CHAPTER 9

Diagonalization: Eigenvalues and Eigenvectors

9.1 Introduction

The ideas in this chapter can be discussed from two points of view.

Matrix Point of View

Suppose an n-square matrix A is given. The matrix A is said to be *diagonalizable* if there exists a nonsingular matrix P such that

$$B = P^{-1}AP$$

is diagonal. This chapter discusses the diagonalization of a matrix A. In particular, an algorithm is given to find the matrix P when it exists.

Linear Operator Point of View

Suppose a linear operator $T: V \to V$ is given. The linear operator T is said to be *diagonalizable* if there exists a basis S of V such that the matrix representation of T relative to the basis S is a diagonal matrix D. This chapter discusses conditions under which the linear operator T is diagonalizable.

Equivalence of the Two Points of View

The above two concepts are essentially the same. Specifically, a square matrix A may be viewed as a linear operator F defined by

$$F(X) = AX$$

where X is a column vector, and $B = P^{-1}AP$ represents F relative to a new coordinate system (basis) S whose elements are the columns of P. On the other hand, any linear operator T can be represented by a matrix A relative to one basis and, when a second basis is chosen, T is represented by the matrix

$$B = P^{-1}AP$$

where P is the change-of-basis matrix.

Most theorems will be stated in two ways: one in terms of matrices A and again in terms of linear mappings T.

Role of Underlying Field K

The underlying number field K did not play any special role in our previous discussions on vector spaces and linear mappings. However, the diagonalization of a matrix A or a linear operator T will depend on the roots of a polynomial $\Delta(t)$ over K, and these roots do depend on K. For example, suppose $\Delta(t) = t^2 + 1$. Then $\Delta(t)$ has no roots if $K = \mathbf{R}$, the real field; but $\Delta(t)$ has roots $\pm i$ if $K = \mathbf{C}$, the complex field. Furthermore, finding the roots of a polynomial with degree greater than two is a subject unto itself (frequently discussed in numerical analysis courses). Accordingly, our examples will usually lead to those polynomials $\Delta(t)$ whose roots can be easily determined.

9.2 Polynomials of Matrices

Consider a polynomial $f(t) = a_n t^n + \cdots + a_1 t + a_0$ over a field K. Recall (Section 2.8) that if A is any square matrix, then we define

$$f(A) = a_n A^n + \cdots + a_1 A + a_0 I$$

where I is the identity matrix. In particular, we say that A is a *root* of $f(t)$ if $f(A) = 0$, the zero matrix.

EXAMPLE 9.1 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$. Let

$$f(t) = 2t^2 - 3t + 5 \qquad \text{and} \qquad g(t) = t^2 - 5t - 2$$

Then

$$f(A) = 2A^2 - 3A + 5I = \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 14 \\ 21 & 37 \end{bmatrix}$$

and

$$g(A) = A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} -5 & -10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, A is a zero of $g(t)$.

The following theorem (proved in Problem 9.7) applies.

THEOREM 9.1: Let f and g be polynomials. For any square matrix A and scalar k,

(i) $(f + g)(A) = f(A) + g(A)$ (iii) $(kf)(A) = kf(A)$
(ii) $(fg)(A) = f(A)g(A)$ (iv) $f(A)g(A) = g(A)f(A)$.

Observe that (iv) tells us that any two polynomials in A commute.

Matrices and Linear Operators

Now suppose that $T: V \to V$ is a linear operator on a vector space V. Powers of T are defined by the composition operation:

$$T^2 = T \circ T, \qquad T^3 = T^2 \circ T, \qquad \ldots$$

Also, for any polynomial $f(t) = a_n t^n + \cdots + a_1 t + a_0$, we define $f(T)$ in the same way as we did for matrices:

$$f(T) = a_n T^n + \cdots + a_1 T + a_0 I$$

where I is now the identity mapping. We also say that T is a *zero* or *root* of $f(t)$ if $f(T) = 0$, the zero mapping. We note that the relations in Theorem 9.1 hold for linear operators as they do for matrices.

Remark: Suppose A is a matrix representation of a linear operator T. Then $f(A)$ is the matrix representation of $f(T)$, and, in particular, $f(T) = 0$ if and only if $f(A) = 0$.

9.3 Characteristic Polynomial, Cayley–Hamilton Theorem

Let $A = [a_{ij}]$ be an n-square matrix. The matrix $M = A - tI_n$, where I_n is the n-square identity matrix and t is an indeterminate, may be obtained by subtracting t down the diagonal of A. The negative of M is the matrix $tI_n - A$, and its determinant

$$\Delta(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n)$$

which is a polynomial in t of degree n and is called the *characteristic polynomial* of A.

We state an important theorem in linear algebra (proved in Problem 9.8).

THEOREM 9.2: (Cayley–Hamilton) Every matrix A is a root of its characteristic polynomial.

Remark: Suppose $A = [a_{ij}]$ is a triangular matrix. Then $tI - A$ is a triangular matrix with diagonal entries $t - a_{ii}$; hence,

$$\Delta(t) = \det(tI - A) = (t - a_{11})(t - a_{22}) \cdots (t - a_{nn})$$

Observe that the roots of $\Delta(t)$ are the diagonal elements of A.

EXAMPLE 9.2 Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. Its characteristic polynomial is

$$\Delta(t) = |tI - A| = \begin{vmatrix} t-1 & -3 \\ -4 & t-5 \end{vmatrix} = (t-1)(t-5) - 12 = t^2 - 6t - 7$$

As expected from the Cayley–Hamilton theorem, A is a root of $\Delta(t)$; that is,

$$\Delta(A) = A^2 - 6A - 7I = \begin{bmatrix} 13 & 18 \\ 24 & 37 \end{bmatrix} + \begin{bmatrix} -6 & -18 \\ -24 & -30 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now suppose A and B are similar matrices, say $B = P^{-1}AP$, where P is invertible. We show that A and B have the same characteristic polynomial. Using $tI = P^{-1}tIP$, we have

$$\begin{aligned} \Delta_B(t) &= \det(tI - B) = \det(tI - P^{-1}AP) = \det(P^{-1}tIP - P^{-1}AP) \\ &= \det[P^{-1}(tI - A)P] = \det(P^{-1}) \det(tI - A) \det(P) \end{aligned}$$

Using the fact that determinants are scalars and commute and that $\det(P^{-1}) \det(P) = 1$, we finally obtain

$$\Delta_B(t) = \det(tI - A) = \Delta_A(t)$$

Thus, we have proved the following theorem.

THEOREM 9.3: Similar matrices have the same characteristic polynomial.

Characteristic Polynomials of Degrees 2 and 3

There are simple formulas for the characteristic polynomials of matrices of orders 2 and 3.

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then

$$\Delta(t) = t^2 - (a_{11} + a_{22})t + \det(A) = t^2 - \operatorname{tr}(A)\, t + \det(A)$$

Here $\operatorname{tr}(A)$ denotes the trace of A—that is, the sum of the diagonal elements of A.

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$\Delta(t) = t^3 - \operatorname{tr}(A)\, t^2 + (A_{11} + A_{22} + A_{33})t - \det(A)$$

(Here A_{11}, A_{22}, A_{33} denote, respectively, the cofactors of a_{11}, a_{22}, a_{33}.)

EXAMPLE 9.3 Find the characteristic polynomial of each of the following matrices:

(a) $A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$, (b) $B = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix}$, (c) $C = \begin{bmatrix} 5 & -2 \\ 4 & -4 \end{bmatrix}$.

(a) We have $\text{tr}(A) = 5 + 10 = 15$ and $|A| = 50 - 6 = 44$; hence, $\Delta(t) + t^2 - 15t + 44$.

(b) We have $\text{tr}(B) = 7 + 2 = 9$ and $|B| = 14 + 6 = 20$; hence, $\Delta(t) = t^2 - 9t + 20$.

(c) We have $\text{tr}(C) = 5 - 4 = 1$ and $|C| = -20 + 8 = -12$; hence, $\Delta(t) = t^2 - t - 12$.

EXAMPLE 9.4 Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 3 & 9 \end{bmatrix}$.

We have $\text{tr}(A) = 1 + 3 + 9 = 13$. The cofactors of the diagonal elements are as follows:

$$A_{11} = \begin{vmatrix} 3 & 2 \\ 3 & 9 \end{vmatrix} = 21, \qquad A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 9 \end{vmatrix} = 7, \qquad A_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$$

Thus, $A_{11} + A_{22} + A_{33} = 31$. Also, $|A| = 27 + 2 + 0 - 6 - 6 - 0 = 17$. Accordingly,

$$\Delta(t) = t^3 - 13t^2 + 31t - 17$$

Remark: The coefficients of the characteristic polynomial $\Delta(t)$ of the 3-square matrix A are, with alternating signs, as follows:

$$S_1 = \text{tr}(A), \qquad S_2 = A_{11} + A_{22} + A_{33}, \qquad S_3 = \det(A)$$

We note that each S_k is the sum of all principal minors of A of order k.

The next theorem, whose proof lies beyond the scope of this text, tells us that this result is true in general.

THEOREM 9.4: Let A be an n-square matrix. Then its characteristic polynomial is

$$\Delta(t) = t^n - S_1 t^{n-1} + S_2 t^{n-2} + \cdots + (-1)^n S_n$$

where S_k is the sum of the principal minors of order k.

Characteristic Polynomial of a Linear Operator

Now suppose $T: V \to V$ is a linear operator on a vector space V of finite dimension. We define the *characteristic polynomial* $\Delta(t)$ of T to be the characteristic polynomial of any matrix representation of T. Recall that if A and B are matrix representations of T, then $B = P^{-1}AP$, where P is a change-of-basis matrix. Thus, A and B are similar, and by Theorem 9.3, A and B have the same characteristic polynomial. Accordingly, the characteristic polynomial of T is independent of the particular basis in which the matrix representation of T is computed.

Because $f(T) = 0$ if and only if $f(A) = 0$, where $f(t)$ is any polynomial and A is any matrix representation of T, we have the following analogous theorem for linear operators.

THEOREM 9.2′: (Cayley–Hamilton) A linear operator T is a zero of its characteristic polynomial.

9.4 Diagonalization, Eigenvalues and Eigenvectors

Let A be any n-square matrix. Then A can be represented by (or is similar to) a diagonal matrix $D = \text{diag}(k_1, k_2, \ldots, k_n)$ if and only if there exists a basis S consisting of (column) vectors $u_1, u_2, \ldots, u_n$ such that

$$\begin{aligned} Au_1 &= k_1u_1 \\ Au_2 &= \quad k_2u_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Au_n &= \qquad\quad k_nu_n \end{aligned}$$

In such a case, A is said to be *diagonizable*. Furthermore, $D = P^{-1}AP$, where P is the nonsingular matrix whose columns are, respectively, the basis vectors $u_1, u_2, \ldots, u_n$.

The above observation leads us to the following definition.

DEFINITION: Let A be any square matrix. A scalar λ is called an *eigenvalue* of A if there exists a nonzero (column) vector v such that

$$Av = \lambda v$$

Any vector satisfying this relation is called an *eigenvector* of A *belonging* to the eigenvalue λ.

We note that each scalar multiple kv of an eigenvector v belonging to λ is also such an eigenvector, because

$$A(kv) = k(Av) = k(\lambda v) = \lambda(kv)$$

The set E_λ of all such eigenvectors is a subspace of V (Problem 9.19), called the *eigenspace* of λ. (If $\dim E_\lambda = 1$, then E_λ is called an *eigenline* and λ is called a *scaling factor*.)

The terms *characteristic value* and *characteristic vector* (or *proper value* and *proper vector*) are sometimes used instead of eigenvalue and eigenvector.

The above observation and definitions give us the following theorem.

THEOREM 9.5: An n-square matrix A is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors. In this case, the diagonal elements of D are the corresponding eigenvalues and $D = P^{-1}AP$, where P is the matrix whose columns are the eigenvectors.

Suppose a matrix A can be diagonalized as above, say $P^{-1}AP = D$, where D is diagonal. Then A has the extremely useful *diagonal factorization*:

$$A = PDP^{-1}$$

Using this factorization, the algebra of A reduces to the algebra of the diagonal matrix D, which can be easily calculated. Specifically, suppose $D = \text{diag}(k_1, k_2, \ldots, k_n)$. Then

$$A^m = (PDP^{-1})^m = PD^mP^{-1} = P\,\text{diag}(k_1^m, \ldots, k_n^m)P^{-1}$$

More generally, for any polynomial $f(t)$,

$$f(A) = f(PDP^{-1}) = Pf(D)P^{-1} = P\,\text{diag}(f(k_1), f(k_2), \ldots, f(k_n))P^{-1}$$

Furthermore, if the diagonal entries of D are nonnegative, let

$$B = P\,\text{diag}(\sqrt{k_1}, \sqrt{k_2}, \ldots, \sqrt{k_n})\,P^{-1}$$

Then B is a *nonnegative square root* of A; that is, $B^2 = A$ and the eigenvalues of B are nonnegative.

EXAMPLE 9.5 Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ and let $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$Av_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = v_1 \qquad \text{and} \qquad Av_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4v_2$$

Thus, v_1 and v_2 are eigenvectors of A belonging, respectively, to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$. Observe that v_1 and v_2 are linearly independent and hence form a basis of $\mathbf{R}^2$. Accordingly, A is diagonalizable. Furthermore, let P be the matrix whose columns are the eigenvectors v_1 and v_2. That is, let

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \qquad \text{and so} \qquad P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Then A is similar to the diagonal matrix

$$D = P^{-1}AP = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

As expected, the diagonal elements 1 and 4 in D are the eigenvalues corresponding, respectively, to the eigenvectors v_1 and v_2, which are the columns of P. In particular, A has the factorization

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Accordingly,

$$A^4 = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 171 & 85 \\ 170 & 86 \end{bmatrix}$$

Moreover, suppose $f(t) = t^3 - 5t^2 + 3t + 6$; hence, $f(1) = 5$ and $f(4) = 2$. Then

$$f(A) = Pf(D)P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$$

Last, we obtain a ''positive square root'' of A. Specifically, using $\sqrt{1} = 1$ and $\sqrt{4} = 2$, we obtain the matrix

$$B = P\sqrt{D}P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

where $B^2 = A$ and where B has positive eigenvalues 1 and 2.

Remark: Throughout this chapter, we use the following fact:

$$\text{If } P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{then } P^{-1} = \begin{bmatrix} d/|P| & -b/|P| \\ -c/|P| & a/|P| \end{bmatrix}.$$

That is, P^{-1} is obtained by interchanging the diagonal elements a and d of P, taking the negatives of the nondiagonal elements b and c, and dividing each element by the determinant $|P|$.

Properties of Eigenvalues and Eigenvectors

Example 9.5 indicates the advantages of a diagonal representation (factorization) of a square matrix. In the following theorem (proved in Problem 9.20), we list properties that help us to find such a representation.