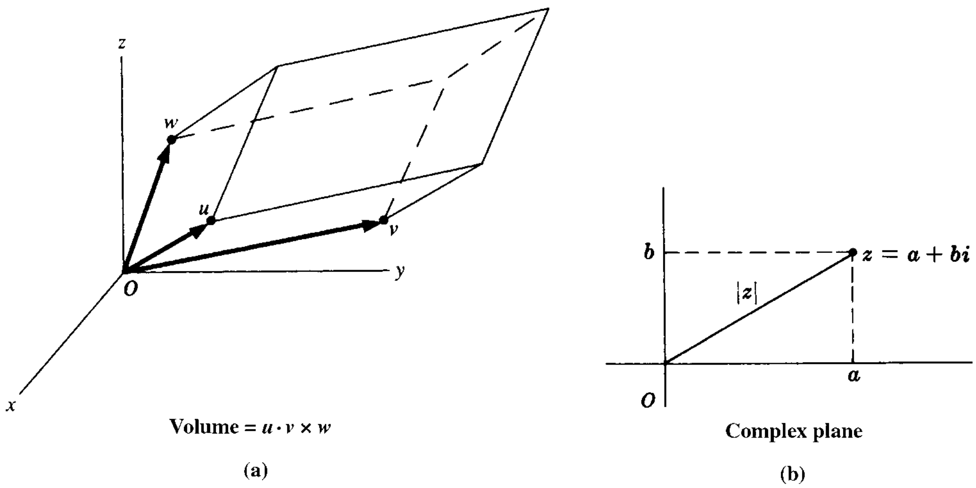

Figure 1-4

THEOREM 1.5: Let u, v, w be vectors in $\mathbf{R}^3$.

(a) The vector $u \times v$ is orthogonal to both u and v.

(b) The absolute value of the "triple product"

$$u \cdot v \times w$$

represents the volume of the parallelopiped formed by the vectors u, v, w. [See Fig. 1-4(a).]

We note that the vectors $u, v, u \times v$ form a right-handed system, and that the following formula gives the magnitude of $u \times v$:

$$\|u \times v\| = \|u\| \|v\| \sin\theta$$

where θ is the angle between u and v.

1.7 Complex Numbers

The set of complex numbers is denoted by $\mathbf{C}$. Formally, a complex number is an ordered pair (a, b) of real numbers where equality, addition, and multiplication are defined as follows:

$$(a, b) = (c, d) \quad \text{if and only if } a = c \text{ and } b = d$$
$$(a, b) + (c, d) = (a + c,\ b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd,\ ad + bc)$$

We identify the real number a with the complex number $(a, 0)$; that is,

$$a \leftrightarrow (a, 0)$$

This is possible because the operations of addition and multiplication of real numbers are preserved under the correspondence; that is,

$$(a, 0) + (b, 0) = (a + b,\ 0) \qquad \text{and} \qquad (a, 0) \cdot (b, 0) = (ab, 0)$$

Thus we view $\mathbf{R}$ as a subset of $\mathbf{C}$, and replace $(a, 0)$ by a whenever convenient and possible.

We note that the set $\mathbf{C}$ of complex numbers with the above operations of addition and multiplication is a *field* of numbers, like the set $\mathbf{R}$ of real numbers and the set $\mathbf{Q}$ of *rational numbers*.

The complex number $(0, 1)$ is denoted by i. It has the important property that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1 \qquad \text{or} \qquad i = \sqrt{-1}$$

Accordingly, any complex number $z = (a, b)$ can be written in the form

$$z = (a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0) \cdot (0, 1) = a + bi$$

The above notation $z = a + bi$, where $a \equiv \text{Re}\, z$ and $b \equiv \text{Im}\, z$ are called, respectively, the *real* and *imaginary parts* of z, is more convenient than (a, b). In fact, the sum and product of complex numbers $z = a + bi$ and $w = c + di$ can be derived by simply using the commutative and distributive laws and $i^2 = -1$:

$$z + w = (a + bi) + (c + di) = a + c + bi + di = (a + b) + (c + d)i$$
$$zw = (a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

We also define the *negative* of z and subtraction in **C** by

$$-z = -1z \qquad \text{and} \qquad w - z = w + (-z)$$

Warning: The letter i representing $\sqrt{-1}$ has no relationship whatsoever to the vector $\mathbf{i} = [1, 0, 0]$ in Section 1.6.

Complex Conjugate, Absolute Value

Consider a complex number $z = a + bi$. The *conjugate* of z is denoted and defined by

$$\bar{z} = \overline{a + bi} = a - bi$$

Then $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$. Note that z is real if and only if $\bar{z} = z$.

The *absolute value* of z, denoted by $|z|$, is defined to be the nonnegative square root of $z\bar{z}$. Namely,

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

Note that $|z|$ is equal to the norm of the vector (a, b) in $\mathbf{R}^2$.

Suppose $z \neq 0$. Then the inverse z^{-1} of z and division in **C** of w by z are given, respectively, by

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \qquad \text{and} \qquad \frac{w}{z} - \frac{w\bar{z}}{z\bar{z}} = wz^{-1}$$

EXAMPLE 1.10 Suppose $z = 2 + 3i$ and $w = 5 - 2i$. Then

$$z + w = (2 + 3i) + (5 - 2i) = 2 + 5 + 3i - 2i = 7 + i$$
$$zw = (2 + 3i)(5 - 2i) = 10 + 15i - 4i - 6i^2 = 16 + 11i$$
$$\bar{z} = \overline{2 + 3i} = 2 - 3i \qquad \text{and} \qquad \bar{w} = \overline{5 - 2i} = 5 + 2i$$
$$\frac{w}{z} = \frac{5 - 2i}{2 + 3i} = \frac{(5 - 2i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{4 - 19i}{13} = \frac{4}{13} - \frac{19}{13}i$$
$$|z| = \sqrt{4 + 9} = \sqrt{13} \qquad \text{and} \qquad |w| = \sqrt{25 + 4} = \sqrt{29}$$

Complex Plane

Recall that the real numbers **R** can be represented by points on a line. Analogously, the complex numbers **C** can be represented by points in the plane. Specifically, we let the point (a, b) in the plane represent the complex number $a + bi$ as shown in Fig. 1-4(b). In such a case, $|z|$ is the distance from the origin O to the point z. The plane with this representation is called the *complex plane*, just like the line representing **R** is called the *real line*.

1.8 Vectors in C^n

The set of all n-tuples of complex numbers, denoted by $\mathbf{C}^n$, is called *complex* n-*space*. Just as in the real case, the elements of $\mathbf{C}^n$ are called *points* or *vectors*, the elements of $\mathbf{C}$ are called *scalars*, and vector addition in $\mathbf{C}^n$ and scalar multiplication on $\mathbf{C}^n$ are given by

$$[z_1, z_2, \ldots, z_n] + [w_1, w_2, \ldots, w_n] = [z_1 + w_1,\ z_2 + w_2,\ \ldots,\ z_n + w_n]$$
$$z[z_1, z_2, \ldots, z_n] = [zz_1, zz_2, \ldots, zz_n]$$

where the z_i, w_i, and z belong to $\mathbf{C}$.

EXAMPLE 1.11 Consider vectors $u = [2 + 3i,\ 4 - i,\ 3]$ and $v = [3 - 2i,\ 5i,\ 4 - 6i]$ in $\mathbf{C}^3$. Then

$$u + v = [2 + 3i,\ 4 - i,\ 3] + [3 - 2i,\ 5i,\ 4 - 6i] = [5 + i,\ 4 + 4i,\ 7 - 6i]$$
$$(5 - 2i)u = [(5 - 2i)(2 + 3i),\ (5 - 2i)(4 - i),\ (5 - 2i)(3)] = [16 + 11i,\ 18 - 13i,\ 15 - 6i]$$

Dot (Inner) Product in C^n

Consider vectors $u = [z_1, z_2, \ldots, z_n]$ and $v = [w_1, w_2, \ldots, w_n]$ in $\mathbf{C}^n$. The *dot* or *inner product* of u and v is denoted and defined by

$$u \cdot v = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

This definition reduces to the real case because $\bar{w}_i = w_i$ when w_i is real. The norm of u is defined by

$$\|u\| = \sqrt{u \cdot u} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |v_n|^2}$$

We emphasize that $u \cdot u$ and so $\|u\|$ are real and positive when $u \neq 0$ and 0 when $u = 0$.

EXAMPLE 1.12 Consider vectors $u = [2 + 3i,\ 4 - i,\ 3 + 5i]$ and $v = [3 - 4i,\ 5i,\ 4 - 2i]$ in $\mathbf{C}_3$. Then

$$\begin{aligned} u \cdot v &= (2 + 3i)(\overline{3 - 4i}) + (4 - i)(\overline{5i}) + (3 + 5i)(\overline{4 - 2i}) \\ &= (2 + 3i)(3 + 4i) + (4 - i)(-5i) + (3 + 5i)(4 + 2i) \\ &= (-6 + 13i) + (-5 - 20i) + (2 + 26i) = -9 + 19i \\ u \cdot u &= |2 + 3i|^2 + |4 - i|^2 + |3 + 5i|^2 = 4 + 9 + 16 + 1 + 9 + 25 = 64 \\ \|u\| &= \sqrt{64} = 8 \end{aligned}$$

The space $\mathbf{C}^n$ with the above operations of vector addition, scalar multiplication, and dot product, is called *complex Euclidean* n-*space*. Theorem 1.2 for $\mathbf{R}^n$ also holds for $\mathbf{C}^n$ if we replace $u \cdot v = v \cdot u$ by

$$u \cdot v = \overline{u \cdot v}$$

On the other hand, the Schwarz inequality (Theorem 1.3) and Minkowski's inequality (Theorem 1.4) are true for $\mathbf{C}^n$ with no changes.

SOLVED PROBLEMS

Vectors in R^n

1.1. Determine which of the following vectors are equal:

$$u_1 = (1, 2, 3), \qquad u_2 = (2, 3, 1), \qquad u_3 = (1, 3, 2), \qquad u_4 = (2, 3, 1)$$

Vectors are equal only when corresponding entries are equal; hence, only $u_2 = u_4$.

1.2. Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, $w = (0, 5, -8)$. Find:

(a) $3u - 4v$,

(b) $2u + 3v - 5w$.

First perform the scalar multiplication and then the vector addition.

(a) $3u - 4v = 3(2, -7, 1) - 4(-3, 0, 4) = (6, -21, 3) + (12, 0, -16) = (18, -21, -13)$

(b) $2u + 3v - 5w = (4, -14, 2) + (-9, 0, 12) + (0, -25, 40) = (-5, -39, 54)$

1.3. Let $u = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Find:

(a) $5u - 2v$,

(b) $-2u + 4v - 3w$.

First perform the scalar multiplication and then the vector addition:

(a) $5u - 2v = 5\begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \\ -20 \end{bmatrix} + \begin{bmatrix} 2 \\ -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 27 \\ 5 \\ -24 \end{bmatrix}$

(b) $-2u + 4v - 3w = \begin{bmatrix} -10 \\ -6 \\ 8 \end{bmatrix} + \begin{bmatrix} -4 \\ 20 \\ 8 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -23 \\ 17 \\ 22 \end{bmatrix}$

1.4. Find x and y, where: (a) $(x, 3) = (2, x + y)$, (b) $(4, y) = x(2, 3)$.

(a) Because the vectors are equal, set the corresponding entries equal to each other, yielding

$$x = 2, \qquad 3 = x + y$$

Solve the linear equations, obtaining $x = 2$, $y = 1$.

(b) First multiply by the scalar x to obtain $(4, y) = (2x, 3x)$. Then set corresponding entries equal to each other to obtain

$$4 = 2x, \qquad y = 3x$$

Solve the equations to yield $x = 2$, $y = 6$.

1.5. Write the vector $v = (1, -2, 5)$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1)$.

We want to express v in the form $v = xu_1 + yu_2 + zu_3$ with x, y, z as yet unknown. First we have

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x + y + 2z \\ x + 2y - z \\ x + 3y + z \end{bmatrix}$$

(It is more convenient to write vectors as columns than as rows when forming linear combinations.) Set corresponding entries equal to each other to obtain

$$\begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 2y - z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + 2z &= 1 \\ y - 3z &= -3 \\ 5z &= 10 \end{aligned}$$

This unique solution of the triangular system is $x = -6, y = 3, z = 2$. Thus, $v = -6u_1 + 3u_2 + 2u_3$.

1.6. Write $v = (2, -5, 3)$ as a linear combination of

$$u_1 = (1, -3, 2), u_2 = (2, -4, -1), u_3 = (1, -5, 7).$$

Find the equivalent system of linear equations and then solve. First,

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ -3x - 4y - 5z \\ 2x - y + 7z \end{bmatrix}$$

Set the corresponding entries equal to each other to obtain

$$\begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ -5y + 5z &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ 0 &= 3 \end{aligned}$$

The third equation, $0x + 0y + 0z = 3$, indicates that the system has no solution. Thus, v cannot be written as a linear combination of the vectors u_1, u_2, u_3.

Dot (Inner) Product, Orthogonality, Norm in R^n

1.7. Find $u \cdot v$ where:

(a) $u = (2, -5, 6)$ and $v = (8, 2, -3)$,

(b) $u = (4, 2, -3, 5, -1)$ and $v = (2, 6, -1, -4, 8)$.

Multiply the corresponding components and add:

(a) $u \cdot v = 2(8) - 5(2) + 6(-3) = 16 - 10 - 18 = -12$

(b) $u \cdot v = 8 + 12 + 3 - 20 - 8 = -5$

1.8. Let $u = (5, 4, 1)$, $v = (3, -4, 1)$, $w = (1, -2, 3)$. Which pair of vectors, if any, are perpendicular (orthogonal)?

Find the dot product of each pair of vectors:

$$u \cdot v = 15 - 16 + 1 = 0, \qquad v \cdot w = 3 + 8 + 3 = 14, \qquad u \cdot w = 5 - 8 + 3 = 0$$

Thus, u and v are orthogonal, u and w are orthogonal, but v and w are not.

1.9. Find k so that u and v are orthogonal, where:

(a) $u = (1, k, -3)$ and $v = (2, -5, 4)$,

(b) $u = (2, 3k, -4, 1, 5)$ and $v = (6, -1, 3, 7, 2k)$.

Compute $u \cdot v$, set $u \cdot v$ equal to 0, and then solve for k:

(a) $u \cdot v = 1(2) + k(-5) - 3(4) = -5k - 10$. Then $-5k - 10 = 0$, or $k = -2$.

(b) $u \cdot v = 12 - 3k - 12 + 7 + 10k = 7k + 7$. Then $7k + 7 = 0$, or $k = -1$.

1.10. Find $\|u\|$, where: (a) $u = (3, -12, -4)$, (b) $u = (2, -3, 8, -7)$.

First find $\|u\|^2 = u \cdot u$ by squaring the entries and adding. Then $\|u\| = \sqrt{\|u\|^2}$.

(a) $\|u\|^2 = (3)^2 + (-12)^2 + (-4)^2 = 9 + 144 + 16 = 169$. Then $\|u\| = \sqrt{169} = 13$.

(b) $\|u\|^2 = 4 + 9 + 64 + 49 = 126$. Then $\|u\| = \sqrt{126}$.