(iii) If A is triangular (i.e., A has zeros above or below the diagonal), then $|A| =$ product of diagonal elements. Thus, in particular, $|I| = 1$, where I is the identity matrix.

The next theorem (proved in Problems 8.23 and 8.25) shows how the determinant of a matrix is affected by the elementary row and column operations.

THEOREM 8.3: Suppose B is obtained from A by an elementary row (column) operation.

(i) If two rows (columns) of A were interchanged, then $|B| = -|A|$.
(ii) If a row (column) of A were multiplied by a scalar k, then $|B| = k|A|$.
(iii) If a multiple of a row (column) of A were added to another row (column) of A, then $|B| = |A|$.

Major Properties of Determinants

We now state two of the most important and useful theorems on determinants.

THEOREM 8.4: The determinant of a product of two matrices A and B is the product of their determinants; that is,

$$\det(AB) = \det(A)\,\det(B)$$

The above theorem says that the determinant is a multiplicative function.

THEOREM 8.5: Let A be a square matrix. Then the following are equivalent:

(i) A is invertible; that is, A has an inverse A^{-1}.
(ii) $AX = 0$ has only the zero solution.
(iii) The determinant of A is not zero; that is, $\det(A) \neq 0$.

Remark: Depending on the author and the text, a nonsingular matrix A is defined to be an invertible matrix A, or a matrix A for which $|A| \neq 0$, or a matrix A for which $AX = 0$ has only the zero solution. The above theorem shows that all such definitions are equivalent.

We will prove Theorems 8.4 and 8.5 (in Problems 8.29 and 8.28, respectively) using the theory of elementary matrices and the following lemma (proved in Problem 8.26), which is a special case of Theorem 8.4.

LEMMA 8.6: Let E be an elementary matrix. Then, for any matrix $A, |EA| = |E||A|$.

Recall that matrices A and B are similar if there exists a nonsingular matrix P such that $B = P^{-1}AP$. Using the multiplicative property of the determinant (Theorem 8.4), one can easily prove (Problem 8.31) the following theorem.

THEOREM 8.7: Suppose A and B are similar matrices. Then $|A| = |B|$.

8.7 Minors and Cofactors

Consider an n-square matrix $A = [a_{ij}]$. Let M_{ij} denote the $(n-1)$-square submatrix of A obtained by deleting its ith row and jth column. The determinant $|M_{ij}|$ is called the *minor* of the element a_{ij} of A, and we define the *cofactor* of a_{ij}, denoted by A_{ij}, to be the ''signed'' minor:

$$A_{ij} = (-1)^{i+j}|M_{ij}|$$

Note that the ''signs'' $(-1)^{i+j}$ accompanying the minors form a chessboard pattern with +'s on the main diagonal:

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ \multicolumn{5}{c}{\dotfill} \end{bmatrix}$$

We emphasize that M_{ij} denotes a matrix, whereas A_{ij} denotes a scalar.

Remark: The sign $(-1)^{i+j}$ of the cofactor A_{ij} is frequently obtained using the checkerboard pattern. Specifically, beginning with + and alternating signs:

$$+, -, +, -, \dots,$$

count from the main diagonal to the appropriate square.

EXAMPLE 8.8 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find the following minors and cofactors: (a) $|M_{23}|$ and A_{23}, (b) $|M_{31}|$ and A_{31}.

(a) $|M_{23}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$, and so $A_{23} = (-1)^{2+3}|M_{23}| = -(-6) = 6$

(b) $|M_{31}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$, and so $A_{31} = (-1)^{1+3}|M_{31}| = +(-3) = -3$

Laplace Expansion

The following theorem (proved in Problem 8.32) holds.

THEOREM 8.8: (Laplace) The determinant of a square matrix $A = [a_{ij}]$ is equal to the sum of the products obtained by multiplying the elements of any row (column) by their respective cofactors:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}$$

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

The above formulas for $|A|$ are called the *Laplace expansions* of the determinant of A by the ith row and the jth column. Together with the elementary row (column) operations, they offer a method of simplifying the computation of $|A|$, as described below.

8.8 Evaluation of Determinants

The following algorithm reduces the evaluation of a determinant of order n to the evaluation of a determinant of order $n - 1$.

ALGORITHM 8.1: (Reduction of the order of a determinant) The input is a nonzero n-square matrix $A = [a_{ij}]$ with $n > 1$.

Step 1. Choose an element $a_{ij} = 1$ or, if lacking, $a_{ij} \neq 0$.

Step 2. Using a_{ij} as a pivot, apply elementary row (column) operations to put 0's in all the other positions in the column (row) containing a_{ij}.

Step 3. Expand the determinant by the column (row) containing a_{ij}.

The following remarks are in order.

Remark 1: Algorithm 8.1 is usually used for determinants of order 4 or more. With determinants of order less than 4, one uses the specific formulas for the determinant.

Remark 2: Gaussian elimination or, equivalently, repeated use of Algorithm 8.1 together with row interchanges can be used to transform a matrix A into an upper triangular matrix whose determinant is the product of its diagonal entries. However, one must keep track of the number of row interchanges, because each row interchange changes the sign of the determinant.

EXAMPLE 8.9 Use Algorithm 8.1 to find the determinant of $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{bmatrix}$.

Use $a_{23} = 1$ as a pivot to put 0's in the other positions of the third column; that is, apply the row operations ''Replace R_1 by $-2R_2 + R_1$,'' ''Replace R_3 by $3R_2 + R_3$,'' and ''Replace R_4 by $R_2 + R_4$.'' By Theorem 8.3(iii), the value of the determinant does not change under these operations. Thus,

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$$

Now expand by the third column. Specifically, neglect all terms that contain 0 and use the fact that the sign of the minor M_{23} is $(-1)^{2+3} = -1$. Thus,

$$|A| = -\begin{vmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = -(4 - 18 + 5 - 30 - 3 + 4) = -(-38) = 38$$

8.9 Classical Adjoint

Let $A = [a_{ij}]$ be an $n \times n$ matrix over a field K and let A_{ij} denote the cofactor of a_{ij}. The *classical adjoint* of A, denoted by adj A, is the transpose of the matrix of cofactors of A. Namely,

$$\text{adj } A = [A_{ij}]^T$$

We say ''classical adjoint'' instead of simply ''adjoint'' because the term ''adjoint'' is currently used for an entirely different concept.

EXAMPLE 8.10 Let $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$. The cofactors of the nine elements of A follow:

$$A_{11} = +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18, \qquad A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2, \qquad A_{13} = +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$$

$$A_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11, \qquad A_{22} = +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14, \qquad A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5$$

$$A_{31} = +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10, \qquad A_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4, \qquad A_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$$

The transpose of the above matrix of cofactors yields the classical adjoint of A; that is,

$$\text{adj } A = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

The following theorem (proved in Problem 8.34) holds.

THEOREM 8.9: Let A be any square matrix. Then

$$A(\text{adj } A) = (\text{adj } A)A = |A|I$$

where I is the identity matrix. Thus, if $|A| \neq 0$,

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

EXAMPLE 8.11 Let A be the matrix in Example 8.10. We have

$$\det(A) = -40 + 6 + 0 - 16 + 4 + 0 = -46$$

Thus, A does have an inverse, and, by Theorem 8.9,

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix} = \begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}$$

8.10 Applications to Linear Equations, Cramer's Rule

Consider a system $AX = B$ of n linear equations in n unknowns. Here $A = [a_{ij}]$ is the (square) matrix of coefficients and $B = [b_i]$ is the column vector of constants. Let A_i be the matrix obtained from A by replacing the ith column of A by the column vector B. Furthermore, let

$$D = \det(A), \qquad N_1 = \det(A_1), \qquad N_2 = \det(A_2), \qquad \ldots, \qquad N_n = \det(A_n)$$

The fundamental relationship between determinants and the solution of the system $AX = B$ follows.

THEOREM 8.10: The (square) system $AX = B$ has a solution if and only if $D \neq 0$. In this case, the unique solution is given by

$$x_1 = \frac{N_1}{D}, \qquad x_2 = \frac{N_2}{D}, \qquad \ldots, \qquad x_n = \frac{N_n}{D}$$

The above theorem (proved in Problem 8.10) is known as *Cramer's rule* for solving systems of linear equations. We emphasize that the theorem only refers to a system with the same number of equations as unknowns, and that it only gives the solution when $D \neq 0$. In fact, if $D = 0$, the theorem does not tell us whether or not the system has a solution. However, in the case of a homogeneous system, we have the following useful result (to be proved in Problem 8.54).

THEOREM 8.11: A square homogeneous system $AX = 0$ has a nonzero solution if and only if $D = |A| = 0$.

EXAMPLE 8.12 Solve the system using determinants $\begin{cases} x + y + z = 5 \\ x - 2y - 3z = -1 \\ 2x + y - z = 3 \end{cases}$

First compute the determinant D of the matrix of coefficients:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = 2 - 6 + 1 + 4 + 3 + 1 = 5$$

Because $D \neq 0$, the system has a unique solution. To compute N_x, N_y, N_z, we replace, respectively, the coefficients of x, y, z in the matrix of coefficients by the constant terms. This yields

$$N_x = \begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix} = 20, \qquad N_y = \begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix} = -10, \qquad N_z = \begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 15$$

Thus, the unique solution of the system is $x = N_x/D = 4$, $y = N_y/D = -2$, $z = N_z/D = 3$; that is, the vector $u = (4, -2, 3)$.

8.11 Submatrices, Minors, Principal Minors

Let $A = [a_{ij}]$ be a square matrix of order n. Consider any r rows and r columns of A. That is, consider any set $I = (i_1, i_2, \ldots, i_r)$ of r row indices and any set $J = (j_1, j_2, \ldots, j_r)$ of r column indices. Then I and J define an $r \times r$ submatrix of A, denoted by $A(I; J)$, obtained by deleting the rows and columns of A whose subscripts do not belong to I or J, respectively. That is,

$$A(I; J) = [a_{st} : s \in I, t \in J]$$

The determinant $|A(I; J)|$ is called a *minor* of A of order r and

$$(-1)^{i_1 + i_2 + \cdots + i_r + j_1 + j_2 + \cdots + j_r} |A(I; J)|$$

is the corresponding signed minor. (Note that a minor of order $n - 1$ is a minor in the sense of Section 8.7, and the corresponding signed minor is a cofactor.) Furthermore, if I' and J' denote, respectively, the remaining row and column indices, then

$$|A(I'; J')|$$

denotes the *complementary minor*, and its sign (Problem 8.74) is the same sign as the minor.

EXAMPLE 8.13 Let $A = [a_{ij}]$ be a 5-square matrix, and let $I = \{1, 2, 4\}$ and $J = \{2, 3, 5\}$. Then $I' = \{3, 5\}$ and $J' = \{1, 4\}$, and the corresponding minor $|M|$ and complementary minor $|M'|$ are as follows:

$$|M| = |A(I; J)| = \begin{vmatrix} a_{12} & a_{13} & a_{15} \\ a_{22} & a_{23} & a_{25} \\ a_{42} & a_{43} & a_{45} \end{vmatrix} \qquad \text{and} \qquad |M'| = |A(I'; J')| = \begin{vmatrix} a_{31} & a_{34} \\ a_{51} & a_{54} \end{vmatrix}$$

Because $1 + 2 + 4 + 2 + 3 + 5 = 17$ is odd, $-|M|$ is the signed minor, and $-|M'|$ is the signed complementary minor.

Principal Minors

A minor is *principal* if the row and column indices are the same, or equivalently, if the diagonal elements of the minor come from the diagonal of the matrix. We note that the sign of a principal minor is always $+1$, because the sum of the row and identical column subscripts must always be even.