\ell}\cdot\hat{u}\right).\tag{5.78}$$

Thus the components of $\vec{\ell}$ form the generators of infinitesimal rotations.

Now the state space for a collection of such particles can be considered the direct or tensor product of the state spaces associated with each one. Since operators from different spaces commute with each other, the unitary operator $U_R^{(1)}$ that describes rotations of one particle will commute with those of another. It is not difficult to see that under these circumstances the operator that rotates the entire state vector $|\psi\rangle$ is the product of the rotation operators for each particle. Suppose, e.g., that $|\psi\rangle$ is a direct product state, i.e.,

$$|\psi\rangle=|\psi_1,\psi_2,\cdots,\psi_N\rangle.$$

Under a rotation $\mathbf{R}$, the state vector $|\psi\rangle$ is taken onto the state vector

$$\begin{aligned}|\psi'\rangle &= |\psi_1',\psi_2',\cdots,\psi_N'\rangle\\ &= U_R^{(1)}|\psi_1'\rangle U_R^{(2)}|\psi_2'\rangle\cdots U_R^{(N)}|\psi_N'\rangle\\ &= U_R^{(1)}U_R^{(2)}\cdots U_R^{(N)}|\psi_1,\psi_2,\cdots,\psi_N\rangle\\ &= U_R|\psi\rangle\end{aligned}$$

where

$$U_R=U_R^{(1)}U_R^{(2)}\cdots U_R^{(N)}$$

is a product of rotation operators for each part of the space, all corresponding to the same rotation $\mathbf{R}$. Because these individual operators can all be written in the same form, i.e.,

$$U_R^{(\beta)}=\exp\left(-i\alpha\vec{\ell}_\beta\cdot\hat{u}\right),$$

where $\vec{\ell}_\beta$ it the orbital angular momentum for particle β, it follows that the total rotation operator for the space takes the form

$$U_R=\prod_\beta\exp\left(-i\alpha\vec{\ell}_\beta\cdot\hat{u}\right)=\exp\left(-i\alpha\vec{L}\cdot\hat{u}\right)$$

where

$$\vec{L} = \sum_\beta \vec{\ell}_\beta.$$

Thus, we are led naturally to the point of view that the generator of rotations for the whole system is the sum of the generators for each part thereof, hence for a collection of spinless particles the total angular momentum $\vec{J}$ coincides with the total orbital angular momentum $\vec{L}$, as we would expect.

Clearly, the generators for any composite system formed from the direct product of other subsystems is always the sum of the generators for each subsystem being combined. That is, for a general direct product space in which the operators $\vec{J}_1, \vec{J}_2, \cdots \vec{J}_N$ are the vector operators whose components are the generators of rotations for each subspace, the corresponding generators of rotation for the combined space is obtained as a sum

$$\vec{J} = \sum_{\alpha=1}^{N} \vec{J}_\alpha$$

of those for each space, and the total rotation operator takes the form

$$U_{\hat{u}}(\alpha) = \exp\left(-i\alpha\vec{J}\cdot\hat{u}\right).$$

For particles with spin, the individual particle spaces can themselves be considered direct products of a spatial part and a spin part. Thus for a single particle of spin $\vec{S}$ the generator of rotations are the components of the vector operator $\vec{J} = \vec{L} + \vec{S}$, where $\vec{L}$ takes care of rotations on the spatial part of the state and $\vec{S}$ does the same for the spin part.

5.6 Eigenstates and Eigenvalues of Angular Momentum Operators

Having explored the relationship between rotations and angular momenta, we now undertake a systematic study of the eigenstates and eigenvalues of a vector operator $\vec{J}$ obeying angular momentum commutation relations of the type that we have derived. As we will see, the process for obtaining this information is very similar to that used to determine the spectrum of the eigenstates of the harmonic oscillator. We consider, therefore, an arbitrary angular momentum operator $\vec{J}$ whose components satisfy the relations

$$[J_i, J_j] = i\sum_k \varepsilon_{ijk}J_k \qquad \left[\vec{J}, J^2\right] = 0. \tag{5.79}$$

We note, as we did for the orbital angular momentum $\vec{L}$, that, since the components J_i do not commute with one another, $\vec{J}$ cannot possess an ONB of eigenstates, i.e., states which are simultaneous eigenstates of all three of its operator components. In fact, one can show that the only possible eigenstates of $\vec{J}$ are those for which the angular momentum is identically zero (an s-state in the language of spectroscopy). Nonetheless, since, according to (5.79), each component of $\vec{J}$ commutes with J^2, it *is* possible to find an ONB of eigenstates common to J^2 and to the component of $\vec{J}$ along any chosen direction. Usually the component of $\vec{J}$ along the z-axis is chosen, because of the simple form taken by the differential operator representing that component of orbital angular momentum in spherical coordinates. Note, however, that due to the cyclical nature of the commutation relations, anything deduced about the spectrum and eigenstates of J^2 and J_z must also apply to the eigenstates common to J^2 and to any other component of $\vec{J}$. Thus the spectrum of J_z must be the same as that of J_x, J_y, or $J_u = \vec{J}\cdot\hat{u}$.

We note also, that, as with l^2, the operator $J^2 = \sum_i J_i^2$ is Hermitian and positive definite and thus its eigenvalues must be greater than or equal to zero. For the moment, we will ignore other quantum numbers and simply denote a common eigenstate of J^2 and J_z as $|j,m\rangle$, where, by definition,

$$
\begin{aligned}
J_z|j,m\rangle &= m|j,m\rangle &(5.80)\\
J^2|j,m\rangle &= j(j+1)|j,m\rangle. &(5.81)
\end{aligned}
$$

which implies that m is the associated eigenvalue of the operator J_z, while the label j is intended to simply identify the corresponding eigenvalue $j(j+1)$ of J^2. The justification for writing the eigenvalues of J^2 in this fashion is only that it simplifies the algebra and the final results obtained. At this point, there is no obvious harm in expressing things in this fashion since for real values of j the corresponding values of $j(j+1)$ include *all* values between 0 and ∞, and so any possible eigenvalue of J^2 can be represented in the form $j(j+1)$ for *some* value of j. Moreover, it is easy to show that any non-negative value of $j(j+1)$ can be obtained using a value of j which is itself non-negative. Without loss of generality, therefore, we assume that $j \geq 0$. We will also, in the interest of brevity, refer to a vector $|j,m\rangle$ satisfying the eigenvalue equations (5.80) and (5.81) as a "vector of angular momentum (j,m)".

To proceed further, it is convenient to introduce the operator

$$J_+ = J_x + iJ_y \tag{5.82}$$

formed from the components of $\vec{J}$ along the x and y axes. The adjoint of J_+ is the operator

$$J_- = J_x - iJ_y, \tag{5.83}$$

in terms of which we can express the original operators

$$J_x = \frac{1}{2}\left(J_+ + J_-\right) \qquad\qquad J_y = \frac{i}{2}\left(J_- - J_+\right). \tag{5.84}$$

Thus, in determining the spectrum and common eigenstates of J^2 and J_z ,we will find it convenient to work with the set of operators $\left\{J_+, J_-, J^2, J_z\right\}$ rather than the set $\left\{J_x, J_y, J_z, J^2\right\}$. In the process, we will require commutation relations for the operators in this new set. We note first that $J_\pm$, being a linear combination of J_x and J_y, must by (5.79) commute with J^2. The commutator of $J_\pm$ with J_z is also readily established; we find that

$$[J_z, J_\pm] = [J_z, J_x] \pm i\,[J_z, J_y] = iJ_y \pm J_x \tag{5.85}$$

or

$$[J_z, J_\pm] = \pm J_\pm. \tag{5.86}$$

Similarly, the commutator of J_+ and J_- is

$$
\begin{aligned}
[J_+, J_-] &= [J_x, J_x] + [iJ_y, J_x] - [J_x, iJ_y] - [iJ_y, iJ_y]\\
&= 2J_z. &(5.87)
\end{aligned}
$$

Thus the commutation relations of interest take the form

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_z \qquad \left[J^2, J_\pm\right] = 0 = \left[J^2, J_z\right]. \tag{5.88}$$

It will also be necessary in what follows to express the operator J^2 in terms of the new "components" $\{J_+, J_-, J_z\}$ rather than the old components $\{J_x, J_y, J_z\}$. To this end we note that $J^2 - J_z^2 = J_x^2 + J_y^2$, and so

$$
\begin{aligned}
J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i\,[J_x, J_y]\\
&= J_x^2 + J_y^2 + J_z = J^2 - J_z(J_z - 1) &(5.89)
\end{aligned}
$$

and

$$\begin{aligned} J_- J_+ &= (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y] \\ &= J_x^2 + J_y^2 - J_z = J^2 - J_z(J_z + 1). \end{aligned} \tag{5.90}$$

These two results imply the relation

$$J^2 = \frac{1}{2}[J_+ J_- + J_- J_+] + J_z^2. \tag{5.91}$$

With these relations we now deduce allowed values in the spectrum of J^2 and J_z. We assume, at first, the existence of at least one nonzero eigenvector $|j, m\rangle$ of J^2 and J_z with angular momentum (j, m), where, consistent with our previous discussion, the eigenvalues of J^2 satisfy the inequalities

$$j(j+1) \geq 0 \qquad\qquad j \geq 0. \tag{5.92}$$

Using this, and the commutation relations, we now prove that the eigenvalue m must lie, for a given value of j, in the range

$$j \geq m \geq -j. \tag{5.93}$$

To show this, we consider the vectors $J_+|j, m\rangle$ and $J_-|j, m\rangle$, whose squared norms are

$$||J_+|j, m\rangle||^2 = \langle j, m|J_- J_+|j, m\rangle \tag{5.94}$$

$$||J_-|j, m\rangle||^2 = \langle j, m|J_+ J_-|j, m\rangle. \tag{5.95}$$

Using (5.89) and (5.90) these last two equations can be written in the form

$$||J_+|j, m\rangle||^2 = \langle j, m|J^2 - J_z(J_z + 1)|j, m\rangle = [j(j+1) - m(m+1)]\,\langle j, m|j, m\rangle \tag{5.96}$$

$$||J_-|j, m\rangle||^2 = \langle j, m|J^2 - J_z(J_z - 1)|j, m\rangle = [j(j+1) - m(m-1)]\,\langle j, m|j, m\rangle \tag{5.97}$$

Now, adding and subtracting a factor of jm from each parenthetical term on the right, these last expressions can be factored into

$$||J_+|j, m\rangle||^2 = (j - m)(j + m + 1)|||j, m\rangle||^2 \tag{5.98}$$

$$||J_-|j, m\rangle||^2 = (j + m)(j - m + 1)|||j, m\rangle||^2. \tag{5.99}$$

For these quantities to remain positive definite, we must have

$$(j - m)(j + m + 1) \geq 0 \tag{5.100}$$

and

$$(j + m)(j - m + 1) \geq 0. \tag{5.101}$$

For positive j, the first inequality requires that

$$j \geq m \qquad \text{and} \qquad m \geq -(j+1) \tag{5.102}$$

and the second that

$$m \geq -j \qquad \text{and} \qquad j + 1 \geq m. \tag{5.103}$$

All four inequalities are satisfied if and only if

$$j \geq m \geq -j, \tag{5.104}$$

which shows, as required, that the eigenvalue m of J_z must lie between $\pm j$.

Having narrowed the range for the eigenvalues of J_z, we now show that the vector $J_+|j,m\rangle$ vanishes if and only if $m = j$, and that otherwise $J_+|j,m\rangle$ is an eigenvector of J^2 and J_z with angular momentum $(j, m+1)$, i.e., it is associated with the same eigenvalue $j(j+1)$ of J^2, but is associated with an eigenvalue of J_z increased by one, as a result of the action of J_+.

To show the first half of the statement, we note from (5.98), that

$$||J_+|j,m\rangle|| = (j-m)(j+m+1)|||j,m\rangle||^2. \tag{5.105}$$

Given the bounds on m, it follows that $J_+|j,m\rangle$ vanishes if and only if $m = j$. To prove the second part, we use the commutation relation $[J_z, J_+] = J_+$ to write $J_zJ_+ = J_+J_z + J_+$ and thus

$$J_zJ_+|j,m\rangle = (J_+J_z + J_+)|j,m\rangle = (m+1)J_+|j,m\rangle, \tag{5.106}$$

showing that $J_+|j,m\rangle$ is an eigenvector of J_z with eigenvalue $m+1$. Also, because $[J^2, J_+] = 0$,

$$J^2J_+|j,m\rangle = J_+J^2|j,m\rangle = j(j+1)J_+|j,m\rangle \tag{5.107}$$

showing that if $m \neq j$, then the vector $J_+|j,m\rangle$ is an eigenvector of J^2 with eigenvalue $j(j+1)$.

In a similar fashion, using (5.99) we find that

$$||J_-|j,m\rangle|| = (j+m)(j-m+1)|||j,m\rangle||^2. \tag{5.108}$$

Given the bounds on m, this proves that the vector $J_-|j,m\rangle$ vanishes if and only if $m = -j$, while the commutation relations $[J_z, J_-] = -J_-$ and $[J^2, J_-] = 0$ imply that

$$J_zJ_-|j,m\rangle = (J_-J_z - J_z)|j,m\rangle = (m-1)J_-|j,m\rangle \tag{5.109}$$

$$J^2J_-|j,m\rangle = J_-J^2|j,m\rangle = j(j+1)J_-|j,m\rangle. \tag{5.110}$$

Thus, when $m \neq -j$, the vector $J_-|j,m\rangle$ is an eigenvector of J^2 and J_z with angular momentum $(j, m-1)$.

Thus, J_+ is referred to as the raising operator, since it acts to increase the component of angular momentum along the z-axis by one unit and J_- is referred to as the lowering operator. Neither operator has any effect on the total angular momentum of the system, as represented by the quantum number j labeling the eigenvalues of J^2.

We now proceed to restrict even further the spectra of J^2 and J_z. We note, e.g., from our preceding analysis that, given any vector $|j,m\rangle$ of angular momentum (j,m) we can produce a sequence

$$J_+|j,m\rangle, J_+^2|j,m\rangle, J_+^3|j,m\rangle, \cdots \tag{5.111}$$

of mutually orthogonal eigenvectors of J^2 with eigenvalue $j(j+1)$ and of J_z with eigenvalues

$$m, (m+1), (m+2), \cdots. \tag{5.112}$$

This sequence must terminate, or else produce eigenvectors of J_z with eigenvalues violating the upper bound in Eq. (5.93). Termination occurs when J_+ acts on the last nonzero vector of the sequence, $J_+^n|j,m\rangle$ say, and takes it on to the null vector. But, as we have shown, this can only occur if $m + n = j$, i.e., if $J_+^n|j,m\rangle$ is an eigenvector of angular momentum $(j, m+n) = (j,j)$. Thus, there must exist an integer n such that

$$n = j - m. \tag{5.113}$$

This, by itself, does not require that m or j be integers, only that the values of m change by an integer amount, so that the difference between m and j be an integer. But similar arguments can be made for the sequence

$$J_-|j,m\rangle, J_-^2|j,m\rangle, J_-^3|j,m\rangle, \cdots \tag{5.114}$$

which will be a series of mutually orthogonal eigenvectors of J^2 with eigenvalue $j(j+1)$ and of J_z with eigenvalues

$$m, (m-1), (m-2), \cdots . \tag{5.115}$$

Again, termination is required to now avoid producing eigenvectors of J_z with eigenvalues violating the lower bound in Eq. (5.93). Thus, the action of J_- on the last nonzero vector of the sequence, $J_+^{n'}|j,m\rangle$ say, is to take it onto the null vector. But this only occurs if $m-n'=-j$, i.e., if $J_+^{n'}|j,m\rangle$ is an eigenvector of angular momentum $(j, m-n')=(j,-j)$. Thus there exists an integer n' such that

$$n' = j + m. \tag{5.116}$$

Adding these two relations, we deduce that there exists an integer $N = n + n'$ such that $2j = n + n' = N$, or

$$j = \frac{N}{2}. \tag{5.117}$$

Thus, j must be either an integer or a half-integer. If N is an even integer, then j is itself an integer and must be contained in the set $j \in \{0, 1, 2, \cdots\}$. For this situation, the results of the proceeding analysis indicate that m must also be an integer and, for a given integer value of j, the values m must take on each of the $2j+1$ integer values $m = 0, \pm 1, \pm 2, \cdots \pm j$. In this case, j is said to be an integral value of angular momentum.

If N is an odd integer, then j differs from an integer by $1/2$, i.e., it is contained in the set $j \in \{1/2, 3/2, 5/2, \cdots\}$, and is said to be half-integral (short for half-odd-integral). For a given half-integral value of j, the values of m must then take on each of the $2j+1$ half-odd-integer values $m = \pm 1/2, \cdots, \pm j$.

Thus, we have deduced the values of j and m that are consistent with the commutation relations (5.79). In particular, the allowed values of j that can occur are the non-negative integers and the positive half-odd-integers. For each value of j, there are always (at least) $2j+1$ fold mutually-orthogonal eigenvectors

$$\{|j,m\rangle \mid m = -j, -j+1, \cdots, j\} \tag{5.118}$$

of J^2 and J_z corresponding to the same eigenvalue $j(j+1)$ of J^2, but different eigenvalues m of J_z (the orthogonality of the different vectors in the set follows from the fact that they are eigenvectors of the Hermitian observable J_z corresponding to different eigenvalues.)

In any given problem involving an angular momentum $\vec{J}$ it must be determined which of the allowed values of j and how many subspaces for each such value actually occur. All of the integer values of angular momentum do, in fact, arise in the study of the orbital angular momentum of a single particle, or of a group of particles. Half-integral values of angular momentum, on the other hand, are invariably found to have their source in the half-integral angular momenta associated with the internal or spin degrees of freedom of particles that are anti-symmetric under exchange, i.e., fermions. Bosons, by contrast, are empirically found to have integer spins. This apparently universal relationship between the exchange symmetry of identical particles and their spin degrees of freedom has actually been derived under a rather broad set of assumptions using the techniques of quantum field theory.

The total angular momentum of a system of particles will generally have contributions from both orbital and spin angular momenta and can be either integral or half-integral, depending upon the number and type of particles in the system. Thus, generally speaking, there do exist different systems in which the possible values of j and m deduced above are actually realized. In other words, there appear to be no superselection rules in nature that further restrict the allowed values of angular momentum from those allowed by the fundamental commutation relations.

5.7 Orthonormalization of Angular Momentum Eigenstates

We have seen in the last section that, given any eigenvector $|j,m\rangle$ with angular momentum (j,m), it is possible to construct a set of $2j+1$ common eigenvectors of J^2 and J_z corresponding to the same value of j, but different values of m. Specifically, the vectors obtained by repeated application of J_+ to $|j,m\rangle$ will produce a set of eigenvectors of J_z with eigenvalues $m+1, m+2, \cdots, j$, while repeated application of J_- to $|j,m\rangle$ will produce the remaining eigenvectors of J_z with eigenvalues $m-1, m-2, , \cdots -j$. Unfortunately, even when the original angular momentum eigenstate $|j,m\rangle$ is suitably normalized, the eigenvectors vectors obtained by application of the raising and lowering operators to this state are not. In this section, therefore, we consider the construction of a basis of normalized angular momentum eigenstates. To this end, we restrict our use of the notation $|j,m\rangle$ so that it refers only to normalized states. We can then express the action of J_+ on such a normalized state in the form

$$J_+|j,m\rangle = \lambda_m|j,m+1\rangle \tag{5.119}$$

where λ_m is a constant, and $|j,m+1\rangle$ represents, according to our definition, a normalized state with angular momentum $(j,m+1)$. We can determine the constant λ_m by considering the quantity

$$||J_+|j,m\rangle||^2 = \langle j,m|J_-J_+|j,m\rangle = |\lambda_m|^2 \tag{5.120}$$

which from the analysis following Eqs. (5.98) and (5.99), and the assumed normalization of the state $|j,m\rangle$, reduces to $|\lambda_m|^2 = j(j+1)-m(m+1)$. Choosing λ_m real and positive,this implies the following relation

$$J_+|j,m\rangle = \sqrt{j(j+1)-m(m+1)}\ |j,m+1\rangle \tag{5.121}$$

between normalized eigenvectors of J^2 and J_z differing by one unit of angular momentum along the z axis. A similar analysis applied to the operator J_- leads to the relation

$$J_-|j,m\rangle = \sqrt{j(j+1)-m(m-1)}\ |j,m-1\rangle. \tag{5.122}$$

These last two relations can also be written in the sometimes more convenient form

$$|j,m+1\rangle = \frac{J_+|j,m\rangle}{\sqrt{j(j+1)-m(m+1)}} \tag{5.123}$$

$$|j,m-1\rangle = \frac{J_-|j,m\rangle}{\sqrt{j(j+1)-m(m-1)}}. \tag{5.124}$$

or

$$|j,m\pm 1\rangle = \frac{J_\pm|j,m\rangle}{\sqrt{j(j+1)-m(m\pm 1)}}. \tag{5.125}$$

Now, if J^2 and J_z do not comprise a complete set of commuting observables for the space on which they are defined, then other commuting observables (e.g., the energy) will be required to form a uniquely labeled basis of orthogonal eigenstates, i.e., to distinguish between different eigenvectors of J^2 and J_z having the same angular momentum (j, m). If we let τ denote the collection of quantum numbers (i.e., eigenvalues) associated with the other observables needed along with J^2 and J_z to form a complete set of observables, then the normalized basis vectors of such a representation can be written in the form $\{|\tau, j, m\rangle\}$. Typically, many representations of this type are possible, since we can always form linear combinations of vectors with the same values of j and m to generate a new basis set. Thus, in general, basis vectors having the same values of τ and j, but different values of m need not obey the relationships derived above involving the raising and lowering operators. However, as we show below, it is always possible to *construct* a so-called **standard representation**, in which the relationships (5.123) and (5.124) are maintained.

To construct such a standard representation, it suffices to work within each eigensubspace $S(j)$ of J^2 with fixed j, since states with different values of j are automatically orthogonal (since J^2 is Hermitian). Within any such subspace $S(j)$ of J^2, there are always contained even smaller eigenspaces $S(j, m)$ spanned by the vectors $\{|\tau, j, m\rangle\}$ of fixed j and fixed m. We focus in particular on the subspace $S(j, j)$ containing eigenvectors of J_z for which m takes its highest value $m = j$, and denote by $\{|\tau, j, j\rangle\}$ a complete set of normalized basis vectors for this subspace, with the index τ distinguishing between different orthogonal basis vectors with angular momentum (j, j). By assumption, then, for the states in this set,

$$\langle \tau, j, j|\tau', j, j\rangle = \delta_{\tau,\tau'} \tag{5.126}$$

For each member $|\tau, j, j\rangle$ of this set, we now construct the natural sequence of $2j+1$ basis vectors by repeated application of J_-, i.e., using (5.124) we set

$$|\tau, j, j-1\rangle = \frac{J_-|\tau, j, j\rangle}{\sqrt{j(j+1) - j(j-1)}} = \frac{J_-|\tau, j, j\rangle}{\sqrt{2j}} \tag{5.127}$$

and the remaining members of the set according to the relation

$$|\tau, j, m-1\rangle = \frac{J_-|\tau, j, m\rangle}{\sqrt{j(j+1) - m(m-1)}}, \tag{5.128}$$

terminating the sequence with the vector $|\tau, j, -j\rangle$. Since the members $|\tau, j, m\rangle$ of this set (with τ and j fixed and $m = -j, \cdots, j$) are eigenvectors of J_z corresponding to different eigenvalues, they are mutually orthogonal and, by construction, properly normalized. It is also straightforward to show that the vectors $|\tau, j, m\rangle$ generated in this way from the basis vector $|\tau, j, j\rangle$ of $S(j, j)$ are orthogonal to the vectors $|\tau', j, m\rangle$ generated from a different basis vector $|\tau', j, j\rangle$ of $S(j, j)$. To see this, we consider the inner product $\langle \tau, j, m-1|\tau', j, m-1\rangle$ and, using the adjoint of (5.128),

$$\langle \tau, j, m-1| = \frac{\langle \tau, j, m|J_+}{\sqrt{j(j+1) - m(m-1)}}, \tag{5.129}$$

we find that

$$\langle \tau, j, m-1|\tau', j, m-1\rangle = \frac{\langle \tau, j, m|J_+J_-|\tau', j, m\rangle}{j(j+1) - m(m-1)} = \langle \tau, j, m|\tau', j, m\rangle \tag{5.130}$$

where we have used (5.99) to evaluate the matrix element of J_+J_-. This shows that, if $|\tau', j, m\rangle$ and $|\tau, j, m\rangle$ are orthogonal, then so are the states generated from them by application of J_-. Since the basis states $|\tau, j, j\rangle$ and $|\tau', j, j\rangle$ used to start each sequence are orthogonal, by construction, so, it follows, are any two sequences of basis vectors so produced, and so also are the subspaces $S(\tau, j)$ and $S(\tau', j)$ spanned by those basis vectors. Proceeding in this way for all values of j a standard representation of basis vectors for the entire space is produced. Indeed, the entire space can be written as a direct sum of the subspaces $S(\tau, j)$ so formed for each j, i.e., we can write

$$S = S(j) \oplus S(j') \oplus S(j'') + \cdots$$

with a similar decomposition

$$S(j) = S(\tau, j) \oplus S(\tau', j) \oplus S(\tau'', j) + \cdots$$

for the eigenspaces $S(j)$ of j^2. The basis vectors for this representation satisfy the obvious orthonormality and completeness relations

$$\langle \tau', j', m' | \tau, j, m \rangle = \delta_{\tau',\tau} \delta_{j',j} \delta_{m',m} \tag{5.131}$$

$$\sum_{\tau,j,m} |\tau, j, m\rangle\langle \tau, j, m| = \mathbf{1}. \tag{5.132}$$

The matrices representing the components of the angular momentum operators are easily computed in any standard representation. In particular, it is easily verified that the matrix elements of J^2 are given in any standard representation by the expression

$$\langle \tau', j', m' | J^2 | \tau, j, m \rangle = j(j+1) \delta_{\tau',\tau} \delta_{j',j} \delta_{m',m} \tag{5.133}$$

while the components of $\vec{J}$ have the following matrix elements

$$\langle \tau', j', m' | J_z | \tau, j, m \rangle = m \delta_{\tau',\tau} \delta_{j',j} \delta_{m',m} \tag{5.134}$$

$$\langle \tau', j', m' | J_\pm | \tau, j, m \rangle = \sqrt{j(j+1) - m(m \pm 1)} \delta_{\tau',\tau} \delta_{j',j} \delta_{m',m\pm 1}. \tag{5.135}$$

The matrices representing the Cartesian components J_x and J_y can then be constructed from the matrices for J_+ and J_-, using relations (5.84), i.e.

$$\begin{aligned}\langle \tau', j', m' | J_x | \tau, j, m \rangle &= \frac{1}{2} \delta_{\tau',\tau} \delta_{j',j} \Big[\sqrt{j(j+1) - m(m-1)} \delta_{m',m-1} \\ &\quad + \sqrt{j(j+1) - m(m+1)} \delta_{m',m+1} \Big]\end{aligned} \tag{5.136}$$

$$\begin{aligned}\langle \tau', j', m' | J_y | \tau, j, m \rangle &= \frac{i}{2} \delta_{\tau',\tau} \delta_{j',j} \Big[\sqrt{j(j+1) - m(m-1)} \delta_{m',m-1} \\ &\quad - \sqrt{j(j+1) - m(m+1)} \delta_{m',m+1} \Big]\end{aligned} \tag{5.137}$$

Clearly, the simplest possible subspace of fixed total angular momentum j is one corresponding to $j = 0$, which according to the results derived above must be of dimension $2j+1 = 1$. Thus, the one basis vector $|\tau, 0, 0\rangle$ in such a space is a simultaneous eigenvector of J^2 and of J_z with eigenvalue $j(j+1) = m = 0$. Such a state, as it turns out, is also an eigenvector of J_x and J_y (indeed of $\vec{J}$ itself). The 1×1 matrices representing J^2, J_z, $J_\pm$, J_x, and J_- in such a space are all identical to the null operator.

The next largest possible angular momentum subspace corresponds to the value $j = 1/2$, which coincides with the $2j+1 = 2$ dimensional spin space of electrons, protons, and neutrons, i.e., particles of spin 1/2. In general, the full Hilbert space of a single particle of spin s can be considered the direct product

$$S = S_{\text{spatial}} \otimes S_{\text{spin}} \tag{5.138}$$

of the Hilbert space describing the particle's motion through space (a space spanned, e.g., by the position states $|\vec{r}\rangle$) and a finite dimensional space describing its internal spin degrees of freedom. The spin space S_s of a particle of spin s is by definition a $2s+1$ dimensional space having as its fundamental observables the components S_x, S_y, and S_z of a spin angular momentum vector $\vec{S}$. In keeping with the analysis of Sec. (5.3), these operators characterize the way that the spin part of the state vector transforms under rotations and, as such, satisfy the standard angular momentum commutation relations, i.e.,

$$[S_i, S_j] = i\sum_k \varepsilon_{ijk} S_k. \tag{5.139}$$

Thus, e.g., an ONB for the space of one such particle consists of the states $|\vec{r}, m_s\rangle$, which are eigenstates of the position operator $\vec{R}$, the total square of the spin angular momentum S^2, and the component of spin S_z along the z-axis according to the relations

$$\vec{R}|\vec{r}, m_s\rangle = \vec{r}|\vec{r}, m_s\rangle \tag{5.140}$$

$$S^2|\vec{r}, m_s\rangle = s(s+1)|\vec{r}, m_s\rangle \tag{5.141}$$

$$S_z|\vec{r}, m_s\rangle = m_s|\vec{r}, m_s\rangle. \tag{5.142}$$

(As is customary, in these last expressions the label s indicating the eigenvalue of S^2 has been suppressed, since for a given class of particle s does not change.) The state vector $|\psi\rangle$ of such a particle, when expanded in such a basis, takes the form

$$|\psi\rangle = \sum_{m_s=-s}^{s} \int d^3r\ |\vec{r}, m_s\rangle\langle\vec{r}, m_s|\psi\rangle = \sum_{m_s=-s}^{s} \int d^3r\ \psi_{m_s}(\vec{r})|\vec{r}, m_s\rangle \tag{5.143}$$

and thus has a "wave function" with $2s+1$ components $\psi_{m_s}(\vec{r})$. As one would expect from the definition of the direct product, operators from the spatial part of the space have no effect on the spin part and vice versa. Thus, spin and spatial operators automatically commute with each other. In problems dealing only with the spin degrees of freedom, therefore, it is often convenient to simply ignore the part of the space associated with the spatial degrees of freedom (as the spin degrees of freedom are often generally ignored in exploring the basic features of quantum mechanics in real space).

Thus, the spin space of a particle of spin $s = 1/2$, is spanned by two basis vectors, often designated $|+\rangle$ and $|-\rangle$, with

$$S^2|\pm\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)|\pm\rangle = \frac{3}{4}|\pm\rangle \tag{5.144}$$

and

$$S_z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle. \tag{5.145}$$

The matrices representing the different components of $\vec{S}$ within a standard representation for such a space are readily computed from (5.133)-(5.137),

$$S^2 = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.146}$$

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.147}$$

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5.148}$$

The Cartesian components S_x, S_y, and S_z are often expressed in terms of the so-called Pauli σ matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.149}$$

in terms of which $S_i = \frac{1}{2}\sigma_i$. The σ matrices have a number of interesting properties that follow from their relation to the angular momentum operators.

5.8 Orbital Angular Momentum Revisited

As an additional example of the occurrance of angular momentum subspaces with integral values of j we consider again the orbital angular momentum of a single particle as defined by the operator $\vec{\ell} = \vec{R}\times\vec{K}$ with Cartesian components $\ell_i = \sum_{j,k} \varepsilon_{ijk} X_j K_k$. In the position representation these take the form of differential operators

$$\ell_i = -i \sum_{j,k} \varepsilon_{ijk} x_j \frac{\partial}{\partial x_k}. \tag{5.150}$$

As it turns out, in standard spherical coordinates (r, θ, ϕ), where

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r\cos\theta, \tag{5.151}$$

the components of $\vec{\ell}$ take a form which is independent of the radial variable r. Indeed, using the chain rule it is readily found that

$$\ell_x = i\left(\sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right) \tag{5.152}$$

$$\ell_y = i\left(-\cos\phi \frac{\partial}{\partial\theta} + \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right) \tag{5.153}$$

$$\ell_z = -i\frac{\partial}{\partial\phi}. \tag{5.154}$$

In keeping with our previous development, it is useful to construct from ℓ_x and ℓ_y the raising and lowering operators

$$\ell_\pm = \ell_x \pm i\ell_y \tag{5.155}$$

which (5.152) and (5.153) reduce to

$$\ell_\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]. \tag{5.156}$$

From these it is also straightforward to construct the differential operator

$$\ell^2 = -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{5.157}$$

representing the total square of the angular momentum.

Now we are interested in finding common eigenstates of ℓ^2 and ℓ_z. Since these operators are independent of r, it suffices to consider only the angular dependence. It is clear, in other words, that the eigenfunctions of these operators can be written in the form

$$\psi_{l,m}(r,\theta,\phi) = f(r)Y_l^m(\theta,\phi) \tag{5.158}$$

where $f(r)$ is any acceptable function of r and the functions $Y_l^m(\theta,\phi)$ are solutions to the eigenvalue equations

$$\begin{aligned} \ell^2 Y_l^m(\theta,\phi) &= l(l+1)Y_l^m(\theta,\phi) \\ \ell_z Y_l^m(\theta,\phi) &= mY_l^m(\theta,\phi)\,. \end{aligned} \tag{5.159}$$

Thus, it suffices to find the functions $Y_l^m(\theta,\phi)$, which are, of course, just the spherical harmonics.

More formally, we can consider the position eigenstates $|\vec{r}\rangle$ as defining direct product states

$$|\vec{r}\rangle = |r,\theta,\phi\rangle = |r\rangle \otimes |\theta,\phi\rangle, \tag{5.160}$$

i.e., the space can be decomposed into a direct product of a part describing the radial part of the wave function and a part describing the angular dependence. The angular part represents the space of functions on the unit sphere, and is spanned by the "angular position eigenstates" $|\theta,\phi\rangle$. In this space, an arbitrary function $\chi(\theta,\phi)$ on the unit sphere is associated with a ket

$$|\chi\rangle = \int d\Omega\; \chi(\theta,\phi)|\theta,\phi\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\;\sin\theta\;\chi(\theta,\phi)|\theta,\phi\rangle \tag{5.161}$$

where $\chi(\theta,\phi) = \langle\theta,\phi|\chi\rangle$ and the integration is over all solid angle, $d\Omega = \sin\theta\; d\theta\; d\phi$. The states $|\theta,\phi\rangle$ form a complete set of states for this space, and so

$$\int d\Omega\; |\theta,\phi\rangle\langle\theta,\phi| = \mathbf{1}. \tag{5.162}$$

The normalization of these states is slightly different than the usual Dirac normalization, however, because of the factor associated with the transformation from Cartesian to spherical coordinates. To determine the appropriate normalization we note that

$$\chi(\theta',\phi') = \langle\theta',\phi'|\chi\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\;\sin\theta\;\chi(\theta,\phi)\langle\theta',\phi'|\theta,\phi\rangle \tag{5.163}$$

which leads to the identification $\delta(\phi-\phi')\delta(\theta-\theta') = \sin\theta\;\langle\theta',\phi'|\theta,\phi\rangle$, or

$$\langle\theta',\phi'|\theta,\phi\rangle = \frac{1}{\sin\theta}\;\delta(\phi-\phi')\delta(\theta-\theta') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta'). \tag{5.164}$$

Clearly, the components of $\vec{\ell}$ are operators defined on this space and so we denote by $|l,m\rangle$ the appropriate eigenstates of the Hermitian operators ℓ^2 and ℓ_z within this space (this assumes that l,m are sufficient to specify each eigenstate, which, of course, turns out to be true). By assumption, then, these states satisfy the eigenvalue equations

$$\begin{aligned} \ell^2|l,m\rangle &= l(l+1)|l,m\rangle \\ \ell_z|l,m\rangle &= m|l,m\rangle \end{aligned} \tag{5.165}$$

and can be expanded in the angular "position representation", i.e.,

$$|l,m\rangle = \int \delta\Omega\; |\theta,\phi\rangle\langle\theta,\phi|l,m\rangle = \int \delta\Omega\; Y_l^m(\theta,\phi)\;|\theta,\phi\rangle \tag{5.166}$$

where the functions

$$Y_l^m(\theta,\phi) = \langle \theta,\phi|l,m\rangle \tag{5.167}$$

are clearly the same as those introduced above, and will turn out to be the spherical harmonics.

To obtain the states $|l,m\rangle$ (or equivalently the functions Y_l^m) we proceed in three stages. First, we determine the general ϕ-dependence of the solution from the eigenvalue equation for ℓ_z. Then, rather than solving the second order equation for ℓ^2 directly, we determine the general form of the solution for the states $|l,l\rangle$ having the largest component of angular momentum along the z-axis consistent with a given value of l. Finally, we use the lowering operator ℓ_- to develop a general prescription for constructing arbitrary eigenstates of ℓ^2 and ℓ_z.

The ϕ-dependence of the eigenfunctions in the position representation follows from the simple form taken by the operator ℓ_z in this representation. Indeed, using (5.154), it follows that

$$\ell_z Y_l^m(\theta,\phi) = -i\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) = mY_l^m(\theta,\phi), \tag{5.168}$$

which has the general solution

$$Y_l^m(\theta,\phi) = F_l^m(\theta)e^{im\phi}. \tag{5.169}$$

Single-valuedness of the wave function in this representation imposes the requirement that $Y_l^m(\theta,\phi) = Y_l^m(\theta,\phi+2\pi)$, which leads to the restriction $m \in \{0,\pm1,\pm2,\cdots\}$. Thus, for the case of orbital angular momentum only integral values of m (and therefore l) can occur. (We have yet to show that all integral values of l do, in fact, occur, however.)

From this result we now proceed to determine the eigenstates $|l,l\rangle$, as represented by the wave functions

$$Y_l^l(\theta,\phi) = F_l^l(\theta)e^{il\phi}. \tag{5.170}$$

To this end, we recall the general result that any such state of maximal angular momentum along the z axis is taken by the *raising operator* onto the null vector, i.e., $\ell_+|l,l\rangle = 0$. In the position representation, using (5.156), this takes the form

$$\begin{aligned}\langle\theta,\phi|\ell_+|l,l\rangle &= e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]Y_l^l(\theta,\phi) \\ &= e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]F_l^l(\theta)e^{il\phi} = 0.\end{aligned} \tag{5.171}$$

Performing the ϕ derivative reduces this to a first order equation for $F_l^l(\theta)$, i.e.,

$$\frac{dF_l^l(\theta)}{d\theta} = l\cot\theta F_l^l(\theta) \tag{5.172}$$

$$\frac{dF_l^l}{F_l^l} = l\frac{d\,(\sin\theta)}{\sin\theta} \tag{5.173}$$

which integrates to give, up to an overall multiplicative constant, a single linearly independent solution

$$F_l^l(\theta) = c_l\sin^l\theta \tag{5.174}$$

for each allowed value of l. Thus, all values of l consistent with the integer values of m deduced above give acceptable solutions. Up to normalization we have, therefore, for each $l = 0,1,2\cdots$, the functions

$$Y_l^l(\theta,\phi) = c_l\sin^l\theta\;e^{il\phi}. \tag{5.175}$$

The appropriate normalization for these functions follows from the relation

$$\langle l,m|l,m\rangle = 1 \tag{5.176}$$

which in the position representation becomes

$$\begin{aligned}\int d\Omega \, \langle l,m|\theta,\phi\rangle\langle\theta,\phi|l,m\rangle &= \int d\Omega \, \left[Y_l^m(\theta,\phi)\right]^* Y_{l'}^{m'}(\theta,\phi) \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, \left|Y_l^m(\theta,\phi)\right|^2 = 1 \end{aligned}\tag{5.177}$$

Substituting in the function $Y_l^l(\theta,\phi) = c_l \sin^l\theta \, e^{il\phi}$, the magnitude of the constants c_l can be determined by iteration, with the result that

$$|c_l| = \sqrt{\frac{(2l+1)!!}{4\pi \ (2l)!!}} = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}, \tag{5.178}$$

where the double factorial notation is defined on the postive integers as follows:

$$n!! = \begin{cases} n(n-2)(n-4)\cdots(2) & \text{if } n \text{ an even integer} \\ n(n-2)(n-4)\cdots(1) & \text{if } n \text{ an odd integer} \\ 1 & \text{if } n = 0 \end{cases} \tag{5.179}$$

With the phase of c_l, (chosen so that $Y_l^0(0,0)$ is real and positive) given by the relation $c_l = (-1)^l |c_l|$ we have the final form for the spherical harmonic of order (l,l), i.e.

$$Y_l^l(\theta,\phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}} \sin^l\theta \, e^{il\phi}. \tag{5.180}$$

From this, the remaining spherical harmonics of the same order l can be generated through application of the lowering operator, e.g., through the relation

$$|l,m-1\rangle = \frac{\ell_-|l,m\rangle}{\sqrt{l(l+1)-m(m-1)}} \tag{5.181}$$

which, in the position representation takes the form

$$Y_l^{m-1}(\theta,\phi) = e^{-i\phi}\left[-\frac{\partial}{\partial\theta} - m\cot\theta\right] Y_l^m(\theta,\phi). \tag{5.182}$$

In fact, it straightforward to derive the following expression

$$|l,m\rangle = \sqrt{\frac{(l+m)!}{(2l)!(l+m)!}}\,[\ell_-]^{l-m}\,|l,l\rangle \tag{5.183}$$

relating an arbitrary state $|l,m\rangle$ to the state $|l,l\rangle$ which we have explicitly found. This last relation is straightforward to prove by induction. We first assume that it holds for some value of m and then consider

$$|l,m-1\rangle = \frac{\ell_-|l,m\rangle}{\sqrt{l(l+1)-m(m-1)}} = \frac{1}{\sqrt{l(l+1)-m(m-1)}}\sqrt{\frac{(l-m)!}{(2l)!(l+m)!}}\,[\ell_-]^{l-m+1}\,|l,l\rangle. \tag{5.184}$$

But, as we have noted before, $l(l+1)-m(m-1)=(l+m)\,(l-m+1)$. Substituting this into the radical on the right and canceling the obvious terms which arise we find that

$$|l,m-1\rangle=\sqrt{\frac{[l-(m-1)]!}{(2l)!(l+m-1)!}}\,[\ell_-]^{l-(m-1)}\,|l,l\rangle \tag{5.185}$$

which is of the same form as the expression we are trying to prove with $m\rightarrow m-1$. Thus, if it is true for any m it is true for all values of m less than the original. We then note that the expression is trivially true for $m=l$, which completes the proof. Thus, the spherical harmonic of order (l,m) can be expressed in terms of the one of order (l,l) in the form

$$Y_l^m(\theta,\phi)=\sqrt{\frac{[l-(m-1)]!}{(2l)!(l+m-1)!}}e^{-i(l-m)\phi}\left[-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right]^{l-(m-1)}Y_l^l(\theta,\phi). \tag{5.186}$$

It is not our intention to provide here a complete derivation of the properties of the spherical harmonics, but rather to show how they fit into the general scheme we have developed regarding angular momentum eigenstates in general. To round things out a bit we mention without proof a number of their useful properties.

1. **Parity -** The parity operator Π acts on the eigenstates of the position representation and inverts them through the origin, i.e., $\Pi|\vec{r}\rangle=|-\vec{r}\rangle$. It is straightforward to show that in the position representation this takes the form $\Pi\psi(\vec{r})=\psi(-\vec{r})$. In spherical coordinates it is also easily verified that under the parity operation $r\rightarrow r$, $\theta\rightarrow\pi-\theta$, and $\phi\rightarrow\phi+\pi$. Thus, for functions on the unit sphere, $\Pi f(\theta,\phi)=f(\pi-\theta,\phi+\pi)$. The parity operator commutes with the components of $\vec{\ell}$ and with ℓ^2. Indeed, the states $|l,m\rangle$ are eigenstates of parity and satisfy the eigenvalue equation

$$\Pi|l,m\rangle=(-1)^l\,|l,m\rangle \tag{5.187}$$

which implies for the spherical harmonics that

$$\Pi Y_l^m(\theta,\phi)=Y_l^m(\pi-\theta,\phi+\pi)=(-1)^l\,Y_l^m(\theta,\phi). \tag{5.188}$$

2. **Complex Conjugation** - It is straightforward to show that

$$[Y_l^m(\theta,\phi)]^*=(-1)^mY_l^{-m}(\theta,\phi). \tag{5.189}$$

This allows spherical harmonics with $m<0$ to be obtained very simply from those with $m>0$.

3. **Relation to Legendre Functions** - The spherical harmonics with $m=0$ are directly related to the Legendre polynomials

$$P_l(u)=\frac{(-1)^l}{2^l l!}\frac{d^l}{du^l}\left(1-u^2\right)^l \tag{5.190}$$

through the relation

$$Y_l^0(\theta,\phi)=\sqrt{\frac{(2l+1)!}{4\pi}}P_l(\cos\theta). \tag{5.191}$$

The other spherical harmonics with $m>0$ are related to the associated Legendre functions

$$P_l^m(u)=\sqrt{(1-u^2)^m}\,\frac{d^mP_l(u)}{du^m}, \tag{5.192}$$

through the relation

$$Y_l^m(\theta,\phi)=(-1)^m\sqrt{\frac{(2l+1)!(l-m)!}{4\pi\,(l+m)!}}P_l^m(\cos\theta)e^{im\phi}. \tag{5.193}$$

5.9 Rotational Invariance

As we have seen, the behavior exhibited by a quantum system under rotations can usually be connected to a property related to the angular momentum of the system. In this chapter we consider in some detail the consequences of *rotational invariance*, that is, we explore just what is implied by the statement that a certain physical state or quantity is unchanged as a result of rotations imposed upon the system. To begin the discussion we introduce the important idea of rotationally invariant subspaces.

5.9.1 Irreducible Invariant Subspaces

In our discussion of commuting or compatible observables we encountered the idea of global invariance and saw, e.g., that the eigenspaces S_a of an observable A are globally invariant with respect to any operator B that commutes with A. It is useful to extend this idea to apply to more than one operator at a time. We therefore introduce the idea of invariant subspaces.

A subspace S' of the state space S is said to be **invariant** with respect to the action of a set $G = \{R_1, R_{2,} \cdots\}$ of operators if, for every $|\psi\rangle$ in S', the vectors $R_1|\psi\rangle, R_2|\psi\rangle, \cdots$ are all in S' as well. S' is than said to be an invariant subspace of the specified set of operators. The basic idea here is that the operators R_i all respect the boundaries of the subspace S', in the sense that they never take a state in S' onto a state outside of S'.

With this definition, we consider a quantum mechanical system with state space S, characterized by total angular momentum $\vec{J}$. It is always possible to express the state space S as a direct sum of orthogonal eigenspaces associated with any observable. (Recall that a space can be decomposed into a direct sum of two or more orthogonal subspaces if any vector in the space can be written as a linear combination of vectors from each subspace.) In the present context we consider the decomposition

$$S = S(j) \oplus S(j') \oplus S(j'') + \cdots \tag{5.194}$$

of our original space S into eigenspaces $S(j)$ of the operator J^2. Now each one of the spaces $S(j)$ associated with a particular eigenvalue $j(j+1)$ of J^2 can, itself, be decomposed into a direct sum

$$S(j) = S(\tau, j) \oplus S(\tau', j) \oplus S(\tau'', j) + \cdots \tag{5.195}$$

of $2j+1$ dimensional subspaces $S(\tau, j)$ associated with a standard representation for the space S, i.e., the vectors in $S(\tau, j)$ comprise all linear combinations

$$|\psi\rangle = \sum_{m=-j}^{j} \psi_m |\tau, j, m\rangle \tag{5.196}$$

of the $2j+1$ basis states $|\tau, j, m\rangle$ with fixed τ and j. We now show that each of these spaces $S(\tau, j)$ is invariant under the action of the operator components $\left\{J_u = \vec{J} \cdot \hat{u}\right\}$ of the total angular momentum. This basically follows from the the way that such a standard representation is constructed. We just need to show that the action of any component of $\vec{J}$ on such a vector takes it onto another vector in the same space, i.e., onto a linear combination of the same basis vectors. But the action of the operator $J_u = \vec{J}\cdot\hat{u} = \sum_i u_i J_i$ is completely determined by the action of the three Cartesian components of the vector operator $\vec{J}$. Clearly, however, the vector

$$J_z|\psi\rangle = \sum_{m=-j}^{j} m\psi_m |\tau, j, m\rangle \tag{5.197}$$

lies in the same subspace $S(\tau, j)$ as $|\psi\rangle$. Moreover, J_x and J_y are simple linear combinations of $J_{\pm}$, which take such a state onto

$$J_{\pm}|\psi\rangle = \sum_{m=-j}^{j} \sqrt{j(j+1) - m(m \pm 1)}\psi_m|\tau, j, m \pm 1\rangle \tag{5.198}$$

which is also in the subspace $S(\tau, j)$. Thus it is clear the the components of $\vec{J}$ cannot take an arbitrary state $|\psi\rangle$ in $S(\tau, j)$ outside the subspace. Hence $S(\tau, j)$ is an invariant subspace of the angular momentum operators. Indeed, it is precisely this invariance that leads to the fact the the matrices representing the components of $\vec{J}$ are *block-diagonal* in any standard representation.

It is not difficult to see that this invariance with respect to the action of the operator J_u extends to any operator function $F(J_u)$. In particular, it extends to the unitary rotation operators $U_{\hat{u}}(\alpha) = \exp(-i\alpha J_u)$ which are simple exponential functions of the components of $\vec{J}$. Thus, we conclude that the spaces $S(\tau, j)$ are also invariant subspaces of the *group* of operators $\{U_{\hat{u}}(\alpha)\}$. We say that each subspace $S(\tau, j)$ is an *invariant subspace of the rotation group*, or that $S(\tau, j)$ is rotationally invariant. We note that the eigenspace $S(j)$ is also an an invariant subspace of the rotation group, since any vector in it is a linear combination of vectors from the invariant subspaces $S(\tau, j)$ and so the action of $U_{\hat{u}}(\alpha)$ on an arbitrary vector in S_j is to take it onto another vector in $S(j)$, with the different parts in each subspace $S(\tau, j)$ staying within that respective subspace. It is clear, however, that although $S(j)$ is invariant with respect to the operators of the rotation group, it can often be reduced or decomposed into lower dimensional invariant parts $S(\tau, j)$. It is is reasonable in light of this decomposability exhibited by $S(j)$, to ask whether the rotationally invariant subspaces $S(\tau, j)$ of which $S(j)$ is formed are *similarly* decomposable. To answer this question we are led to the idea of irreducibility.

An invariant subspace S' of a group of operators $G = \{R_1, R_2, \cdots\}$ is said to be **irreducible** with respect to G (or is an irreducible invariant subspace of G) if, for every non-zero vector $|\psi\rangle$ in S', the vectors $\{R_i|\psi\rangle\}$ span S'. Conversely, S' is **reducible** if there exists a nonzero vector $|\psi\rangle$ in S' for which the vectors $\{R_i|\psi\rangle\}$ fail to span the space. Clearly, in the latter case the vectors spanned by the set $\{R_i|\psi\rangle\}$ form a subspace of S' that is itself invariant with respect to G.

We now answer the question we posed above, and show explicitly that any invariant subspace $S(\tau, j)$ associated with a standard representation for the state space S is, in fact, an *irreducible invariant subspace* of the rotation group $\{U_{\hat{u}}(\alpha)\}$, i.e., $S(\tau, j)$ cannot be decomposed into smaller invariant subspaces. To prove this requires several steps. To begin, we let

$$|\psi\rangle = \sum_{m=-j}^{j} \psi_m|\tau, j, m\rangle \tag{5.199}$$

again be an arbitrary (nonzero) vector in $S(\tau, j)$ and we formally denote by S_R the subspace of $S(\tau, j)$ spanned by the vectors $\{U_{\hat{u}}(\alpha)|\psi\rangle\}$, that is, S_R is the subspace of all vectors that can be written as a linear combination of vectors obtained through a rotation of the state $|\psi\rangle$. It is clear that S_R is contained in $S(\tau, j)$, since the latter is invariant under rotations; the vectors $\{U_{\hat{u}}(\alpha)|\psi\rangle\}$ must all lie inside $S(\tau, j)$. We wish to show that in fact $S_R = S(\tau, j)$. To do this we show that S_R contains a basis for $S(\tau, j)$ and hence the two spaces are equivalent.

To this end we note that the vectors $J_u|\psi\rangle$ are all contained in S_R. This follows from the form of infinitesimal rotations $U_{\hat{u}}(\delta\alpha) = 1 - i\delta\alpha J_u$, which imply that

$$J_u = \frac{1}{i\alpha}\left[1 - U_{\hat{u}}(\delta a)\right] = \frac{1}{i\alpha}\left[U_{\hat{u}}(0) - U_{\hat{u}}(\delta a)\right]. \tag{5.200}$$

Thus,

$$J_u|\psi\rangle = \frac{1}{i\alpha}\left[U_{\hat{u}}(0) - U_{\hat{u}}(\delta a)\right]|\psi\rangle = \frac{1}{i\alpha}\left[U_{\hat{u}}(0)|\psi\rangle - U_{\hat{u}}(\delta a)|\psi\rangle\right] \tag{5.201}$$

which is, indeed, a linear combination of the vectors $\{U_{\hat{u}}(\alpha)|\psi\rangle\}$, and thus is in the space S_R spanned by such vectors. By a straightforward extension, it follows, that the vectors

$$\begin{aligned} J_{\pm}|\psi\rangle &= (J_x \pm iJ_y)|\psi\rangle \\ &= \frac{1}{i\alpha}\left[U_x(0) - U_x(\delta a) \pm iU_y(0) \mp iU_y(\delta a)\right]|\psi\rangle \end{aligned} \tag{5.202}$$

are also in S_R. In fact, any vector of the form

$$(J_-)^q \, (J_+)^P \, |\psi\rangle \tag{5.203}$$

will be expressable as a linear combination of various products of $U_x(0)$, $U_y(0)$, $U_x(\delta\alpha)$, and $U_y(\delta\alpha)$ acting on $|\psi\rangle$. Since the product of any two such rotation operators is itself a rotation, the result will be a linear combination of the vectors $\{U_{\hat{u}}(\alpha)|\psi\rangle\}$, and so will also lie in S_R.

We now note, that since $|\psi\rangle$ is a linear combination of the vectors $|\tau, j, m\rangle$, each of which is raised or annihilated by the operator J_+, there exists an integer $P < 2j$ for which $(J_+)^P|\psi\rangle = \lambda|\tau, j, j\rangle$ (in other words, we keep raising the components of $|\psi\rangle$ up and annhilating them till the component that initially had the smallest value of J_z is the only one left.) It follows, therefore, that the basis state $|\tau, j, j\rangle$ lies in the subspace S_R. We are now home free, since we can now repeatedly apply the lowering operator, remaining within S_R with each application, to deduce that all of the basis vectors $|\tau, j, m\rangle$ lie in the subspace $S_{R.}$ Thus S_R is a subspace of $S(\tau, j)$ that contains a basis for $S(\tau, j)$. The only way this can happen is if $S_R = S(\tau, j)$.

It follows that the under the unitary transformation associated with a general rotation, the basis vectors $\{|\tau, j, m\rangle\}$ of $S(\tau, j)$ are transformed into new basis vectors for the *same* invariant subspace. Indeed, it is not hard to see, based upon our earlier description of the rotation process, that a rotation R that takes the unit vector $\hat{z}$ onto a new direction $\hat{z}'$ will take the basis kets $|\tau, j, m_z\rangle$, which are eigenstates of J^2 and J_z onto a new set of basis kets $|\tau, j, m_{z'}\rangle$ for the same space that are now eigenstates of J^2 and the component $J_{z'}$ of $\vec{J}$ along the new direction. These new vectors can, of course, be expressed as linear combinations of the original ones. The picture that emerges is that, under rotations, the vectors $|\tau, j, m\rangle$ transform (irreducibly) into linear combinations of themselves. This transformation is essentially geometric in nature and is analogous to the way that normal basis vectors in R^3 transform into linear combinations of one another. Indeed, by analogy, the coefficients of this linear transformation are identical in any subspace of a standard representation having the same value of j, since the basis vectors of such a representation have been constructed using the angular momentum operators in precisely the same fashion. This leads to the concept of **rotation matrices**, i.e., a set of standard matrices representing the rotation operators $U_{\hat{u}}(\alpha)$ in terms of their effect on the vectors within any irreducible invariant subspace $S(\tau, j)$. Just as with the matrices representing the components of $\vec{J}$ within any irreducible subspace $S(\tau, j)$, the elements of the rotation matrices will depend upon j and m but are indepndent of τ. Thus, e.g., a rotation U_R of a basis ket $|\tau, j, m\rangle$ of $S(\tau, j)$ results in a linear combination of such states of the form

$$U_R|\tau, j, m\rangle = \sum_{m'=-j}^{j} |\tau, j, m'\rangle\langle\tau, j, m'|U_R|\tau, j, m\rangle$$
$$= \sum_{m'=-j}^{j} |\tau, j, m'\rangle R^{(j)}_{m',m} \tag{5.204}$$

where, as the notation suggests, the elements

$$R^{(j)}_{m',m} = \langle\tau, j, m'|U_R|\tau, j, m\rangle = \langle j, m'|U_R|, j, m\rangle \tag{5.205}$$

of the $2j+1$ dimensional rotation matrix are independent of τ. In terms of the elements $R^{(j)}_{m',m}$ of the rotation matrices, the invariance of the subspaces $S(\tau, j)$ under rotations leads to the general relation

$$\langle\tau', j', m'|U_R|\tau, j, m\rangle = \delta_{j',j}\delta_{\tau',\tau}R^{(j)}_{m',m}. \tag{5.206}$$

These matrices are straightforward to compute for low dimensional subspaces, and general formulas have been developed for calculating the matrices for rotations associated with the Euler angles (α, β, γ). For rotations about the z axis the matrices take a particulalrly simple form, since the rotation operator in this case is a simple function of the operator J_z of which the states $|j, m\rangle$ are eigenstates. Thus, e.g.,

$$R^{(j)}_{m'm}(z, \alpha) = \langle j, m'|e^{-i\alpha J_z}|j, m\rangle = e^{-im\alpha}\delta_{m,m'} \tag{5.207}$$

In a subspace with $j = 1/2$, for example, this takes the form

$$\hat{R}^{(1/2)}(z, \alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{+i\alpha/2} \end{pmatrix}, \tag{5.208}$$

while in a space with $j = 1$ we have

$$\hat{R}^{(1)}(z, \alpha) = \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix}.$$

The point is that, once the rotation matrices have been worked out for a given value of j, they can be used for a standard representation of any quantum mechanical system. Thus, e.g., we can deduce a transformation law associated with rotations of the spherical harmonics, i.e.,

$$R\left[Y_l^m(\theta, \phi)\right] = \langle\theta, \phi|U_R|l, m\rangle$$
$$= \sum_{m'=-l}^{l} Y_l^{m'}(\theta, \phi)R^{(l)}_{m',m}. \tag{5.209}$$

For rotations about the z-axis, this takes the form

$$\begin{aligned} R_z(\alpha)\left[Y_l^m(\theta, \phi)\right] &= \langle\theta, \phi|e^{-i\alpha\ell_z}|l, m\rangle \\ &= e^{-im\alpha}Y_l^m(\theta, \phi) = e^{-im\alpha}F_l^m(\theta)e^{im\phi} \\ &= F_l^m(\theta)e^{im(\phi-\alpha)} \\ &= Y_l^m(\theta, \phi - \alpha) \end{aligned} \tag{5.210}$$

which is readily confirmed from simple geometric arguments.

5.9.2 Rotational Invariance of States

We now consider a physical state of the system that is invariant under rotations, i.e., that has the property that

$$U_R|\psi\rangle = |\psi\rangle \tag{5.211}$$

for all rotations U_R. This can be expressed in terms of the angular momentum of the system by noting that invariance under infinitesimal rotations

$$U_{\hat{u}}(\delta\alpha)|\psi\rangle = |\psi\rangle - i\delta\alpha J_u|\psi\rangle = |\psi\rangle,$$

requires that

$$J_u|\psi\rangle = 0 \tag{5.212}$$

for all $\hat{u}$, which also implies that $J^2|\psi\rangle = 0$. Thus, a state $|\psi\rangle$ is rotationally invariant if and only if it has zero angular momentum.

5.9.3 Rotational Invariance of Operators

If an observable Q is rotationally invariant it is, by our earlier definition, a scalar with respect to rotations, and we can deduce the following:

1. $[U_R, Q] = 0$
2. $[J_u, Q] = 0 = [J^2, Q]$
3. There exists an ONB of eigenstates states $\{|\tau, q, j, m\rangle\}$ common to J_z, J^2, and Q.
4. The eigenvalues of q of Q within any irreducible subspace $S(j)$ are (at least) $2j + 1$ fold degenerate.

This degeneracy, referred to as a rotational or essential degeneracy, is straightfoward to show. Suppose that $|q\rangle$ is an eigenstate of Q, so that $Q|q\rangle = q|q\rangle$. Then

$$QU_R|q\rangle = U_RQ|q\rangle = qU_R|q\rangle. \tag{5.213}$$

This shows that $U_R|q\rangle$ is an eigenstate of Q with the same eigenvalue. Of course not all the states $\{U_R|q\rangle\}$ are linearly independent. From this set of states we can form linear combinations which are also eigenstates of J^2 and J_z, and which can be partitioned into irreducible invariant subspaces of well defined j. The $2j + 1$ linearly independent basis states associated with each such irreducible subspace $S(q, j)$ are then $2j+1$ fold degenerate eigenstates of Q.

As an important special case, suppose that the Hamiltonian of a quantum system is a scalar with respect to rotations. We can then conclude that

1. H is rotationally invariant.
2. $[U_R, H] = [J_u, H] = [J^2, H] = 0$.
3. The components of $\vec{J}$ are constants of the motion, since
$$\frac{d}{dt}\langle J_u\rangle = \frac{i}{\hbar}\langle [H, J_u]\rangle = 0. \tag{5.214}$$
4. The equations of motion are invariant under rotations. Thus, if $|\psi(t)\rangle$ is a solution to
$$\left(i\hbar\frac{d}{dt} - H\right)|\psi(t)\rangle = 0 \tag{5.215}$$

then

$$U_R\left(i\hbar\frac{d}{dt}-H\right)|\psi(t)\rangle=\left(i\hbar\frac{d}{dt}-H\right)U_R|\psi(t)\rangle=0 \qquad (5.216)$$

which shows that the rotated state $U_R|\psi(t)\rangle$ is also a solution to the equations of motion.

5. There exists an ONB of eigenstates states $\{|E,\tau,j,m\rangle\}$ common to J_z, J^2, and H.

6. The eigenvalues E of H within any irreducible subspace are (at least) $2j+1$ fold degenerate. For the case of the Hamiltonian, these degeneracies are referred to as multiplets. Thus, a nondegenerate subspace associated with a state of zero angular momentum is referred to as a singlet, a doubly-degenerate state associated with a two-fold degenerate $j=1/2$ state is a doublet, and a three-fold degenerate state associated with angular momentum $j=1$ is referred to as a triplet.

5.10 Addition of Angular Momenta

Let S_1 and S_2 be two quantum mechanical state spaces associated with angular momentum $\vec{J}_1$ an $\vec{J}_2$, respectively. Let $\{|\tau_1,j_1,m_1\rangle\}$ denote the basis vectors of a standard representation for S_1, which is decomposable into corresponding irreducible subspaces $S_1(\tau_1,j_1)$, and denote by $\{|\tau_2,j_2,m_2\rangle\}$ the basis vectors of a standard representation for S_2, decomposable into irreducible subspaces $S_2(\tau_2,j_2)$.

The combined quantum system formed from S_1 and S_2 is an element of the direct product space

$$S=S_1\otimes S_2. \qquad (5.217)$$

The direct product states

$$|\tau_1,j_1,m_1;\tau_2,j_2,m_2\rangle=|\tau_1,j_1,m_1\rangle|\tau_2,j_2,m_2\rangle \qquad (5.218)$$

form an orthonormal basis for S. As we will see, however, these direct product states do not define a standard representation for S. Indeed, for this combined space the *total* angular momentum vector is the sum

$$\vec{J}=\vec{J}_1+\vec{J}_2 \qquad (5.219)$$

of those assocated with each "factor space". What this means is that the rotation operators for the combined space are just the products of the rotation operators for each individual space

$$\begin{aligned} U_{\hat{u}}(\alpha) &= U_{\hat{u}}^{(1)}(\alpha)U_{\hat{u}}^{(2)}(\alpha) \\ &= e^{-i\alpha\vec{J}_1\cdot\hat{u}}e^{-i\alpha\vec{J}_2\cdot\hat{u}}=e^{-i\alpha\left(\vec{J}_1+\vec{J}_2\right)\cdot\hat{u}} \qquad (5.220)\\ e^{-i\alpha\vec{J}\cdot\hat{u}} &= e^{-i\alpha\left(\vec{J}_1+\vec{J}_2\right)\cdot\hat{u}} \qquad (5.221)\end{aligned}$$

so that the generators of rotations for each factor space simply add. At this point, we make no specific identification of the nature of the two subspaces involved. Accordingly, the results that we will derive will apply equally well to the description of two spinless particles (for which $\vec{J}=\vec{L}=\vec{L}_1+\vec{L}_2$ is the total orbital angular momentum of the pair), to the description of a single particle with spin (for which $\vec{J}=\vec{L}+\vec{S}$ is the sum of the orbital and spin angular momenta of the particle), or even to a collection of particles (where $\vec{J}=\vec{L}+\vec{S}$ is again the sum of the orbital and spin angular momentum, but where now the latter represent the corresponding orbital and spin angular momenta $\vec{L}=\sum_\alpha\vec{L}_\alpha$ and $\vec{S}=\sum_\alpha\vec{S}_\alpha$ for the entire collection of particles).

In either of these cases the problem of interest is to construct a standard representation $|\tau, j, m\rangle$ of common eigenstates of J^2 and J_z associated with the total angular momentum vector $\vec{J}$ of the system as linear combinations of the direct product states $|\tau_1, j_1, m_1; \tau_2, j_2, m_2\rangle$. As it turns out, the latter are eigenstates of

$$\begin{aligned} J_z &= (\vec{J}_1 + \vec{J}_2) \cdot \hat{z} \\ &= J_{1z} + J_{2z} \end{aligned} \tag{5.222}$$

since they are individually eigenstates of J_{1z} and J_{2z}, i.e.,

$$\begin{aligned} J_z|\tau_1, j_1, m_1; \tau_2, j_2, m_2\rangle &= (J_{1z} + J_{2z})|\tau_1, j_1, m_1; \tau_2, j_2, m_2\rangle \\ &= (m_1 + m_2)|\tau_1, j_1, m_1; \tau_2, j_2, m_2\rangle \\ &= m|\tau_1, j_1, m_1; \tau_2, j_2, m_2\rangle \end{aligned} \tag{5.223}$$

where $m = m_1 + m_2$. The problem is that these direct product states are generally not eigenstates of

$$\begin{aligned} J^2 &= (\vec{J}_1 + \vec{J}_2) \cdot (\vec{J}_1 + \vec{J}_2) \\ &= J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 \end{aligned} \tag{5.224}$$

because, although they are eigenstates of J_1^2 and J_2^2, they are not eigenstates of

$$\vec{J}_1 \cdot \vec{J}_2 = \sum_i J_{1i} J_{2i} \tag{5.225}$$

due to the presence in this latter expression of operator components of $\vec{J}_1$ and $\vec{J}_2$ perpendicular to the z axis.

Using the language of invariant subspaces, another way of expressing the problem at hand is as follows: determine how the direct product space $S = S_1 \otimes S_2$ can be decomposed into its own irreducible invariant subspaces $S(\tau, j)$. This way of thinking about the problem actually leads to a simplification. We note that since S_1 and S_2 can each be written as a direct sum

$$S_1 = \sum_{\tau_1, j_1} S_1(\tau_1, j_1) \qquad\qquad S_2 = \sum_{\tau_2, j_2} S_2(\tau_2, j_2) \tag{5.226}$$

of rotationally invariant subspaces, the direct product of S_1 and S_2 can also be writtten as a direct sum

$$\begin{aligned} S &= S_1 \otimes S_2 \\ &= \sum_{\tau_1, j_1, \tau_2, j_2} S_1(\tau_1, j_1) \otimes S_2(\tau_2, j_2) \\ &= \sum_{\tau_1, j_1, \tau_2, j_2} S(\tau_1, \tau_2, j_1, j_2) \end{aligned} \tag{5.227}$$

of direct product subspaces $S(\tau_1, \tau_2, j_1, j_2) = S_1(\tau_1, j_1) \otimes S_2(\tau_2, j_2)$.

Now, because $S_1(\tau_1, j_1)$ and $S_2(\tau_2, j_2)$ are rotationally invariant, so is their direct product, i.e., any vector $|\tau_1 j_1 m_1; \tau_2 j_2 m_2\rangle$ in this space will be take by an arbitrary rotation onto the direct product of two other vectors, one from $S_1(\tau_1, j_1)$ and one from $S_2(\tau_2, j_2)$; it will remain inside $S(\tau_1, \tau_2, j_1, j_2)$. On the other hand, although $S(\tau_1, \tau_2, j_1, j_2)$ is rotationally invariant there is no reason to expect it that it is also irreducible. However, in decomposing S into irreducible invariant subspaces, we can use the fact that we already have a natural decomposition of that space into smaller invariant subspaces. To completely reduce the space S we just need to break these smaller parts into even smaller *irreducible* parts.

Within any such space $S(\tau_1, \tau_2, j_1, j_2)$ the values of $\tau_1, \tau_2, j_1,$ and j_2 are fixed. Thus we can simplify our notation in accord with the simpler problem at hand which can be stated thusly: find the irreducible invariant subspaces of a direct product space $S(\tau_1, \tau_2, j_1, j_2) = S_1(\tau_1, j_1) \otimes S_2(\tau_2, j_2)$ with fixed values of j_1 and j_2. While working in this subspace we suppress any constant labels, and so denote by

$$S(j_1, j_2) = S(\tau_1, \tau_2, j_1, j_2) = S_1(\tau_1, j_1) \otimes S_2(\tau_2, j_2) \tag{5.228}$$

the subspace of interest and by

$$|m_1, m_2\rangle = |\tau_1, j_1, m_1; \tau_2, j_2, m_2\rangle \tag{5.229}$$

the original direct product states within this subspace. These latter are eigenvectors of J_1^2 and J_2^2 with eigenvalues $j_1(j_1+1)$ and $j_2(j_2+1)$, respectively, and of J_{1z} and J_{2z} with eigenvalues m_1 and m_2. We will denote the sought-after common eigenstates of J^2 and J_z in this subspace by the vectors

$$|j, m\rangle = |\tau_1, j_1, \tau_2, j_2; j, m\rangle \tag{5.230}$$

which are to be formed as linear combinations of the states $|m_1, m_2\rangle$.

With this notation we now proceed to prove the main result, referred to as

> **The addition theorem**: The $(2j_1 + 1)(2j_2 + 1)$ dimensional space $S(j_1, j_2)$ contains exactly one irreducible subspace $S(j)$ for each value of j in the sequence
>
> $$j = j_1 + j_2, j_1 + j_2 - 1, \cdots, |j_1 - j_2|. \tag{5.231}$$
>
> In other words, the subspace $S(j_1, j_2)$ can be reduced into a direct sum
>
> $$S(j_1, j_2) = S(j_1 + j_2) \oplus S(j_1 + j_2 - 1) \oplus \cdots S(|j_1 - j_2|) \tag{5.232}$$
>
> of irreducible invariant subspaces of the rotation group, where each space $S(j)$ is spanned by $2j + 1$ basis vectors $|j, m\rangle$.

To prove this result we begin with a few general observations, and then follow up with what is essentially a proof-by-construction. First, we note that since the space $S(j_1, j_2)$ contains states $|m_1, m_2\rangle$ with

$$j_1 \geq m_1 \geq -j_1 \qquad \text{and} \qquad j_2 \geq m_2 \geq -j_2 \tag{5.233}$$

the corresponding eigenvalues $m = m_1 + m_2$ of J_z within this subspace can only take on values in the range

$$j_1 + j_2 \geq m \geq -(j_1 + j_2). \tag{5.234}$$

This implies that the eigenvalues of J^2 must, themselves be labeled by values of j satisfying the bound

$$j_1 + j_2 \geq j. \tag{5.235}$$

Moreover, it is not hard to see that the sum of m_1 and m_2 will result in integral values of m if m_1 and $m_2$are both integral or both half-integral and will result in half-integral values of m if one is integral and the other half-integral. Since the integral character of m_1 and m_2 is determined by the character of j_1 and j_2, we deduce that

$$j = \begin{cases} \text{integral if } j_1, j_2 \text{ are both integral or both half-integral} \\ \\ \text{half-integral otherwise.} \end{cases} \tag{5.236}$$

With these preliminary observations out of the way, we now proceed to observe that in the subspace $S(j_1, j_2)$ there is only one direct product state $|m_1, m_2\rangle$ in which the value of $m = m_1 + m_2$ takes on its largest value of $j_1 + j_2$, namely that vector in which m_1 and m_2 both individually take on their largest values, j_1 and j_2. This fact, we assert, implies that the vector, $|m_1, m_2\rangle = |j_1, j_2\rangle$ must be also be an eigenvector of J^2 and J_z with angular momentum $(j, m) = (j_1 + j_2, j_1 + j_2)$. In other words, it is that vector of an irreducible subspace $S(j)$ with $j = j_1 + j_2$ having the maximum possible component of angular momentum along the z axis consistent with that value of j. To prove this assertion, we note that if this were not the case, we could act on this vector with the raising operator

$$J_+ = J_{1+} + J_{2+} \tag{5.237}$$

and produce an eigenstate of J_z with eigenvalue $m = j_1 + j_2 + 1$. This state would have to be in $S(j_1, j_2)$ because the latter is invariant under the action of the components of $\vec{J}$. But no vector exists in this space with m larger than $j_1 + j_2$. Thus, when J_+ acts on $|j_1, j_2\rangle$ it must take it onto the null vector. The only states having this property are those of the form $|j, m\rangle$ with $j = m$, which proves the assertion.

Since there is only one such state in $S(j_1, j_2)$ with this value of m, moreover, there can be only one irreducible subspace $S(j)$ with $j = j_1 + j_2$ (in general there would be one such vector starting the sequence of basis vectors for each such subspace). Thus we identify

$$|j_1 + j_2, j_1 + j_2\rangle = |j_1, j_2\rangle,$$

where the left side of this expression indicates the $|j, m\rangle$ state, the right side indicates the original direct product state $|m_1, m_2\rangle$. The remaining basis states $|j, m\rangle$ in this irreducible space $S(j)$ with $j = j_1 + j_2$ can now, in principle, be produced by repeated application of the lowering operator

$$J_- = J_{1-} + J_{2-}. \tag{5.238}$$

For example, we note that, in the $|j, m\rangle$ representation,

$$\begin{aligned} J_-|j_1 + j_2, j_1 + j_2\rangle &= \\ & \sqrt{(j_1 + j_2)(j_1 + j_2 + 1) - (j_1 + j_2)(j_1 + j_2 - 1)}\; |j_1 + j_2, j_1 + j_2 - 1\rangle \\ &= \sqrt{2(j_1 + j_2)}\; |j_1 + j_2, j_1 + j_2 - 1\rangle. \end{aligned} \tag{5.239}$$

But $J_- = J_{1-} + J_{2-}$, so this same expression can be written in the $|m_1, m_2\rangle$ representation, after some manipulation, as

$$(J_{1-} + J_{2-})|j_1, j_2\rangle = \sqrt{2j_1}\; |j_1 - 1, j_2\rangle + \sqrt{2j_2}\; |j_1, j_2 - 1\rangle. \tag{5.240}$$

Equating these last two results then gives

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_2}{j_1 + j_2}}\; |j_1, j_2 - 1\rangle + \sqrt{\frac{j_1}{j_1 + j_2}}\; |j_1 - 1, j_2\rangle. \tag{5.241}$$

This procedure can obviously be repeated for the remaining basis vectors with this value of j.

We now proced to essentially repeat the argument, by noticing that there are exactly *two* direct product states $|m_1, m_2\rangle$ in which the value of $m = m_1 + m_2$ takes the next largest value possible, i.e., $m = j_1 + j_2 - 1$, namely, the states

$$\begin{aligned} |m_1, m_2\rangle &= |j_1, j_2 - 1\rangle \\ |m_1, m_2\rangle &= |j_1 - 1, j_2\rangle. \end{aligned} \tag{5.242}$$

From these two orthogonal states we can produce any eigenvectors of J_z in $S(j_1, j_2)$ having eigenvalue $m = j_1 + j_2 - 1$. In particular, we can form the state (5.241), which is an eigenvector of J^2 with $j = j_1 + j_2$. But we can also produce from these two direct product states a vector orthogonal to (5.241), e.g., the vector

$$\sqrt{\frac{j_2}{j_1 + j_2}}\,|j_1, j_2 - 1\rangle - \sqrt{\frac{j_1}{j_1 + j_2}}\,|j_1 - 1, j_2\rangle. \tag{5.243}$$

Analogous to our previous argument we argue that this latter state must be an eigenstate of J^2 and J_z with angular momentum $(j, m) = (j_1 + j_2 - 1, j_1 + j_2 - 1)$. In other words, it is that vector of an irreducible subspace $S(j)$ with $j = j_1 + j_2 - 1$ having the maximum possible component of angular momentum along the z axis consistent with that value of j. To prove this assertion, assume it were not the case. We could then act both on this vector and on (5.241) with the raising operator and produce in $S(j_1, j_2)$ two orthogonal eigenstates of J_z with eigenvalue $m = j_1 + j_2$ (since, as we have seen the raising and lowering operators preserve the orthogonality of such sequences). But there is only *one* such state with this value of m, and *it* is obtained by applying the raising operator to (5.241). Thus, application of J_+ to (5.243) must take it onto the null vector, and hence it must be a state of the type asserted. Since there are no other orthogonal states of this type that can be constructed, we deduce that there is exactly one irreducible invariant subspace $S(j)$ with $j = j_1 + j_2 - 1$, and we identify (5.243) with the state heading the sequence of basis vectors for that space. As before, the remaining basis vectors $|j, m\rangle$ for this value of j can then be generated by applying the lowering operator $J_- = J_{1-} + J_{2-}$ to (5.243).

The next steps, we hope, are clear: repeat this procedure until we run out of vectors. The basic idea is that as we move down through each value of j we always encounter just enough linearly-independent direct product states with a given value of m to form by linear combination the basis vectors associated with those irreducible spaces already generated with higher values of j, as well as one additional vector which is to be constructed orthogonal to those from the other irreducible spaces. This remaining state forms the beginning vector for a new sequence of $2j + 1$ basis vectors associated with the present value of j.

Thus, e.g., for small enough n, we find that there are exactly $n+1$ direct product states $|m_1, m_2\rangle$ in which the value of $m = m_1 + m_2$ takes the value $m = j_1 + j_2 - n$, namely, the states with

$$\begin{array}{ll} m_1 = & m_2 = \\ j_1 - n & j_2 \\ j_1 - n + 1 & j_2 - 1 \\ \vdots & \vdots \\ j_1 & j_2 - n \end{array}. \tag{5.244}$$

If, at this stage, there has been exactly one irreducible invariant subspace $S(j)$ for all values of j greater than $j_1 + j_2 - n$, than we can form from these $n + 1$ vectors those

n vectors having this m value that have already been obtained using J_- from spaces with higher values of j. We can then form exactly one additional vector orthogonal to these (e.g., by the Gram-Schmidt procedure) which cannot be associated with any of the subspaces $S(j)$ already constructed and so, by elimination, must be associated with the one existing subspace $S(j)$ having $j = j_1 + j_2 - n$. This state must, moreover, be that state of this space in which $m = j$, (if it were not we could use the raising operator to produce $n+1$ orthogonal states with m having the next higher value which exceeds the number of orthogonal states of this type). Thus, the remaining basis vectors of this space can be constructed using the lowering operator on this state. We have inductively shown, therefore, that if there is exactly one irreducible invariant subspace $S(j)$ for all values of j greater than $j_1 + j_2 - n$, then there is exactly one such subspace for $j = j_1 + j_2 - n$. This argument proceeds until we reach a value of n for which there are not $n+1$ basis vectors with the required value of m. From the table above, we see that this occurs when the values of m_1 or m_2 exceed there natural lower bounds of $-j_1$ and $-j_2$. Conversely, the induction proof holds for all values of n such that

$$j_1 - n \geq -j_1 \text{ and } j_2 - n \geq -j_2 \tag{5.245}$$

or

$$2j_1 \geq n \text{ and } 2j_2 \geq n \tag{5.246}$$

With $j = j_1 + j_2 - n$, this implies that

$$j \geq j_2 - j_1 \text{ and } j \geq j_1 - j_2 \tag{5.247}$$

or more simply

$$j \geq |j_1 - j_2|. \tag{5.248}$$

Thus, the arguments above prove the basic statement of the addition theorem, namely that there exists in $S(j_1, j_2)$ exactly one space $S(j)$ for values of j starting at $j_1 + j_2$ and stepping down one unit at a time to the value of $|j_1 - j_2|$. Moreover, the method of proof contains an outline of the basic procedure used to actually construct the subspaces of interest. Of course there remains the logical possibility that there exist other irreducible subspaces in $S(j_1, j_2)$ that are simply not accessible using the procedure outlined. It is straightforward to show that this is not the case, however, by simply counting the number of vectors produced through the procedure outlined. Indeed, for each value of $j = j_1 + j_2, \cdots, |j_1 - j_2|$ the procedure outlined above generates $2j+1$ basis vectors. The total number of such basis vectors is then represented by the readily computable sum

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1) \tag{5.249}$$

showing that they are sufficient in number to generate a space of dimension equal to the original.

Thus, the states $|j, m\rangle = |\tau_1, j_1, \tau_2, j_2; j, m\rangle$ formed in this way comprise an orthonormal basis for the subspace $S(j_1, j_2)$. Implicitly, therefore, there is within this subspace a unitary transformation between the original direct product states $|m_1, m_2\rangle = |\tau_1, j_1, \tau_2, j_2; m_1, m_2\rangle$ and the basis states $|j, m\rangle$ assocated with the total angular momentum $\vec{J}$. By an appropriate choice of phase, the expansion coefficients (i.e., the matrix elements of the unitary transformation between these two sets) can be chosen independent of τ and τ' and dependent only on the values $j, m, j_1, j_2, m_1,$ and m_2. Thus, e.g., the new basis states can be written as linear combinations of the direct product states in the usual way, i.e.,

$$|j,m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1,j_2,m_1,m_2\rangle\langle j_1,j_2,m_1,m_2|j,m\rangle \tag{5.250}$$

where we have included the quantum numbers j_1 and j_2 in the expansion to explicitly indicate the subspaces that are being combined. Similarly, the direct product states can be written as linear combinations of the new basis states $|j,m\rangle$, i.e.,

$$|j_1,j_2,m_1,m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j,m\rangle\langle j,m|j_1,j_2,m_1,m_2\rangle. \tag{5.251}$$

These expansions are completely determined once we know the corresponding expansion coefficients $\langle j_1,j_2,m_1,m_2|j,m\rangle$, which are referred to as Clebsch-Gordon coefficients (or CG coefficients). Different authors denote these expansion coefficients in different ways, e.g.,

$$C^{j_1,j_2,m_1,m_2}_{j,m} = \langle j,m|j_1,j_2,m_1,m_2\rangle. \tag{5.252}$$

It is straightforward, using the procedure outlined above, to generate the CG coefficients for given values of j_1, j_2, and j. They obey certain properties that follow from their definition and from the way in which they are constructed. We enumerate some of these properties below.

1. **Restrictions on j and m** - It is clear from the proof of the addition theorem detailed above that the CG coefficient $\langle j_1,j_2,m_1,m_2|j,m\rangle$ must vanish unless the two states in the innner product have the same z component of total angular momentum. In addition, we must have the value of total j on the right lie within the range produced by the angular momenta j_1 and j_2. Thus we have the restriction

$$\begin{aligned} \langle j_1,j_2,m_1,m_2|j,m\rangle &= 0 \qquad \text{unless} \qquad m = m_1+m_2 \\ \langle j_1,j_2,m_1,m_2|j,m\rangle &= 0 \qquad \text{unless} \qquad j_1+j_2 \geq j \geq |j_1-j_2|. \end{aligned} \tag{5.253}$$

   The restriction on j is referred to as the triangle inequality, since it is equivalent to the condition that the positive numbers j, j_1, and j_2 be able to represent the lengths of the three sides of some triangle. As such, it is easily shown to apply to any permutation of these three numbers, i.e., its validity also implies that $j_1 + j \geq j_2 \geq |j_1 - j|$ and that $j + j_2 \geq j_1 \geq |j - j_2|$.

2. **Phase convention** - From the process outlined above, the only ambiguity involved in constructing the states $|j,m\rangle$ from the direct product states $|m_1,m_2\rangle$ is at the point where we construct the maximally aligned vector $|j,j\rangle$ for each subspace $S(j)$. This vector can always be constructed orthogonal to the states with the same value of m associated with higher values of j, but the phase of the state so constructed can, in principle, take any value. To unambiguously specify the CG coefficients, therefore, this phase must be unambiguously specified. This is done by defining the phase of this state relative to a particular direct product state. In particular, we define the CG coefficients so that the coefficient

$$\langle j_1,j_2,j_1,j-j_1|j,j\rangle = \langle j,j|j_1,j_2,j_1,j-j_1\rangle \geq 0 \tag{5.254}$$

   is real and positive. Since the remaining states $|j,m\rangle$ are constructed from $|j,j\rangle$ using the lowering operator this choice makes all of the CG coefficients real

$$\langle j_1,j_2,m_1,m_2|j,m\rangle = \langle j,m|j_1,j_2,m_1,m_2\rangle, \tag{5.255}$$

although not necessarily positive (one can show, for example, that the sign of $\langle j_1, j_2, m_1, m_2 | j, j\rangle$ is $(-1)^{j_1 - m_1}$.)

3. **Orthogonality and completeness relations** - Being eigenstates of Hermitian operators the two sets of states $\{|j,m\rangle\}$ and $\{|m_1,m_2\rangle\}$ each form an ONB for the subspace $S(j_1, j_2)$. Orthonormality implies that

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 m_1' m_2'\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'} \tag{5.256}$$

$$\langle j,m|j',m'\rangle = \delta_{j,j'}\delta_{m,m'} \tag{5.257}$$

While completeness of each set within this subspace implies that

$$\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{m=-j}^{j} |j,m\rangle\langle j,m| = 1 \qquad \text{within } S(j_1,j_2) \tag{5.258}$$

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} |j_1,j_2,m_1,m_2\rangle\langle j_1,j_2,m_1,m_2| = 1 \qquad \text{within } S(j_1,j_2) \tag{5.259}$$

inserting the completeness relations into the orthonormality relations gives corresponding orthonormality conditions for the CG coefficients, i.e.,

$$\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{m=-j}^{j} \langle j_1 j_2 m_1 m_2|j,m\rangle\langle j,m|j_1 j_2 m_1' m_2'\rangle = \delta_{m_1,m_1'} \tag{5.260}$$

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} \langle j,m|j_1,j_2,m_1,m_2\rangle\langle j_1,j_2,m_1,m_2|j',m'\rangle = \delta_{j,j'}\delta_{m,m'}. \tag{5.261}$$

4. **Recursion relation** - The states $|j,m\rangle$ in each irreducible subspace $S(j)$ are formed from the state $|j,j\rangle$ by application of the lowering operator. It is possible, as a result, to use the lowering operator to obtain recursion relations for the Clebsch-Gordon coefficients associated with fixed values of j, j_1, and j_2. To develop these relations we consider the matrix element of $J_\pm$ between the states $|j,m\rangle$ and the states $|j_1,j_2,m_1,m_2\rangle$, i.e., we consider

$$\langle j,m|J_\pm|j_1,j_2,m_1,m_2\rangle = \langle j,m|(J_{1\pm}+J_{2\pm})|j_1,j_2,m_1,m_2\rangle \tag{5.262}$$

On the left hand side of this expression we let $J_\pm$ act on the bra $\langle j,m|$. Since this is the adjoint of $J_\mp|j,m\rangle$ the role of the raising and lowering operators is reversed, i.e.,

$$\langle j,m|J_\pm = \sqrt{j(j+1)-m(m\mp 1)}\ \langle j,m\mp 1|. \tag{5.263}$$

Substituting this in above and letting $J_{1\pm}$ and $J_{2\pm}$ act to the right we obtain the relations

$$\begin{aligned} &\sqrt{j(j+1)-m(m\mp 1)}\ \langle j,m\mp 1|j_1,j_2,m_1,m_2\rangle \\ =\ &\sqrt{j_1(j_1+1)-m_1(m_1\pm 1)}\ \langle j,m|j_1,j_2,m_1\pm 1,m_2\rangle \\ &+\sqrt{j_2(j_2+1)-m_2(m_2\pm 1)}\ \langle j,m|j_1,j_2,m_1,m_2\pm 1\rangle. \end{aligned} \tag{5.264}$$

These relations allow all CG coefficients for fixed j, j_1, and j_2, to be obtained from a single one, e.g., $\langle j_1,j_2,j_1,j-j_1|j,j\rangle$.

5. **Clebsch-Gordon series** - As a final property of the CG coefficients we derive a relation that follows from the fact that the space $S(j_1, j_2)$ is formed from the direct product of irreducible invariant subspaces $S_1(j_1)$ and $S_2(j_2)$. In particular, we know that in the space $S_1(j_1)$ the basis vectors $|j_1, m_1\rangle$ transform under rotations into linear combinations of the themselves according to the relation

$$U_R^{(1)}|j_1 m_1\rangle = \sum_{m_1'=-j_1}^{j_1} |j_1, m_1'\rangle R^{(j_1)}_{m_1',m_1}, \tag{5.265}$$

where $R^{(j_1)}_{m_1',m_1}$ is the rotation matrix associated with an irreducible invariant subspace with angular momentum j_1. Similarly, for the states $|j_2, m_2\rangle$ of $S_2(j_2)$ we have the relation

$$U_R^{(2)}|j_2 m_2\rangle = \sum_{m_2'=-j_2}^{j_2} |j_2, m_2'\rangle R^{(j_2)}_{m_2',m_2}. \tag{5.266}$$

It follows that in the combined space the direct product states $|j_1, j_2, m_1, m_2\rangle$ transform under rotations as follows

$$\begin{aligned} U_R|j_1, j_2, m_1, m_2\rangle &= U_R^{(1)}U_R^{(2)}|j_1, j_2, m_1, m_2\rangle \\ &= \sum_{m_1',m_2'} |j_1, j_2, m_1', m_2'\rangle R^{(j_1)}_{m_1',m_1} R^{(j_2)}_{m_2',m_2}. \end{aligned} \tag{5.267}$$

On the other hand, we can also express the states $|j_1, j_2, m_1, m_2\rangle$ in terms of the states $|j, m\rangle$, i.e.,

$$U_R|j_1, j_2, m_1, m_2\rangle = \sum_{j,m} U_R|j, m\rangle\langle j, m|j_1, j_2, m_1, m_2\rangle. \tag{5.268}$$

But the states $|j, m\rangle$ are the basis vectors of an irreducible invariant subspace $S(j)$ of the combined subspace, and so tranform accordingly,

$$U_R|j, m\rangle = \sum_{m'} |j, m'\rangle R^{(j)}_{m',m}. \tag{5.269}$$

Thus, we deduce that

$$U_R|j_1, j_2, m_1, m_2\rangle = \sum_{j,m,m'} |j, m'\rangle R^{(j)}_{m',m}\langle j, m|j_1, j_2, m_1, m_2\rangle \tag{5.270}$$

$$= \sum_{m_1',m_2'} \sum_{j,m,m'} |j_1, j_2, m_1', m_2'\rangle\langle j_1, j_2, m_1', m_2'|j, m'\rangle R^{(j)}_{m',m}\langle j, m|j_1, j_2, m_1, m_2\rangle \tag{5.271}$$

where in the second line we have transformed the states $|j, m'\rangle$ back to the direct product representation. Comparing coefficients in (5.267) and (5.271), we deduce a relation between matrix elements of the rotation matrices for different values of j, namely,

$$R^{(j_1)}_{m_1',m_1} R^{(j_2)}_{m_2',m_2} = \sum_{j,m,m'} \langle j_1, j_2, m_1', m_2'|j, m'\rangle R^{(j)}_{m',m}\langle j, m|j_1, j_2, m_1, m_2\rangle \tag{5.272}$$

which is referred to as the Clebsch-Gordon series. Note that in this last expression the CG coefficients actually allow the sum over m and m' to both collapse to a single term with $m = m_1 + m_2$ and $m' = m'_1 + m'_2$, making it equivalent to

$$R^{(j_1)}_{m'_1,m_1} R^{(j_2)}_{m'_2,m_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \langle j_1, j_2, m'_1, m'_2 | j, m'_1{+}m'_2 \rangle R^{(j)}_{m'_1+m'_2, m_1+m_2} \langle j, m_1{+}m_2 | j_1, j_2, m_1, m_2 \rangle \tag{5.273}$$

5.11 Reducible and Irreducible Tensor Operators

We have seen that it is possible to classify observables of a system in terms of their transformation properties. Thus, scalar observables are, by definition, invariant under rotations. The components of a vector observable, on the other hand, transform into well defined linear combinations of one another under an arbitrary rotation. As it turns out, these two examples constitute special cases of a more general classification scheme involving the concept of **tensor operators**. By definition, a collection of n operators $\{Q_1, Q_2, \cdots, Q_n\}$ comprise an n-component rotational tensor operator $\mathbf{Q}$ if they transform under rotations into linear combinations of each other, i.e., if for each rotation R there are a set of coefficients $D_{ji}(R)$ such that

$$R\,[Q_i] = U_R Q_i U_r^+ = \sum_j Q_j D_{ji}(R). \tag{5.274}$$

It is straightforward to show that under these circumstances the matrices $D(R)$ with matrix elements $D_{ji}(R)$ form a representation for the rotation group.

As an example, we note that the Cartesian components $\{V_x, V_y, V_z\}$ of a vector operator $\vec{V}$ form the components of a 3-component tensor $\mathbf{V}$. Indeed, under an arbitrary rotation, the operator $V_u = \vec{V} \cdot \hat{u}$ is transformed into

$$R\,[V_u] = V_{u'} = \vec{V} \cdot \hat{u}' \tag{5.275}$$

where $\hat{u}' = A_R \hat{u}$ indicates the direction obtained by performing the rotation R on the vector $\hat{u}$. Thus,

$$u'_i = \sum_j A_{ij} u_j. \tag{5.276}$$

If $\hat{u}$ corresponds to the Cartesian unit vector $\hat{x}_k$, then $u_j = \delta_{j,k}$ and $u'_i = A_{ik}$, so that

$$R\,[V_k] = \vec{V} \cdot \hat{x}'_k = \sum_i V_i A_{ik} \tag{5.277}$$

which shows that the components of $\vec{V}$ transform into linear combinations of one another under rotations, with coefficients given by the 3×3 rotation matrices A_R.

As a second example, if $\vec{V}$ and $\vec{W}$ are vector operators and Q is a scalar operator, then the operators $\{Q, V_x, V_y, V_z, W_x, W_y, W_z\}$ form a seven component tensor, since under an arbitrary rotation R they are taken onto

$$Q \quad \to \quad Q + \sum_j 0 \cdot V_j + \sum_j 0 \cdot W_j \tag{5.278}$$

$$V_i \quad \to \quad \sum_j V_j A_{ji} + \sum_j 0 \cdot W_j + 0 \cdot Q \tag{5.279}$$

$$W_i \quad \to \quad \sum_j W_j A_{ji} + \sum_j 0 \cdot V_j + 0 \cdot Q \tag{5.280}$$

which satisfies the definition. Clearly in this case, however, the set of seven components can be partitioned into 3 separate sets of operators $\{Q\}$, $\{V_x, V_y, V_z\}$, and $\{W_x, W_y, W_z\}$, which independently transform into linear combinations of one another, i.e., the tensor can be reduced into two 3-component tensors and one tensor having just one component (obviously any scalar operator constitutes a one-component tensor). Note also that in this circumstance the matrices $D(R)$ governing the transformation of these operators is block diagonal, i.e., it has the form

$$D(R) = \begin{pmatrix} 1 & & \\ & \begin{pmatrix} & A_R & \end{pmatrix} & \\ & & \begin{pmatrix} & A_R & \end{pmatrix} \end{pmatrix}. \tag{5.281}$$

Thus this representation of the rotation group is really a combination of three separate representations, one of which is 1 dimensional and two of which are 3 dimensional.

This leads us to the idea of **irreducible tensors**. A tensor operator $\mathbf{T}$ is said to be reducible if its components $\{T_1, T_2, \cdots, T_n\}$, or any set of linear combinations thereof, can be partitioned into tensors having a smaller number of components. If a tensor cannot be so partitioned it is said to be irreducible.

For example, if $\vec{V}$ is a vector operator, the set of operators $\{V_x, V_y\}$ do *not* form a two component tensor, because a rotation of V_x about the y axis through $\pi/2$ takes it onto V_z, which cannot be expressed as a linear combination of V_x and V_y. Thus, a vector operator *cannot* be reduced into smaller subsets of operators. It follows that *all vector operators are irreducible.*

It is interesting to note that the language that we are using here to describe the components of tensor operators is clearly very similar to that describing the behavior of the basis vectors associated with rotationally invariant subspaces of a quantum mechanical Hilbert space. Indeed, in a certain sense the basic reducibility problem has alread been solved in the context of combining angular momenta. We are led quite naturally, therefore, to introduce a very useful class of irreducible tensors referred to as **spherical tensors**.

By definition, a collection of $2j+1$ operators $\{T_j^m | m = -j, \cdots, j\}$ form the components of an irreducible spherical tensor $\mathbf{T}_j$ of rank j if they transform under rotations into linear combinations of one another in the same way as the basis vectors $|j, m\rangle$ of an irreducible invariant subspace $S(j)$. Specifically, this means that under a rotation R, the operator T_j^m is taken onto

$$R\left[T_j^m\right] = U_R T_j^m U_R^+ = \sum_{m'} T_j^{m'} R_{m',m}^{(j)} \tag{5.282}$$

where $R_{m',m}^{(j)}$ represents the rotation matrix associated with an eigenspace of J^2 with this value of j. Since the basis vectors $|j, m\rangle$ transform irreducibly, it is not hard to see that the components of spherical tensors of this sort do so as well.

It is not hard to see that, according to this definition, a scalar observable $Q = Q_0^0$ is an irreducible spherical tensor of rank zero, i.e., its transformation law

$$R\left[Q_0^0\right] = Q_0^0 \tag{5.283}$$

is the same as that of the single basis vector $|0, 0\rangle$ associated with a subspace of zero angular momentum, for which

$$R\left[|0, 0\rangle\right] = |0, 0\rangle. \tag{5.284}$$

Similary, a vector operator $\vec{V}$, which defines an irreducible three component tensor, can be viewed as a spherical tensor $\mathbf{V}_1$ of rank one. By definition, the spherical tensor components $\{V_1^m | m = 1, 0, -1\}$ of a vector operator $\vec{V}$ are given as the following linear combinations

$$V_1^1 = -\frac{(V_x + iV_y)}{\sqrt{2}} \qquad V_1^0 = V_z \qquad V_1^{-1} = \frac{V_x - iV_y}{\sqrt{2}} \tag{5.285}$$

of its Cartesian components. In this representation we see that the raising and lowering operators can be expressed in terms of the spherical tensor components of the angular momentum operator $\vec{J}$ through the relation

$$J_\pm = \mp\sqrt{2}J_1^{\pm 1}. \tag{5.286}$$

To see that the spherical components of a vector define an irreducible tensor of unit rank we must show that they transform appropriately. To this end it suffices to demonstrate the transformation properties for any vector operator, since the transformation law will clearly be the same for all vector operators (as it is for the Cartesian components). Consider, then, in the space of a single particle the vector operator $\hat{R}$ which has the effect in the position representation of multiplying the wavefunction at $\vec{r}$ by the radial unit vector $\hat{r}$, i.e.,

$$\hat{R}|\vec{r}\rangle = \hat{r}|\vec{r}\rangle = \frac{\vec{r}}{|\vec{r}|}|\vec{r}\rangle \tag{5.287}$$

$$\langle\vec{r}|\hat{R}|\psi\rangle = \hat{r}\psi(\vec{r}) = \frac{\vec{r}\psi(\vec{r})}{|\vec{r}|}. \tag{5.288}$$

In the position representation the Cartesian components of this operator can be written in spherical coordinates (r, θ, ϕ) in the usual way, i.e.,

$$\hat{R}_x = \frac{x}{r} = \cos\phi\sin\theta \tag{5.289}$$

$$\hat{R}_y = \frac{y}{r} = \sin\phi\sin\theta \tag{5.290}$$

$$\hat{R}_z = \frac{z}{r} = \cos\theta \tag{5.291}$$

In this same representation the spherical components of this vector operator take the form

$$\hat{R}_1^1 = -\frac{1}{\sqrt{2}}(\cos\phi\sin\theta + i\sin\phi\sin\theta) = -\frac{1}{\sqrt{2}}\,e^{i\phi}\sin\theta \tag{5.292}$$

$$\hat{R}_1^0 = \cos\theta \tag{5.293}$$

$$\hat{R}_1^{-1} = \frac{1}{\sqrt{2}}(\cos\phi\sin\theta - i\sin\phi\sin\theta) = \frac{1}{\sqrt{2}}\,e^{-i\phi}\sin\theta \tag{5.294}$$

Aside from an overall constant, these are equivalent to the spherical harmonics of order one, i.e.,

$$\hat{R}_1^1 = \sqrt{\frac{4\pi}{3}}Y_1^1(\theta, \phi) \tag{5.295}$$

$$\hat{R}_1^0 = \sqrt{\frac{4\pi}{3}}Y_1^0(\theta, \phi) \tag{5.296}$$

$$\hat{R}_1^{-1} = \sqrt{\frac{4\pi}{3}}Y_1^{-1}(\theta, \phi) \tag{5.297}$$

which, as we have seen, transform as the basis vectors of an irreducible subspace with $j = 1$

$$R\left[Y_1^m\right] = \sum_{m'=-1}^{1} Y_1^{m'} R_{m',m}^{(1)} \tag{5.298}$$

from which it follows that the spherical components of $\hat{R}$ transform in the same way, i.e.,

$$R\left[\hat{R}_1^m\right] = \sum_{m'=-1}^{1} \hat{R}_1^{m'} R_{m',m}^{(1)} \tag{5.299}$$

Thus, this vector operator (and hence all vector operators) define a spherical tensor of rank one.

As a natural extension of this, it is not hard to see that the spherical harmonics of a given order l define the components of a tensor operator in (e.g.) the position representation. Thus, we can define an irreducible tensor operator $\mathbf{Y}_l$ with $2l+1$ components $\{Y_l^m|m = -l, \cdots, l\}$ which have the following effect in the position representation

$$\langle \vec{r}|Y_l^m|\psi\rangle = Y_l^m(\theta,\phi)\psi(r,\theta,\phi). \tag{5.300}$$

We note in passing that the components of this tensor operator arise quite naturally in the multipole expansion of electrostatic and magnetostatic fields.

The product of two tensors is, itself, generally a tensor. For example, if $\mathbf{T}_{j_1}$ and $\mathbf{Q}_{j_2}$ represent spherical tensors of rank j_1 and j_2, respectively, then the set of products $\left\{T_{j_1}^{m_1} Q_{j_2}^{m_2}\right\}$ form a $(2j_1+1)(2j_2+1)$ component tensor. In general, however, such a tensor is reducible into tensors of smaller rank. Indeed, since the components $T_{j_1}^{m_1}$, $Q_{j_2}^{m_2}$ of each tensor transform as the basis vectors $|j_1, m_1\rangle, |j_2, m_2\rangle$ of an irreducible subspace $S(j_1), S(j_2)$ the process of reducing the product of two spherical tensors into irreducible components is essentially identical to the process of reducing a direct product space $S(j_1, j_2)$ into its irreducible components. Specifically, from the components $\left\{T_{j_1}^{m_1} Q_{j_2}^{m_2}\right\}$ we can form, for each $j = j_1 + j_2, \cdots, |j_1 - j_2|$ an irreducible tensor $\mathbf{W}_j$ with components

$$W_j^m = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} T_{j_1}^{m_1} Q_{j_2}^{m_2} \langle j_1, j_2, m_1, m_2|j, m\rangle \qquad m = -j, \cdots, j. \tag{5.301}$$

To show that these $2j+1$ components comprise a spherical tensor of rank j we must show that they satisfy the approriate transformation law. Consider

$$\begin{aligned} R\left[W_j^m\right] &= U_R W_j^m U_R^+ = \sum_{m_1,m_2} U_R T_{j_1}^{m_1} Q_{j_2}^{m_2} U_R^+ \, \langle j_1, j_2, m_1, m_2|j, m\rangle \\ &= \sum_{m_1,m_2} U_R T_{j_1}^{m_1} U_R^+ U_R Q_{j_2}^{m_2} U_R^+ \, \langle j_1, j_2, m_1, m_2|j, m\rangle \\ &= \sum_{m_1,m_2} \sum_{m_1',m_2'} T_{j_1}^{m_1'} Q_{j_2}^{m_2'} \, R_{m_1',m_1}^{(j_1)} R_{m_2',m_2}^{(j_2)} \, \langle j_1, j_2, m_1, m_2|j, m\rangle \end{aligned} \tag{5.302}$$

Using the Clebsch-Gordon series this last expression can be written

$$\begin{aligned}
R\left[W_j^m\right] &= \sum_{m_1,m_2}\sum_{m_1',m_2'}\sum_{j',m',m''} T_{j_1}^{m_1'} Q_{j_2}^{m_2'}\ \langle j_1,j_2,m_1',m_2'|j',m'\rangle R^{(j')}_{m',m''} \\
&\quad \times\langle j',m''|j_1 j_2,m_1,m_2\rangle\langle j_1,j_2,m_1,m_2|j,m\rangle \\
&= \sum_{m_1',m_2'}\sum_{j,m',m''} T_{j_1}^{m_1'} Q_{j_2}^{m_2'}\ \langle j_1,j_2,m_1',m_2'|j',m'\rangle R^{(j)}_{m',m''}\langle j',m''|j,m\rangle \\
&= \sum_{m'}\sum_{m_1',m_2'} T_{j_1}^{m_1'} Q_{j_2}^{m_2'}\ \langle j_1,j_2,m_1',m_2'|j,m'\rangle R^{(j)}_{m',m} \\
&= \sum_{m'} W_j^{m'} R^{(j)}_{m',m} \qquad (5.303)
\end{aligned}$$

which shows that W_j^m does indeed transform as the mth component of a tensor of rank j, and where we have used the orthonormality and completeness relations associated with the C-G coefficients.

5.12 Tensor Commutation Relations

Just as scalar and vector observables obey characteristic commutation relations

$$[J_i,Q]=0 \qquad (5.304)$$

$$[J_i,V_j]=i\sum_{ijk}\varepsilon_{ijk}V_k \qquad (5.305)$$

with the components of angular momentum, so do the components of a general spherical tensor operator. As with scalars and vectors, these commutation relations follow from the way that these operators transform under infinitesimal rotations. Recall that under an infinitesimal rotation $U_{\hat{u}}(\delta\alpha)=1-i\delta\alpha J_u$, an arbitrary operator Q is transformed into

$$Q'=Q-i\delta\alpha\,[J_u,Q] \qquad (5.306)$$

Thus, the mth component of the spherical tensor $\mathbf{T}_j$ is transformed by $U_{\hat{u}}(\delta\alpha)$ into

$$R_{\hat{u}}(\delta\alpha)[T_j^m]=T_j^m-i\delta\alpha\left[J_u,T_j^m\right] \qquad (5.307)$$

On the other hand, by definition, under any rotation T_j^m is transformed into

$$R_{\hat{u}}(\delta\alpha)[T_j^m]=\sum_{m'=-j}^{j} T_j^{m'} R^{(j)}_{m',m}(\hat{u},\delta\alpha). \qquad (5.308)$$

But

$$\begin{aligned}
R^{(j)}_{m',m}(\hat{u},\delta\alpha) &= \langle j,m'|U_{\hat{u}}(\delta\alpha)|j,m\rangle \\
&= \langle j,m'|1-i\delta\alpha J_u|j,m\rangle \\
&= \delta_{m,m'}-i\delta\alpha\langle j,m'|J_u|j,m\rangle \qquad (5.309)
\end{aligned}$$

which implies that

$$R_{\hat{u}}(\delta\alpha)[T_j^m]=T_j^m-i\delta\alpha\sum_{m'=-j}^{j} T_j^{m'}\langle j,m'|J_u|j,m\rangle. \qquad (5.310)$$

Comparing these we deduce the commutation relations

$$\left[J_u, T_j^m\right] = \sum_{m'=-j}^{j} T_j^{m'} \langle j, m'|J_u|j, m\rangle. \tag{5.311}$$

As special cases of this we have

$$\left[J_z, T_j^m\right] = mT_j^m.$$

and

$$\left[J_\pm, T_j^m\right] = \sqrt{j(j+1) - m(m\pm 1)}\, T_j^{m\pm 1}. \tag{5.312}$$

5.13 The Wigner Eckart Theorem

We now prove and important result regarding the matrix elements of tensor operators between basis states $|\tau, j, m\rangle$ of any standard representation. This result, known as the Wigner-Eckart theorem, illustrates, in a certain sense, the constraints put on the components of tensor operators by the transformation laws that they satisfy. Specifically, we will show that the matrix elements of the components of a spherical tensor operator $\mathbf{T}_J$ between basis states $|\tau, j, m\rangle$ are given by the product

$$\langle \tau, j, m|T_J^M|\tau', j', m'\rangle = \langle j, m|J, j', M, m'\rangle\langle \tau, j||T||\tau' j'\rangle \tag{5.313}$$

of the Clebsch Gordon coefficient $\langle j, m|J, j', M, m'\rangle$ and a quantity $\langle \tau, j||T||\tau' j'\rangle$ that is independent of m, M, and m', referred to as the reduced matrix element. Thus, the "orientational" dependence of the matrix element is completely determined by geometrical considerations. This result is not entirely surprising, given that the two quantities on the right hand side of the matrix element, i.e., the $T_J^M|\tau', j', m'\rangle$ transform under rotations like a direct product ket of the form $|J, M\rangle \otimes |j', m'\rangle$, while the quantity on the left transforms as a bra of total angular momentum (j, m). The reduced matrix element $\langle \tau, j||T||\tau' j'\rangle$ characterizes the extent to which the given tensor operator $\mathbf{T}_J$ mixes the two subspaces $S(\tau, j)$ and $S(\tau', j')$, and is generally different for each tensor operator.

To prove the Wigner-Eckart theorem we will simply show that the matrix elements of interest obey the same recursion relations as the CG coefficients. To this end, we use the simplifying notation

$$C_{j,m}^{j_1,j_2,m_1,m_2} = \langle j, m|j_2, j_1, m_1, m_2\rangle \tag{5.314}$$

for the CG coefficients and denote the matrix elements of interest in a similar fashion, i.e.,

$$T_{j,m}^{j_1,j_2,m_1,m_2} = \langle \tau, j, m|T_{j_1}^{m_2}|\tau', j_2, m_2\rangle. \tag{5.315}$$

This latter quantity is implicitly a function of the labels τ, τ' , but we will suppress this dependence until it is needed. We then recall that the CG coefficients obey a recursion relation that is generated by consideration of the matrix elements

$$\langle j, m|J_\pm|j_2, j_1, m_1, m_2\rangle = \langle j, m|(J_{1\pm} + J_{2\pm})|j_2, j_1, m_1, m_2\rangle \tag{5.316}$$

which leads to the relation

$$\begin{aligned}\sqrt{j(j+1) - m(m\mp 1)}C_{j,m\mp 1}^{j_1,j_2,m_1,m_2} &= \sqrt{j_1(j_1+1) - m_1(m_1\pm 1)}\, C_{j,m}^{j_1,j_2,m_1\pm 1,m_2} \\ &\quad + \sqrt{j_2(j_2+1) - m_2(m_2\pm 1)}C_{j,m}^{j_1,j_2,m_1,m_2\pm 1}\end{aligned} \tag{5.317}$$

To obtain a similar relation for the matrix elements of T_{j_1} we consider an "analogous" matrix element

$$\begin{aligned} \langle \tau, j, m | J_\pm T_{j_1}^{m_1} | \tau', j_2, m_2 \rangle &= \sqrt{j(j+1) - m(m \mp 1)}\, \langle \tau, j, m \mp 1 | T_{j_1}^{m_1} | \tau', j_2, m_2 \rangle \\ &= \sqrt{j(j+1) - m(m \mp 1)}\, T_{j,m\mp 1}^{j_1,j_2,m_1,m_2}. \end{aligned} \tag{5.318}$$

We can evaluate this in a second way by using the commutation relations satisfied by $J_\pm$ and $T_{j_1}^{m_1}$, i.e., we can write

$$\begin{aligned} J_\pm T_{j_1}^{m_1} &= \left[J_\pm, T_{j_1}^{m_1} \right] + T_{j_1}^{m_1} J_\pm \\ &= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\, T_{j_1}^{m_1 \pm 1} + T_{j_1}^{m_1} J_\pm \end{aligned} \tag{5.319}$$

which allows us to express the matrix element above in the form

$$\begin{aligned} \langle \tau, j, m | J_\pm T_{j_1}^{m_1} | \tau', j_2, m_2 \rangle &= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\, \langle \tau, j, m | T_{j_1}^{m_1 \pm 1} | \tau', j_2, m_2 \rangle \\ &\quad + \langle \tau, j, m | T_{j_1}^{m_1} J_\pm | \tau', j_2, m_2 \rangle \end{aligned} \tag{5.320}$$

which reduces to

$$\begin{aligned} \langle \tau, j, m | J_\pm T_{j_1}^{m_1} | \tau', j_2, m_2 \rangle &= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\, T_{j,m}^{j_1,j_2,m_1\pm 1,m_2} \\ &\quad + \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}\, T_{j,m}^{j_1,j_2,m_1,m_2\pm 1} \end{aligned} \tag{5.321}$$

Comparing the two expressions for $\langle \tau, j, m | J_\pm T_{j_1}^{m_1} | \tau', j_2, m_2 \rangle$ we deduce the recursion relation

$$\begin{aligned} \sqrt{j(j+1) - m(m \mp 1)}\, T_{j,m\mp 1}^{j_1,j_2,m_1,m_2} &= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\, T_{j,m}^{j_1,j_2,m_1\pm 1,m_2} \\ &\quad + \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}\, T_{j,m}^{j_1,j_2,m_1,m_2\pm 1} \end{aligned} \tag{5.322}$$

which is precisely the same as that obeyed by the Clebsch-Gordon coefficients $C_{j,m}^{j_1,j_2,m_1,m_2}$. The two sets of number, for given values of j, j_1, and j_2, must be proportional to one another. Introducing the reduced matrix element $\langle \tau, j || T || \tau' j' \rangle$ as the constant of proportionality, we deduce that

$$T_{j,m}^{j_1,j_2,m_1,m_2} = \langle \tau, j || T || \tau' j' \rangle C_{j,m}^{j_1,j_2,m_1,m_2} \tag{5.323}$$

which becomes after a little rearrangement

$$\langle \tau, j, m | T_J^M | \tau', j', m' \rangle = \langle j, m | J, j', M, m' \rangle \langle \tau, j || T || \tau' j' \rangle. \tag{5.324}$$

This theorem is very useful because it leads automatically to certain selection rules. Indeed, because of the CG coefficient on the right hand side we see that the matrix element of T_J^M between two states of this type vanishes unless

$$\Delta m = m - m' = M \tag{5.325}$$

$$j' + J \geq j \geq |j' - J|. \tag{5.326}$$

Thus, for example we see that the matrix elements of a scalar operator Q_0^0 vanish unless $\Delta m = 0$ and $\Delta j = j - j' = 0$. Thus, scalar operators cannot change the angular momentum of any states that they act upon. (They are often said to carry no angular momentum, in contrast to tensor operators of higher rank, which can and do change the angular momentum of the states that they act upon.) Thus the matrix elements for scalar operators take the form

$$\langle \tau, j, m | Q_0^0 | \tau', j', m' \rangle = Q_{\tau,\tau'} \delta_{j,j'} \delta_{m,m'}. \tag{5.327}$$

In particular, it follows that *within* any irreducible subspace $S(\tau, j)$ the matrix representing any scalar Q_0^0 is just a constant Q_τ times the identity matrix for that space (confirming the rotational invariance of scalar observables within any such subspace), i.e.,

$$\langle \tau, j, m | Q_0^0 | \tau, j, m' \rangle = Q_\tau \delta_{m,m'}. \tag{5.328}$$

Application of the Wigner-Eckart theorem to vector operators $\vec{V}$, leads to consideration of the spherical components $\{V_1^m | m = 0, \pm 1\}$ of such an operator. The corresponding matrix elements satisfy the relation

$$\langle \tau, j, m | V_1^M | \tau', j', m' \rangle = \langle j, m | 1, j', M, m' \rangle \langle \tau, j || V || \tau' j' \rangle, \tag{5.329}$$

and vanish unless

$$\Delta m = M \in \{0, \pm 1\}. \tag{5.330}$$

Similarly, application of the triangle inequality to vector operators leads to the selection rule

$$\Delta j = 0, \pm 1. \tag{5.331}$$

Thus, vector operators act as though they impart or take away angular momentum $j = 1$.

The matrix elements of a vector operator within any given irreducible space are proportional to those of any other vector operator, such as the angular momentum operator $\vec{J}$, whose spherical components satisfy

$$\langle \tau, j, m | J_1^M | \tau', j', m' \rangle = \langle j, m | 1, j', M, m' \rangle \langle \tau, j || J || \tau' j' \rangle, \tag{5.332}$$

from which it follows that

$$\langle \tau, j, m | V_1^M | \tau, j, m' \rangle = \alpha(\tau, j) \langle \tau, j, m | J_1^M | \tau, j, m' \rangle \tag{5.333}$$

where $\alpha(\tau, j) = \langle \tau, j || V || \tau, j \rangle / \langle \tau, j || J || \tau, j \rangle$ is a constant. Thus, within any subspace $S_\tau(j)$ all vector operators are proportional, we can write

$$\vec{V} = \alpha \vec{J} \qquad \text{within } S_\tau(j). \tag{5.334}$$

It is a straight forward exercise to compute the constant of proportionality in terms of the scalar observable $\vec{J} \cdot \vec{V}$, the result being what is referred to as the projection theorem, i.e.,

$$\vec{V} = \frac{\langle \vec{J} \cdot \vec{V} \rangle}{j(j+1)} \, \vec{J} \qquad \text{within } S_\tau(j), \tag{5.335}$$

where the mean value $\langle \vec{J} \cdot \vec{V} \rangle$, being a scalar with respect to rotation can be taken with respect to any state in the subspace $S_\tau(j)$.

In a similar manner one finds that the nonzero matrix elements *within* any irreducible subspace are proportional, i.e., for two nonzero tensor operators T_J and W_J of the same rank, it follows that, provided $\langle \tau, j || W || \tau j \rangle \neq 0$,

$$\langle \tau, j, m | T_J^M | \tau', j', m' \rangle = \alpha \langle \tau, j, m | W_J^M | \tau', j', m' \rangle \tag{5.336}$$

where $\alpha = \langle \tau, j || T || \tau' j' \rangle / \langle \tau, j || W || \tau', j' \rangle$. Thus, the orientational dependence of the $(2j + 1)(2J + 1)(2j' + 1)$ matrix elements is completely determined by the transformational properties of the states and the tensors involved.