Miscellaneous Problems

3.43. Let L be a linear combination of the m equations in n unknowns in the system (3.2). Say L is the equation

$$(c_1a_{11} + \cdots + c_m a_{m1})x_1 + \cdots + (c_1 a_{1n} + \cdots + c_m a_{mn})x_n = c_1 b_1 + \cdots + c_m b_m \qquad (1)$$

Show that any solution of the system (3.2) is also a solution of L.

Let $u = (k_1, \ldots, k_n)$ be a solution of (3.2). Then

$$a_{i1}k_1 + a_{i2}k_2 + \cdots + a_{in}k_n = b_i \qquad (i = 1, 2, \ldots, m) \qquad (2)$$

Substituting u in the left-hand side of (1) and using (2), we get

$$\begin{aligned}(c_1a_{11} + \cdots + c_m a_{m1})k_1 \ + \ \cdots \ + \ (c_1 a_{1n} + \cdots + c_m a_{mn})k_n \\ = c_1(a_{11}k_1 + \cdots + a_{1n}k_n) \ + \ \cdots \ + \ c_m(a_{m1}k_1 + \cdots + a_{mn}k_n) \\ = c_1 b_1 + \cdots + c_m b_m\end{aligned}$$

This is the right-hand side of (1); hence, u is a solution of (1).

3.44. Suppose a system $\mathscr{M}$ of linear equations is obtained from a system $\mathscr{L}$ by applying an elementary operation (page 64). Show that $\mathscr{M}$ and $\mathscr{L}$ have the same solutions.

Each equation L in $\mathscr{M}$ is a linear combination of equations in $\mathscr{L}$. Hence, by Problem 3.43, any solution of $\mathscr{L}$ will also be a solution of $\mathscr{M}$. On the other hand, each elementary operation has an inverse elementary operation, so $\mathscr{L}$ can be obtained from $\mathscr{M}$ by an elementary operation. This means that any solution of $\mathscr{M}$ is a solution of $\mathscr{L}$. Thus, $\mathscr{L}$ and $\mathscr{M}$ have the same solutions.

3.45. Prove Theorem 3.4: Suppose a system $\mathscr{M}$ of linear equations is obtained from a system $\mathscr{L}$ by a sequence of elementary operations. Then $\mathscr{M}$ and $\mathscr{L}$ have the same solutions.

Each step of the sequence does not change the solution set (Problem 3.44). Thus, the original system $\mathscr{L}$ and the final system $\mathscr{M}$ (and any system in between) have the same solutions.

3.46. A system $\mathscr{L}$ of linear equations is said to be *consistent* if no linear combination of its equations is a degenerate equation L with a nonzero constant. Show that $\mathscr{L}$ is consistent if and only if $\mathscr{L}$ is reducible to echelon form.

Suppose $\mathscr{L}$ is reducible to echelon form. Then $\mathscr{L}$ has a solution, which must also be a solution of every linear combination of its equations. Thus, L, which has no solution, cannot be a linear combination of the equations in $\mathscr{L}$. Thus, $\mathscr{L}$ is consistent.

On the other hand, suppose $\mathscr{L}$ is not reducible to echelon form. Then, in the reduction process, it must yield a degenerate equation L with a nonzero constant, which is a linear combination of the equations in $\mathscr{L}$. Therefore, $\mathscr{L}$ is not consistent; that is, $\mathscr{L}$ is inconsistent.

3.47. Suppose u and v are distinct vectors. Show that, for distinct scalars k, the vectors $u + k(u - v)$ are distinct.

Suppose $u + k_1(u - v) = u + k_2(u - v)$. We need only show that $k_1 = k_2$. We have

$$k_1(u - v) = k_2(u - v), \qquad \text{and so} \qquad (k_1 - k_2)(u - v) = 0$$

Because u and v are distinct, $u - v \neq 0$. Hence, $k_1 - k_2 = 0$, and so $k_1 = k_2$.

3.48. Suppose AB is defined. Prove

(a) Suppose A has a zero row. Then AB has a zero row.

(b) Suppose B has a zero column. Then AB has a zero column.

(a) Let R_i be the zero row of A, and $C_1, \ldots, C_n$ the columns of B. Then the ith row of AB is

$$(R_iC_1, R_iC_2, \ldots, R_iC_n) = (0, 0, 0, \ldots, 0)$$

(b) B^T has a zero row, and so $B^TA^T = (AB)^T$ has a zero row. Hence, AB has a zero column.

SUPPLEMENTARY PROBLEMS

Linear Equations, 2 × 2 Systems

3.49. Determine whether each of the following systems is linear:

(a) $3x - 4y + 2yz = 8$, (b) $ex + 3y = \pi$, (c) $2x - 3y + kz = 4$

3.50. Solve (a) $\pi x = 2$, (b) $3x + 2 = 5x + 7 - 2x$, (c) $6x + 2 - 4x = 5 + 2x - 3$

3.51. Solve each of the following systems:

(a) $\begin{aligned} 2x + 3y &= 1 \\ 5x + 7y &= 3 \end{aligned}$ (b) $\begin{aligned} 4x - 2y &= 5 \\ -6x + 3y &= 1 \end{aligned}$ (c) $\begin{aligned} 2x - 4 &= 3y \\ 5y - x &= 5 \end{aligned}$ (d) $\begin{aligned} 2x - 4y &= 10 \\ 3x - 6y &= 15 \end{aligned}$

3.52. Consider each of the following systems in unknowns x and y:

(a) $\begin{aligned} x - ay &= 1 \\ ax - 4y &= b \end{aligned}$ (b) $\begin{aligned} ax + 3y &= 2 \\ 12x + ay &= b \end{aligned}$ (c) $\begin{aligned} x + ay &= 3 \\ 2x + 5y &= b \end{aligned}$

For which values of a does each system have a unique solution, and for which pairs of values (a, b) does each system have more than one solution?

General Systems of Linear Equations

3.53. Solve

(a) $\begin{aligned} x + y + 2z &= 4 \\ 2x + 3y + 6z &= 10 \\ 3x + 6y + 10z &= 17 \end{aligned}$ (b) $\begin{aligned} x - 2y + 3z &= 2 \\ 2x - 3y + 8z &= 7 \\ 3x - 4y + 13z &= 8 \end{aligned}$ (c) $\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + 8z &= 4 \\ 5x + 8y + 19z &= 11 \end{aligned}$

3.54. Solve

(a) $\begin{aligned} x - 2y &= 5 \\ 2x + 3y &= 3 \\ 3x + 2y &= 7 \end{aligned}$ (b) $\begin{aligned} x + 2y - 3z + 2t &= 2 \\ 2x + 5y - 8z + 6t &= 5 \\ 3x + 4y - 5z + 2t &= 4 \end{aligned}$ (c) $\begin{aligned} x + 2y + 4z - 5t &= 3 \\ 3x - y + 5z + 2t &= 4 \\ 5x - 4y + 6z + 9t &= 2 \end{aligned}$

3.55. Solve

(a) $\begin{aligned} 2x - y - 4z &= 2 \\ 4x - 2y - 6z &= 5 \\ 6x - 3y - 8z &= 8 \end{aligned}$ (b) $\begin{aligned} x + 2y - z + 3t &= 3 \\ 2x + 4y + 4z + 3t &= 9 \\ 3x + 6y - z + 8t &= 10 \end{aligned}$

3.56. Consider each of the following systems in unknowns x, y, z:

(a) $\begin{aligned} x - 2y &= 1 \\ x - y + az &= 2 \\ ay + 9z &= b \end{aligned}$ (b) $\begin{aligned} x + 2y + 2z &= 1 \\ x + ay + 3z &= 3 \\ x + 11y + az &= b \end{aligned}$ (c) $\begin{aligned} x + y + az &= 1 \\ x + ay + z &= 4 \\ ax + y + z &= b \end{aligned}$

For which values of a does the system have a unique solution, and for which pairs of values (a, b) does the system have more than one solution? The value of b does not have any effect on whether the system has a unique solution. Why?