CHAPTER 1

Vectors in R^n and C^n, Spatial Vectors

1.1 Introduction

There are two ways to motivate the notion of a vector: one is by means of lists of numbers and subscripts, and the other is by means of certain objects in physics. We discuss these two ways below.

Here we assume the reader is familiar with the elementary properties of the field of real numbers, denoted by $\mathbf{R}$. On the other hand, we will review properties of the field of complex numbers, denoted by $\mathbf{C}$. In the context of vectors, the elements of our number fields are called *scalars*.

Although we will restrict ourselves in this chapter to vectors whose elements come from $\mathbf{R}$ and then from $\mathbf{C}$, many of our operations also apply to vectors whose entries come from some arbitrary field K.

Lists of Numbers

Suppose the weights (in pounds) of eight students are listed as follows:

156, 125, 145, 134, 178, 145, 162, 193

One can denote all the values in the list using only one symbol, say w, but with different subscripts; that is,

$$w_1, \quad w_2, \quad w_3, \quad w_4, \quad w_5, \quad w_6, \quad w_7, \quad w_8$$

Observe that each subscript denotes the position of the value in the list. For example,

$$w_1 = 156, \text{ the first number, } w_2 = 125, \text{ the second number, } \ldots$$

Such a list of values,

$$w = (w_1, w_2, w_3, \ldots, w_8)$$

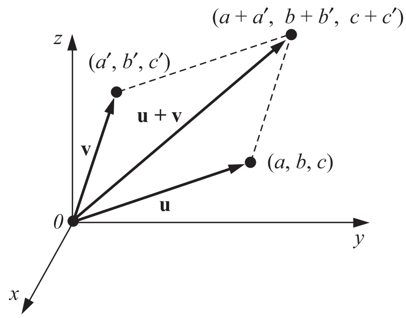
is called a *linear array* or *vector*.

Vectors in Physics

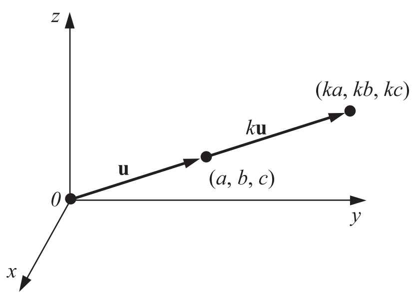
Many physical quantities, such as temperature and speed, possess only "magnitude." These quantities can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, that possess both "magnitude" and "direction." These quantities, which can be represented by arrows having appropriate lengths and directions and emanating from some given reference point O, are called *vectors*.

Now we assume the reader is familiar with the space $\mathbf{R}^3$ where all the points in space are represented by ordered triples of real numbers. Suppose the origin of the axes in $\mathbf{R}^3$ is chosen as the reference point O for the vectors discussed above. Then every vector is uniquely determined by the coordinates of its endpoint, and vice versa.

There are two important operations, vector addition and scalar multiplication, associated with vectors in physics. The definition of these operations and the relationship between these operations and the endpoints of the vectors are as follows.



(a) Vector Addition



(b) Scalar Multiplication

Figure 1-1

(i) ***Vector Addition:*** The resultant $\mathbf{u}+\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is obtained by the *parallelogram law*; that is, $\mathbf{u}+\mathbf{v}$ is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. Furthermore, if (a, b, c) and (a', b', c') are the endpoints of the vectors $\mathbf{u}$ and $\mathbf{v}$, then $(a+a',\ b+b',\ c+c')$ is the endpoint of the vector $\mathbf{u}+\mathbf{v}$. These properties are pictured in Fig. 1-1(a).

(ii) ***Scalar Multiplication:*** The product $k\mathbf{u}$ of a vector $\mathbf{u}$ by a real number k is obtained by multiplying the magnitude of $\mathbf{u}$ by k and retaining the same direction if $k > 0$ or the opposite direction if $k < 0$. Also, if (a, b, c) is the endpoint of the vector $\mathbf{u}$, then (ka, kb, kc) is the endpoint of the vector $k\mathbf{u}$. These properties are pictured in Fig. 1-1(b).

Mathematically, we identify the vector $\mathbf{u}$ with its (a, b, c) and write $\mathbf{u} = (a, b, c)$. Moreover, we call the ordered triple (a, b, c) of real numbers a point or vector depending upon its interpretation. We generalize this notion and call an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers a vector. However, special notation may be used for the vectors in $\mathbf{R}^3$ called *spatial* vectors (Section 1.6).

1.2 Vectors in $\mathbf{R}^n$

The set of all n-tuples of real numbers, denoted by $\mathbf{R}^n$, is called *n-space*. A particular n-tuple in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n)$$

is called a *point* or *vector*. The numbers a_i are called the *coordinates*, *components*, *entries*, or *elements* of u. Moreover, when discussing the space $\mathbf{R}^n$, we use the term *scalar* for the elements of $\mathbf{R}$.

Two vectors, u and v, are *equal*, written $u = v$, if they have the same number of components and if the corresponding components are equal. Although the vectors $(1, 2, 3)$ and $(2, 3, 1)$ contain the same three numbers, these vectors are not equal because corresponding entries are not equal.

The vector $(0, 0, \ldots, 0)$ whose entries are all 0 is called the *zero vector* and is usually denoted by 0.

EXAMPLE 1.1

(a) The following are vectors:

$$(2, -5), \quad (7, 9), \quad (0, 0, 0), \quad (3, 4, 5)$$

The first two vectors belong to $\mathbf{R}^2$, whereas the last two belong to $\mathbf{R}^3$. The third is the zero vector in $\mathbf{R}^3$.

(b) Find x, y, z such that $(x-y,\ x+y,\ z-1) = (4, 2, 3)$.

By definition of equality of vectors, corresponding entries must be equal. Thus,

$$x - y = 4, \qquad x + y = 2, \qquad z - 1 = 3$$

Solving the above system of equations yields $x = 3$, $y = -1$, $z = 4$.

Column Vectors

Sometimes a vector in n-space $\mathbf{R}^n$ is written vertically rather than horizontally. Such a vector is called a *column vector*, and, in this context, the horizontally written vectors in Example 1.1 are called *row vectors*. For example, the following are column vectors with $2, 2, 3$, and 3 components, respectively:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1.5 \\ \frac{2}{3} \\ -15 \end{bmatrix}$$

We also note that any operation defined for row vectors is defined analogously for column vectors.

1.3 Vector Addition and Scalar Multiplication

Consider two vectors u and v in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

Their *sum*, written $u + v$, is the vector obtained by adding corresponding components from u and v. That is,

$$u + v = (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n)$$

The *scalar product* or, simply, *product*, of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k. That is,

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. The sum of vectors with different numbers of components is not defined.

Negatives and subtraction are defined in $\mathbf{R}^n$ as follows:

$$-u = (-1)u \qquad \text{and} \qquad u - v = u + (-v)$$

The vector $-u$ is called the *negative* of u, and $u - v$ is called the *difference* of u and v.

Now suppose we are given vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$ and scalars $k_1, k_2, \ldots, k_m$ in $\mathbf{R}$. We can multiply the vectors by the corresponding scalars and then add the resultant scalar products to form the vector

$$v = k_1 u_1 + k_2 u_2 + k_3 u_3 + \cdots + k_m u_m$$

Such a vector v is called a *linear combination* of the vectors $u_1, u_2, \ldots, u_m$.

EXAMPLE 1.2

(a) Let $u = (2, 4, -5)$ and $v = (1, -6, 9)$. Then

$$\begin{aligned} u + v &= (2 + 1,\ 4 + (-5),\ -5 + 9) = (3, -1, 4) \\ 7u &= (7(2), 7(4), 7(-5)) = (14, 28, -35) \\ -v &= (-1)(1, -6, 9) = (-1, 6, -9) \\ 3u - 5v &= (6, 12, -15) + (-5, 30, -45) = (1, 42, -60) \end{aligned}$$

(b) The zero vector $0 = (0, 0, \ldots, 0)$ in $\mathbf{R}^n$ is similar to the scalar 0 in that, for any vector $u = (a_1, a_2, \ldots, a_n)$.

$$u + 0 = (a_1 + 0,\ a_2 + 0,\ \ldots,\ a_n + 0) = (a_1, a_2, \ldots, a_n) = u$$

(c) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $2u - 3v = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -2 \end{bmatrix}$.