Basic properties of vectors under the operations of vector addition and scalar multiplication are described in the following theorem.

THEOREM 1.1: For any vectors u, v, w in $\mathbf{R}^n$ and any scalars k, k' in $\mathbf{R}$,

(i)	$(u+v)+w = u+(v+w)$,	(v)	$k(u+v) = ku+kv$,
(ii)	$u+0 = u$,	(vi)	$(k+k')u = ku+k'u$,
(iii)	$u+(-u) = 0$,	(vii)	(kk')u=k(k'u),
(iv)	$u+v = v+u$,	(viii)	$1u = u$.

We postpone the proof of Theorem 1.1 until Chapter 2, where it appears in the context of matrices (Problem 2.3).

Suppose u and v are vectors in $\mathbf{R}^n$ for which $u = kv$ for some nonzero scalar k in $\mathbf{R}$. Then u is called a *multiple* of v. Also, u is said to be in the *same* or *opposite direction* as v according to whether $k > 0$ or $k < 0$.

1.4 Dot (Inner) Product

Consider arbitrary vectors u and v in $\mathbf{R}^n$; say,

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

The *dot product* or *inner product* or *scalar product* of u and v is denoted and defined by

$$u \cdot v = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

That is, $u \cdot v$ is obtained by multiplying corresponding components and adding the resulting products. The vectors u and v are said to be *orthogonal* (or *perpendicular*) if their dot product is zero—that is, if $u \cdot v = 0$.

EXAMPLE 1.3

(a) Let $u = (1, -2, 3)$, $v = (4, 5, -1)$, $w = (2, 7, 4)$. Then,

$$u \cdot v = 1(4) - 2(5) + 3(-1) = 4 - 10 - 3 = -9$$
$$u \cdot w = 2 - 14 + 12 = 0, \qquad v \cdot w = 8 + 35 - 4 = 39$$

Thus, u and w are orthogonal.

(b) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $u \cdot v = 6 - 3 + 8 = 11$.

(c) Suppose $u = (1, 2, 3, 4)$ and $v = (6, k, -8, 2)$. Find k so that u and v are orthogonal.

First obtain $u \cdot v = 6 + 2k - 24 + 8 = -10 + 2k$. Then set $u \cdot v = 0$ and solve for k:

$$-10 + 2k = 0 \qquad \text{or} \qquad 2k = 10 \qquad \text{or} \qquad k = 5$$

Basic properties of the dot product in $\mathbf{R}^n$ (proved in Problem 1.13) follow.

THEOREM 1.2: For any vectors u, v, w in $\mathbf{R}^n$ and any scalar k in $\mathbf{R}$:

(i)	$(u+v) \cdot w = u \cdot w + v \cdot w$,	(iii)	$u \cdot v = v \cdot u$,
(ii)	$(ku) \cdot v = k(u \cdot v)$,	(iv)	$u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$.

Note that (ii) says that we can "take k out" from the first position in an inner product. By (iii) and (ii),

$$u \cdot (kv) = (kv) \cdot u = k(v \cdot u) = k(u \cdot v)$$

That is, we can also "take k out" from the second position in an inner product.

The space $\mathbf{R}^n$ with the above operations of vector addition, scalar multiplication, and dot product is usually called *Euclidean n-space*.

Norm (Length) of a Vector

The *norm* or *length* of a vector u in $\mathbf{R}^n$, denoted by $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$. In particular, if $u = (a_1, a_2, \ldots, a_n)$, then

$$\|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

That is, $\|u\|$ is the square root of the sum of the squares of the components of u. Thus, $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$.

A vector u is called a *unit* vector if $\|u\| = 1$ or, equivalently, if $u \cdot u = 1$. For any nonzero vector v in $\mathbf{R}^n$, the vector

$$\hat{v} = \frac{1}{\|v\|} v = \frac{v}{\|v\|}$$

is the unique unit vector in the same direction as v. The process of finding $\hat{v}$ from v is called *normalizing* v.

EXAMPLE 1.4

(a) Suppose $u = (1, -2, -4, 5, 3)$. To find $\|u\|$, we can first find $\|u\|^2 = u \cdot u$ by squaring each component of u and adding, as follows:

$$\|u\|^2 = 1^2 + (-2)^2 + (-4)^2 + 5^2 + 3^2 = 1 + 4 + 16 + 25 + 9 = 55$$

Then $\|u\| = \sqrt{55}$.

(b) Let $v = (1, -3, 4, 2)$ and $w = (\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{1}{6})$. Then

$$\|v\| = \sqrt{1 + 9 + 16 + 4} = \sqrt{30} \qquad \text{and} \qquad \|w\| = \sqrt{\frac{9}{36} + \frac{1}{36} + \frac{25}{36} + \frac{1}{36}} = \sqrt{\frac{36}{36}} = \sqrt{1} = 1$$

Thus w is a unit vector, but v is not a unit vector. However, we can normalize v as follows:

$$\hat{v} = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{30}}, \frac{-3}{\sqrt{30}}, \frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}\right)$$

This is the unique unit vector in the same direction as v.

The following formula (proved in Problem 1.14) is known as the Schwarz inequality or Cauchy–Schwarz inequality. It is used in many branches of mathematics.

THEOREM 1.3 (Schwarz): For any vectors u, v in $\mathbf{R}^n$, $|u \cdot v| \leq \|u\| \|v\|$.

Using the above inequality, we also prove (Problem 1.15) the following result known as the "triangle inequality" or Minkowski's inequality.

THEOREM 1.4 (Minkowski): For any vectors u, v in $\mathbf{R}^n$, $\|u + v\| \leq \|u\| + \|v\|$.

Distance, Angles, Projections

The *distance* between vectors $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ in $\mathbf{R}^n$ is denoted and defined by

$$d(u, v) = \|u - v\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

One can show that this definition agrees with the usual notion of distance in the Euclidean plane $\mathbf{R}^2$ or space $\mathbf{R}^3$.

The *angle* θ between nonzero vectors u, v in $\mathbf{R}^n$ is defined by

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

This definition is well defined, because, by the Schwarz inequality (Theorem 1.3),

$$-1 \leq \frac{u \cdot v}{\|u\|\|v\|} \leq 1$$

Note that if $u \cdot v = 0$, then $\theta = 90^\circ$ (or $\theta = \pi/2$). This then agrees with our previous definition of orthogonality.

The *projection* of a vector u onto a nonzero vector v is the vector denoted and defined by

$$\text{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{u \cdot v}{v \cdot v} v$$

We show below that this agrees with the usual notion of vector projection in physics.

EXAMPLE 1.5

(a) Suppose $u = (1, -2, 3)$ and $v = (2, 4, 5)$. Then

$$d(u, v) = \sqrt{(1-2)^2 + (-2-4)^2 + (3-5)^2} = \sqrt{1 + 36 + 4} = \sqrt{41}$$

To find $\cos\theta$, where θ is the angle between u and v, we first find

$$u \cdot v = 2 - 8 + 15 = 9, \qquad \|u\|^2 = 1 + 4 + 9 = 14, \qquad \|v\|^2 = 4 + 16 + 25 = 45$$

Then

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{9}{\sqrt{14}\sqrt{45}}$$

Also,

$$\text{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{9}{45}(2, 4, 5) = \frac{1}{5}(2, 4, 5) = \left(\frac{2}{5}, \frac{4}{5}, 1\right)$$

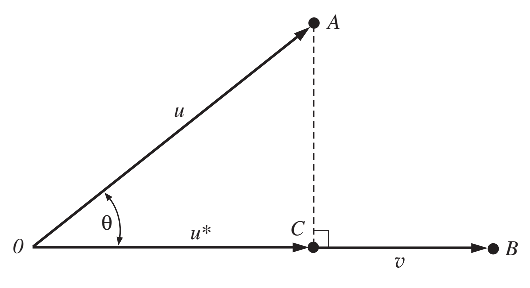
(b) Consider the vectors u and v in Fig. 1-2(a) (with respective endpoints A and B). The (perpendicular) projection of u onto v is the vector u^* with magnitude

$$\|u^*\| = \|u\|\cos\theta = \|u\|\frac{u \cdot v}{\|u\|\|v\|} = \frac{u \cdot v}{\|v\|}$$

To obtain u^*, we multiply its magnitude by the unit vector in the direction of v, obtaining

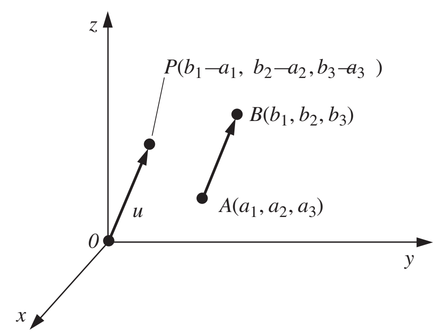
$$u^* = \|u^*\|\frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|}\frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|^2} v$$

This is the same as the above definition of proj(u, v).



Projection u^* of u onto v

(a)



$u = B - A$

(b)

Figure 1-2

1.5 Located Vectors, Hyperplanes, Lines, Curves in $\mathbf{R}^n$

This section distinguishes between an n-tuple $P(a_i) \equiv P(a_1, a_2, \ldots, a_n)$ viewed as a point in $\mathbf{R}^n$ and an n-tuple $u = [c_1, c_2, \ldots, c_n]$ viewed as a vector (arrow) from the origin O to the point $C(c_1, c_2, \ldots, c_n)$.

Located Vectors

Any pair of points $A(a_i)$ and $B(b_i)$ in $\mathbf{R}^n$ defines the *located vector* or *directed line segment* from A to B, written $\overrightarrow{AB}$. We identify $\overrightarrow{AB}$ with the vector

$$u = B - A = [b_1 - a_1, \ b_2 - a_2, \ \ldots, \ b_n - a_n]$$

because $\overrightarrow{AB}$ and u have the same magnitude and direction. This is pictured in Fig. 1-2(b) for the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in $\mathbf{R}^3$ and the vector $u = B - A$ which has the endpoint $P(b_1 - a_1, \ b_2 - a_2, \ b_3 - a_3)$.

Hyperplanes

A *hyperplane* H in $\mathbf{R}^n$ is the set of points $(x_1, x_2, \ldots, x_n)$ that satisfy a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the vector $u = [a_1, a_2, \ldots, a_n]$ of coefficients is not zero. Thus a hyperplane H in $\mathbf{R}^2$ is a line, and a hyperplane H in $\mathbf{R}^3$ is a plane. We show below, as pictured in Fig. 1-3(a) for $\mathbf{R}^3$, that u is orthogonal to any directed line segment $\overrightarrow{PQ}$, where $P(p_i)$ and $Q(q_i)$ are points in H. [For this reason, we say that u is *normal* to H and that H is *normal* to u.]

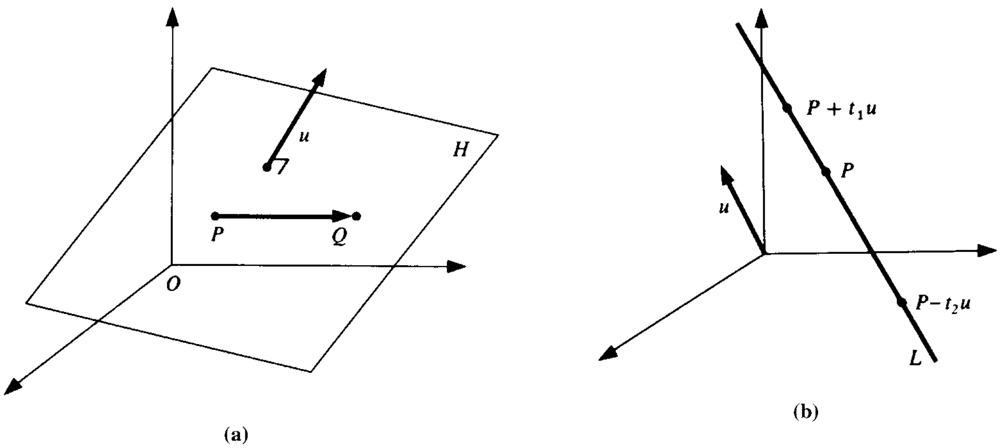


Figure 1-3

Because $P(p_i)$ and $Q(q_i)$ belong to H, they satisfy the above hyperplane equation—that is,

$$a_1p_1 + a_2p_2 + \cdots + a_np_n = b \quad \text{and} \quad a_1q_1 + a_2q_2 + \cdots + a_nq_n = b$$

Let $\quad v = \overrightarrow{PQ} = Q - P = [q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n]$

Then

$$\begin{aligned} u \cdot v &= a_1(q_1 - p_1) + a_2(q_2 - p_2) + \cdots + a_n(q_n - p_n) \\ &= (a_1q_1 + a_2q_2 + \cdots + a_nq_n) - (a_1p_1 + a_2p_2 + \cdots + a_np_n) = b - b = 0 \end{aligned}$$

Thus $v = \overrightarrow{PQ}$ is orthogonal to u, as claimed.

Lines in $\mathbf{R}^n$

The *line* L in $\mathbf{R}^n$ passing through the point $P(b_1, b_2, \ldots, b_n)$ and in the direction of a nonzero vector $u = [a_1, a_2, \ldots, a_n]$ consists of the points $X(x_1, x_2, \ldots, x_n)$ that satisfy

$$X = P + tu \qquad \text{or} \qquad \begin{cases} x_1 = a_1 t + b_1 \\ x_2 = a_2 t + b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = a_n t + b_n \end{cases} \qquad \text{or} \quad L(t) = (a_i t + b_i)$$

where the *parameter* t takes on all real values. Such a line L in $\mathbf{R}^3$ is pictured in Fig. 1-3(b).

EXAMPLE 1.6

(a) Let H be the plane in $\mathbf{R}^3$ corresponding to the linear equation $2x - 5y + 7z = 4$. Observe that $P(1, 1, 1)$ and $Q(5, 4, 2)$ are solutions of the equation. Thus P and Q and the directed line segment

$$v = \overrightarrow{PQ} = Q - P = [5 - 1,\ 4 - 1,\ 2 - 1] = [4, 3, 1]$$

lie on the plane H. The vector $u = [2, -5, 7]$ is normal to H, and, as expected,

$$u \cdot v = [2, -5, 7] \cdot [4, 3, 1] = 8 - 15 + 7 = 0$$

That is, u is orthogonal to v.

(b) Find an equation of the hyperplane H in $\mathbf{R}^4$ that passes through the point $P(1, 3, -4, 2)$ and is normal to the vector $u = [4, -2, 5, 6]$.

The coefficients of the unknowns of an equation of H are the components of the normal vector u; hence, the equation of H must be of the form

$$4x_1 - 2x_2 + 5x_3 + 6x_4 = k$$

Substituting P into this equation, we obtain

$$4(1) - 2(3) + 5(-4) + 6(2) = k \qquad \text{or} \qquad 4 - 6 - 20 + 12 = k \qquad \text{or} \qquad k = -10$$

Thus, $4x_1 - 2x_2 + 5x_3 + 6x_4 = -10$ is the equation of H.

(c) Find the parametric representation of the line L in $\mathbf{R}^4$ passing through the point $P(1, 2, 3, -4)$ and in the direction of $u = [5, 6, -7, 8]$. Also, find the point Q on L when $t = 1$.

Substitution in the above equation for L yields the following parametric representation:

$$x_1 = 5t + 1, \qquad x_2 = 6t + 2, \qquad x_3 = -7t + 3, \qquad x_4 = 8t - 4$$

or, equivalently,

$$L(t) = (5t + 1, 6t + 2, -7t + 3, 8t - 4)$$

Note that $t = 0$ yields the point P on L. Substitution of $t = 1$ yields the point $Q(6, 8, -4, 4)$ on L.

Curves in $\mathbf{R}^n$

Let D be an interval (finite or infinite) on the real line $\mathbf{R}$. A continuous function $F\colon D \to \mathbf{R}^n$ is a *curve* in $\mathbf{R}^n$. Thus, to each point $t \in D$ there is assigned the following point in $\mathbf{R}^n$:

$$F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]$$

Moreover, the derivative (if it exists) of $F(t)$ yields the vector

$$V(t) = \frac{dF(t)}{dt} = \left[\frac{dF_1(t)}{dt}, \frac{dF_2(t)}{dt}, \ldots, \frac{dF_n(t)}{dt}\right]$$

which is tangent to the curve. Normalizing $V(t)$ yields

$$\mathbf{T}(t) = \frac{V(t)}{\|V(t)\|}$$

Thus, $\mathbf{T}(t)$ is the unit tangent vector to the curve. (Unit vectors with geometrical significance are often presented in bold type.)

EXAMPLE 1.7 Consider the curve $F(t) = [\sin t, \cos t, t]$ in $\mathbf{R}^3$. Taking the derivative of $F(t)$ [or each component of $F(t)$] yields

$$V(t) = [\cos t, -\sin t, 1]$$

which is a vector tangent to the curve. We normalize $V(t)$. First we obtain

$$\|V(t)\|^2 = \cos^2 t + \sin^2 t + 1 = 1 + 1 = 2$$

Then the unit tangent vection $\mathbf{T}(t)$ to the curve follows:

$$\mathbf{T}(t) = \frac{V(t)}{\|V(t)\|} = \left[\frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

1.6 Vectors in R³ (Spatial Vectors), ijk Notation

Vectors in $\mathbf{R}^3$, called *spatial vectors*, appear in many applications, especially in physics. In fact, a special notation is frequently used for such vectors as follows:

$\mathbf{i} = [1, 0, 0]$ denotes the unit vector in the x direction.
$\mathbf{j} = [0, 1, 0]$ denotes the unit vector in the y direction.
$\mathbf{k} = [0, 0, 1]$ denotes the unit vector in the z direction.

Then any vector $u = [a, b, c]$ in $\mathbf{R}^3$ can be expressed uniquely in the form

$$u = [a, b, c] = a\mathbf{i} + b\mathbf{j} + c\mathbf{j}$$

Because the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors and are mutually orthogonal, we obtain the following dot products:

$$\mathbf{i}\cdot\mathbf{i} = 1, \quad \mathbf{j}\cdot\mathbf{j} = 1, \quad \mathbf{k}\cdot\mathbf{k} = 1 \qquad \text{and} \qquad \mathbf{i}\cdot\mathbf{j} = 0, \quad \mathbf{i}\cdot\mathbf{k} = 0, \quad \mathbf{j}\cdot\mathbf{k} = 0$$

Furthermore, the vector operations discussed above may be expressed in the **ijk** notation as follows. Suppose

$$u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Then

$$u + v = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \qquad \text{and} \qquad cu = ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k}$$

where c is a scalar. Also,

$$u \cdot v = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{and} \qquad \|u\| = \sqrt{u \cdot u} = a_1^2 + a_2^2 + a_3^2$$

EXAMPLE 1.8 Suppose $u = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $v = 4\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}$.

(a) To find $u + v$, add corresponding components, obtaining $u + v = 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

(b) To find $3u - 2v$, first multiply by the scalars and then add:

$$3u - 2v = (9\mathbf{i} + 13\mathbf{j} - 6\mathbf{k}) + (-8\mathbf{i} + 16\mathbf{j} - 14\mathbf{k}) = \mathbf{i} + 29\mathbf{j} - 20\mathbf{k}$$

(c) To find $u \cdot v$, multiply corresponding components and then add:

$$u \cdot v = 12 - 40 - 14 = -42$$

(d) To find $\|u\|$, take the square root of the sum of the squares of the components:

$$\|u\| = \sqrt{9 + 25 + 4} = \sqrt{38}$$

Cross Product

There is a special operation for vectors u and v in $\mathbf{R}^3$ that is not defined in $\mathbf{R}^n$ for $n \neq 3$. This operation is called the *cross product* and is denoted by $u \times v$. One way to easily remember the formula for $u \times v$ is to use the determinant (of order two) and its negative, which are denoted and defined as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \text{and} \qquad -\begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad$$

Here a and d are called the *diagonal* elements and b and c are the *nondiagonal* elements. Thus, the determinant is the product ad of the diagonal elements minus the product bc of the nondiagonal elements, but vice versa for the negative of the determinant.

Now suppose $u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $v = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then

$$\begin{aligned} u \times v &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} \boxed{a_1} & a_2 & a_3 \\ \boxed{b_1} & b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & \boxed{a_2} & a_3 \\ b_1 & \boxed{b_2} & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 & \boxed{a_3} \\ b_1 & b_2 & \boxed{b_3} \end{vmatrix}\mathbf{i} \end{aligned}$$

That is, the three components of $u \times v$ are obtained from the array

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

(which contain the components of u above the component of v) as follows:

(1) Cover the first column and take the determinant.
(2) Cover the second column and take the negative of the determinant.
(3) Cover the third column and take the determinant.

Note that $u \times v$ is a vector; hence, $u \times v$ is also called the *vector product* or *outer product* of u and v.

EXAMPLE 1.9 Find $u \times v$ where: (a) $u = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, (b) $u = [2, -1, 5]$, $v = [3, 7, 6]$.

(a) Use $\begin{bmatrix} 4 & 3 & 6 \\ 2 & 5 & -3 \end{bmatrix}$ to get $u \times v = (-9 - 30)\mathbf{i} + (12 + 12)\mathbf{j} + (20 - 6)\mathbf{k} = -39\mathbf{i} + 24\mathbf{j} + 14\mathbf{k}$

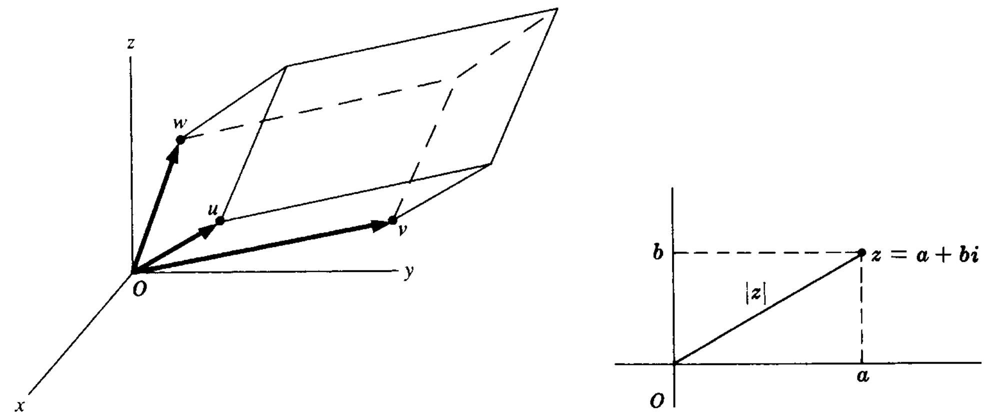
(b) Use $\begin{bmatrix} 2 & -1 & 5 \\ 3 & 7 & 6 \end{bmatrix}$ to get $u \times v = [-6 - 35, 15 - 12, 14 + 3] = [-41, 3, 17]$

Remark: The cross products of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are as follows:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}, & \mathbf{k} \times \mathbf{j} = -\mathbf{i}, & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Thus, if we view the triple $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ as a cyclic permutation, where $\mathbf{i}$ follows $\mathbf{k}$ and hence $\mathbf{k}$ precedes $\mathbf{i}$, then the product of two of them in the given direction is the third one, but the product of two of them in the opposite direction is the negative of the third one.

Two important properties of the cross product are contained in the following theorem.



Volume = $u \cdot v \times w$

(a)

Complex plane

(b)

Figure 1-4

THEOREM 1.5: Let u, v, w be vectors in $\mathbf{R}^3$.

(a) The vector $u \times v$ is orthogonal to both u and v.

(b) The absolute value of the "triple product"

$$u \cdot v \times w$$

represents the volume of the parallelopiped formed by the vectors u, v, w. [See Fig. 1-4(a).]

We note that the vectors $u, v, u \times v$ form a right-handed system, and that the following formula gives the magnitude of $u \times v$:

$$\|u \times v\| = \|u\|\|v\| \sin\theta$$

where θ is the angle between u and v.

1.7 Complex Numbers

The set of complex numbers is denoted by $\mathbf{C}$. Formally, a complex number is an ordered pair (a, b) of real numbers where equality, addition, and multiplication are defined as follows:

$$(a, b) = (c, d) \quad \text{if and only if } a = c \text{ and } b = d$$
$$(a, b) + (c, d) = (a + c,\; b + d)$$
$$(a, b) \cdot (c, d) = (ac - bd,\; ad + bc)$$

We identify the real number a with the complex number $(a, 0)$; that is,

$$a \leftrightarrow (a, 0)$$

This is possible because the operations of addition and multiplication of real numbers are preserved under the correspondence; that is,

$$(a, 0) + (b, 0) = (a + b,\; 0) \qquad \text{and} \qquad (a, 0) \cdot (b, 0) = (ab, 0)$$

Thus we view $\mathbf{R}$ as a subset of $\mathbf{C}$, and replace $(a, 0)$ by a whenever convenient and possible.

We note that the set $\mathbf{C}$ of complex numbers with the above operations of addition and multiplication is a *field* of numbers, like the set $\mathbf{R}$ of real numbers and the set $\mathbf{Q}$ of *rational numbers*.