Chapter 14

Probabilistic Inventory Models

1

(a) Effective lead time L

$= 15-10 = 5$ days

$\mu_L = 100 \times 5 = 500$ units

$\sigma_L = \sqrt{10^2 \times 5} = 22.36$ units

$B \geq 22.36 \times 1.645 \simeq 37$ units

Order 1000 units whenever the inventory level drops to 537 units

(b) Effective lead time $L = 23-20 = 3$ days

$\mu_L = 100 \times 3 = 300$ units

$\sigma_L = \sqrt{10^2 \times 3} = 17.32$ units

$B \geq 17.32 \times 1.645 \simeq 29$ units

Order 1000 units whenever the inventory level drops to 329 units

(c) Effective lead time = 8 days

$\mu_L = 100 \times 8 = 800$ units

$\sigma_L = \sqrt{10^2 \times 8} = 28.28$ units

$B \geq 28.28 \times 1.645 \simeq 47$ units

(d) Effective lead time = 0

$\mu_L = \sigma_L = 0, \quad B \geq 0$

Order 1000 units whenever the inventory level drops to 0 unit.

2

Demand/day = N(200, 20)

$h = \$.04$/day/unit, $K = \$100$, $L = 7$ days

order quantity $= \sqrt{\dfrac{2KD}{h}} = \sqrt{\dfrac{2 \times 100 \times 200}{.04}}$

$= 1000$ units

Cycle length $= \dfrac{1000}{200} = 5$ days

Effective lead time $= 7-5 = 2$ days

$\mu_L = 200 \times 100 = 2.00$ units $\quad K_{.02} = 2.06$

$\sigma_L = \sqrt{20^2 \times 2} = 28.28$

$B \geq 28.28 \times 2.06 = 58.27 = 59$ discs

Order 1000 discs whenever the inventory level drops to 459 units.

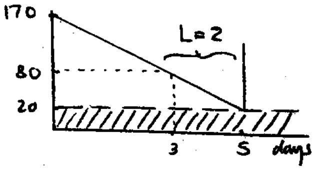
3

Demand/day = N(30, 5)

$h = \$.02$/day/unit, $K = \$30$

(a) $L = \dfrac{80-20}{30}$

$= 2$ days



$\mu_L = 60$ units

$\sigma_L = \sqrt{5^2 \times 2} \simeq 7.07$ units

$P\{\text{demand during } L \geq 80\}$

$= P\left\{z \geq \dfrac{80-60}{7.07}\right\}$

$= P\{z \geq 2.83\}$

$= 1 - .9977 = .0023$

(b) $y = \sqrt{\dfrac{2 \times 30 \times 30}{.02}} = 300$ rolls

Cycle length $= \dfrac{300}{30} = 10$ days

Lead time = 2 days

$\mu_L = 2 \times 30 = 60$ units

$\sigma_L = \sqrt{5^2 \times 2} = 7.07$ units

$K_{.1} = 1.28$

$B \geq 7.07 \times 1.28 \simeq 10$

Order 300 rolls whenever the inventory level drops to 70 rolls.

1

(a) $D/y = \frac{1000}{320} = 3.125$ setups

(b) $\frac{KD}{y} = 100 \times 3.125 = \$\,312.50$ /month

(c) $h\left(\frac{y}{2} + R - E\{x\}\right) = 2\left(\frac{320}{2} + 94 - 50\right)$
$= \$\,408$

(d) $pS = 10 \times .20397 \cong \$\,2.04$

(e) $\int_R^\infty f(x)dx = \int_{94}^{100} \frac{1}{100} dx = \frac{100-94}{100} = .06$

2

$D = 1000$ gallons per month

$K = \$100$, $h = \$2$/gal/month

$p = \$10$/gal.

$f(x) = \frac{1}{50}$, $0 \le x \le 50$, $E\{x\} = 25$

$\hat{y} = \sqrt{\frac{2 \times 1000(100 + 10 \times 25)}{2}} = 591.6$

$\tilde{y} = \frac{PD}{h} = \frac{10 \times 1000}{2} = 5000$

$\tilde{y} > \hat{y} \Rightarrow$ unique solution exists

$S = \int_R^{50} (x-R)\frac{1}{50} dx = \frac{R^2}{100} - R + 25$

$y_i = \sqrt{\frac{2 \times 1000(100 + 10S)}{2}} = \sqrt{100{,}000 + 10{,}000S}$

$\int_{R_i}^{50} \frac{1}{50} dx = \frac{2y_i}{5000} \Rightarrow R_i = 50 - \frac{y_i}{100}$

Iteration 1:

$S = 0$

$y_1 = \sqrt{100{,}000} = 316.23$ gal

$R_1 = 50 - \frac{316.23}{100} = 46.84$ gal

Iteration 2:

$S = \frac{46.84^2}{100} - 46.84 + 25 = .099856$

$y_2 = \sqrt{100{,}000 + 10{,}000 \times .099856} = 317.80$

$R_2 = 50 - \frac{317.80}{100} = 46.82$

Iteration 3:

$S = \frac{46.82^2}{100} - 46.82 + 25 = .101124$

continued...

2 continued

$y_3 = \sqrt{100{,}000 + 10{,}000 \times .101124} = 317.82$

$R_3 = 50 - \frac{317.82}{100} = 46.82$

Optimum solution:

$y^* \cong 318$ gal, $R^* \cong 47$ gal

3

$f(x) = \frac{1}{20}$, $40 \le x \le 60$, $E\{x\} = 50$

$\hat{y} = \sqrt{\frac{2 \times 1000(100 + 10 \times 50)}{2}} = 774.6$ gal

$\tilde{y} = \frac{10 \times 1000}{2} = 5000$ gal

$\tilde{y} > \hat{y} \Rightarrow$ unique solution exists

$S = \int_R^{60} (x-R)\frac{1}{20} dx = \frac{1}{20}\left[\frac{x^2}{2} - Rx\right]_R^{60}$

$= \frac{R^2}{40} - 3R + 90$

$y_i = \sqrt{100{,}000 + 10{,}000S}$

$\int_{R_i}^{60} \frac{1}{20} dx = \frac{2y_i}{10 \times 1000} \Rightarrow R_i = 60 - \frac{y_i}{250}$

Iteration 1:

$S = 0$

$y_1 = \sqrt{100{,}000} = 316.23$ gal

$R_1 = 60 - \frac{316.23}{250} = 58.735$

Iteration 2:

$S = \frac{58.7}{40} - 3 \times 58.735 + 90 = .04$

$y_2 = \sqrt{100{,}000 + 10{,}000 \times .04} = 316.823$

$R_2 = 60 - \frac{316.823}{250} = 58.733$ gal

Optimum solution:

$y^* = 316.85 \simeq 317$ gal.

$R^* = 58.73 \simeq 59$ gal.

R^* in the present model is smaller than R^* in Example because f(x) has a smaller variance, and hence less uncertainty.

4

For the normal distribution, it can be shown that the following approximation holds

$$S = \int_R^\infty (x-R) f(x)\, dx$$
$$\simeq \sqrt{Var\{x\}}\, L(R_s) \qquad (1)$$

where

$Var\{x\}$ = variance of x given $f(x)$

$$R_s = \frac{R - E\{x\}}{\sqrt{Var\{x\}}} \qquad (2)$$

$L(R_s)$ = standard normal loss integral

$$= \int_{R_s}^\infty (z - R_s)\Phi(z)\, dz$$

$\Phi(z)$ is $N(0,1)$. The values of $L(\cdot)$ can be found in standard statistical tables

$$\int_R^\infty f(x)\, dx = \frac{hy}{pD}$$

or

$$\int_R^\infty \Phi(z)\, dz = \frac{hy}{pD} \qquad (3)$$

The steps of the solution algorithm are:

1. Compute first trial
$$y = \sqrt{\frac{2KD}{h}}$$
2. Compute R_s from (3) using the current value of y and the standard normal tables
3. Compute R from (2) using the current value of R_s; that is,
$$R = E\{x\} + R_s\sqrt{var\{x\}}$$

4 continued

If two successive values of R are approximately equal, stop. Otherwise, go to step 4

4. Compute S from (1) using standard normal loss integral tables. Then find
$$y = \sqrt{\frac{2D(K+pS)}{h}}$$
Go to step 2.

1

$$E\{C(y)\} = h\sum_{D=0}^{y}(y-D)f(D) + p\sum_{D=y+1}^{\infty}(D-y)f(D)$$

Consider $E\{C(y)\} \le E\{C(y-1)\}$:

$$E\{C(y-1)\} = h\sum_{D=0}^{y-1}(y-1-D)f(D) + p\sum_{D=y}^{\infty}(D-y+1)f(D)$$

$$= h\sum_{D=0}^{y-1}(y-D)f(D) + p\sum_{D=y}^{\infty}(D-y)f(D) - h\sum_{D=0}^{y-1}f(D) + p\sum_{D=y}^{\infty}f(D) - c$$

$$= E\{C(y)\} + p - (h+p)\sum_{D=0}^{y-1}f(D)$$

Thus,

$$E\{C(y-1)\} - E\{C(y)\} = p - (h+p)P\{D\le y-1\} \ge 0$$

Hence

$$P\{D\le y-1\} \le \frac{p}{p+h}$$

Similarly, it can be shown that

$$P\{D\le y\} \ge \frac{p}{p+h}$$

Thus, y^* must satisfy

$$P\{D\le y^*-1\} \le \frac{p}{p+h} \le P\{D\le y^*\}$$

2

$f(D) = \frac{1}{5}$, $10 \le D \le 15$

$\int_{10}^{y} f(D)dD \le .1$:

$$\int_{10}^{y}\frac{1}{5}dD = \frac{y-10}{5} \le .1 \Rightarrow y \le 10.5$$

$\int_{y}^{15} f(D)dD \le .1$:

$$\int_{y}^{15}\frac{1}{5}dD = \frac{15-y}{5} \le .1 \Rightarrow y \ge 14.5$$

The two conditions cannot be satisfied simultaneously.

3

$$q = \frac{p}{p+h} = \frac{p}{p+1}$$

y	0	1	2	3	4	5	6
$P\{D\le y\}$	.05	.15	.25	.45	.7	.85	.9

↑ $y = 4$

From the CDF,

$P\{D\le 4-1\} = .45$

$P\{D\le 4\} = .7$

Thus, $.45 \le \frac{p}{p+1} \le .7$

or $.43 \le p \le .82$

4

Maximize expected revenue.

$$E\{\text{revenue}\} = -10y + \int_{200}^{y} 25D\,f(D)dD + \int_{y}^{250} 25y\,f(D)dD$$

$$= -10y + \frac{25D^2}{100}\Big|_{200}^{y} + \frac{25y}{50}D\Big|_{y}^{250}$$

$$= -.25y^2 + 115y - 10,000$$

$$\frac{\partial E\{\text{revenue}\}}{\partial y} = -.5y + 115 = 0$$

$y = 230$ copies

5

$$E\{\text{revenue}\} = -7y + \int_{90}^{y}[25D + 5(y-D)]f(D)dD + \int_{y}^{150} 25y\,f(D)dD$$

$$= -\frac{y^2}{6} + 48y - 1350$$

$$\frac{\partial E\{\text{revenue}\}}{\partial y} = \frac{-y}{3} + 48$$

$y = 144$ donuts

Decision: Stock 12 dozens

Set 14.2a

6

Use continuous pdf as an approximation

$$E\{revenue\} =$$

$$-50y + \int_{20}^{y} [110D + 55(y-D)] f(D)\,dD + \int_{y}^{30} 110y\, f(D)\,dD$$

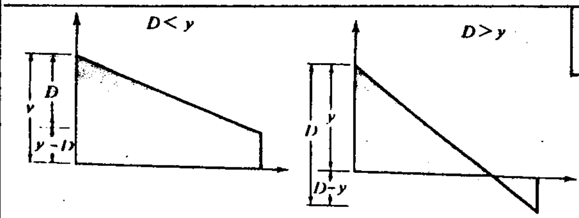
$$= -50y + \frac{1}{10}\left[55yD + \frac{55D^2}{2}\right]_{20}^{y} + 110y\left[\frac{D}{10}\right]_{y}^{30}$$

$$= -2.75y^2 + 175y - 1100$$

$$\frac{\partial E\{revenue\}}{\partial y} = -5.5y + 175 = 0$$

$$y \simeq 32 \text{ coats}$$

7



Average holding inventory $= y - \frac{D}{2}$ Average holding inventory $= \frac{y^2}{2D}$

Average shortage inventory $= 0$ Average shortage inventory $= \frac{(D-y)^2}{2D}$

$$E\{C(y)\} = c(y-x) + h\left\{\int_0^y \left(y-\frac{D}{2}\right) f(D)\,dD + \int_y^\infty \frac{y^2}{2D} f(D)\,dD\right\} + p\int_y^\infty \frac{(D-y)^2}{2D} f(D)\,dD$$

$$\frac{\partial E\{\}}{\partial y} = c + \left(\int_0^y f(D)\,dD + \int_y^\infty \frac{y}{D} f(D)\,dD\right) - p\int_y^\infty \left(\frac{D-y}{D}\right) f(D)\,dD = 0$$

$$\int_0^{y^*} f(D)\,dD + y^* \int_{y^*}^\infty \frac{f(D)}{D}\,dD = \frac{p-c}{p+h}$$

8

$$f(D) = \frac{1}{100}, \quad 0 \le D \le 100$$

$$\int_0^y f(D)\,dD + y\int_y^{100} \frac{f(D)}{D}\,dD = \frac{p-c}{p+h}$$

$$\int_0^y \frac{1}{100}\,dD + y\int_y^{100} \frac{1}{100D}\,dD = \frac{p-c}{p+h}$$

$$\frac{y}{100} + \frac{y}{100}(\ln 100 - \ln y) = \frac{p-c}{p+h}$$

$$.056y - .01y\ln y = \frac{45-30}{45+25} = .2143$$

Trial and error yield $y^* \approx 5.5$ units

1

$$E\{C(s)\} = K + E\{C(S)\}$$

$$.25s^2 - 4.5s + 22.5 = 5 + .25S^2 - 4.5S + 22.5$$

$$.25s^2 - 4.5s + 15.25 = 0 \quad (\text{for } S = 9)$$

Solution: $s = (4.53 \text{ or } 13.47)$

Policy: If $x < 4.53$, order $9 - x$

$x \geq 4.53$, do not order

2

$$E\{R(y)\} = -c(y-x) + \int_0^y [rD - h(y-D)] f(D)dD + \int_y^\infty [ry - p(D-y)] f(D)dD$$

$$\frac{\partial E\{\ \}}{\partial y} = -c - \int_0^y h f(D)dD + ryf(D) + \int_y^\infty (r+p) f(D)dD - ryf(D) = 0$$

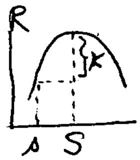
Thus,

$$\int_0^{y^*} f(D)dD = \frac{r+p-c}{r+p-h}$$

In the presence of setup cost, we have an s-S policy. Define s such that

$$E\{R(s)\} = E\{R(S)\} - K$$

For the numeric problem,



$$E\{R(y)\} = .4y^2 + 5y - 20 - 2x$$

$$\int_0^S f(D)dD = \frac{3+4-2}{3+4-1} = .625$$

Thus, $S = 6.25$

Next, $-.4s^2 + 5s - 5.625 = 0$

Thus, $s = 1.25$

Policy:

If $x < 1.25$, order $6.25 - x$

$x \geq 1.25$, do not order

3

$$-\frac{s^2}{6} + 4s - 1350 = -10 - \frac{144^2}{6} + 48 \times 144 - 1350$$

Thus,

$$s^2 - 288s + 20676 = 0$$

$$s = \begin{cases} 136.25 \\ 151.25 \end{cases}$$

Optimal policy

If $x < 136$, order $144 - x$

$x \geq 136$, do not order

1

$$L(y_i) = \int_0^{y_i} (rD - h(y_i - D)) f(D)\,dD + \int_{y_i}^{\infty} (ry_i + (\alpha r' - p)(D - y_i)) f(D)\,dD \quad i = 1,2$$

where
$$r' = \begin{cases} r & i = 1 \\ r - c & i = 2 \end{cases}$$

$$g_2(x_2) = \max_{y_2 \ge x_2} \{-c(y_2 - x_2) + L(y_2)\}$$

$$g_1(x_1) = \max_{y_1 \ge x_1} \{-c(y_1 - x_1) + L(y_1) + \alpha E\{g_2(y_1 - D)\}\}$$

For period 2:

$$\frac{\partial f_2(y_2 | x_2)}{\partial y_2} = -c + L'(y_2^*) = 0$$

or
$$\int_0^{y_2^*} f(D)\,dD = \frac{r + p - c - \alpha(r - c)}{r + p + h - \alpha(r - c)}$$

$$g_2(y_1 - D) = \begin{cases} L_2(y_1 - D), & D \le y_1 - y_2^* \\ -c(y_2^* - y_1 + D) + L(y_2^*), & D \ge y_1 - y_2^* \end{cases}$$

$$E\{\ (y_1 - D)\} = \int_0^{y_1 - y_2^*} L_2(y_1 - D) f(D)\,dD + \int_{y_1 - y_2^*}^{\infty} (-c(y_2^* - y_1 + D) + L(y_2^*)) f(D)\,dD$$

This, when substituted in the expression for $g_1(x_1)$, will yield total expected profit in terms of y_1. Hence, the value of y_1^* can be obtained.

In terms of the given numerical example, we have

$$\frac{1}{10} \int_0^{y_2^*} dD = \frac{2 + 3 - 1 - .8(2 - 1)}{2 + 3 + .1 - .8(2 - 1)} = .75$$

Thus, $y_2^* = 7.5$

$$L(z) = \frac{1}{10} \left\{ \int_0^z (2D - .1(z - D))\,dD + \int_z^{10} (2z + (.8r' - 3)(D - z))\,dD \right\}$$

continued...

1 continued

$$= (.04r' - .255)z^2 + (5 - .8r')z + (4r' - 15)$$

Hence

$$L(y_2) = [.04(2-1) - .255]y_2^2 + [5 - .8(2-1)]y_2 + [4(2-1) - 15]$$

$$= -.215y_2^2 + 4.2y_2 - 11$$

$$L(y_2^*) = L(7.5) = 8.4$$

$$g_2(y_1 - D) = \begin{cases} -.215(y_1 - D)^2 + 4.2(y_1 - D) - 11, & D \le y_1 - 7.5 \\ .9 - y + D, & D \ge y_1 - 7.5 \end{cases}$$

$$E\{g_2(y_1 - D)\} = \frac{1}{10} \left\{ \int_0^{y_1 - 7.5} [-.215(y_1 - D)^2 + 4.2(y_1 - D) - 11]\,dD + \int_{y_1 - 7.5}^{10} (.9 + y - D)\,dD \right\}$$

$$= \frac{1}{10}(-.072y_1^3 + 2.115y_1^2 - 11y_1 - 5 - y_1^2 - 5.4y_1 - 19.625)$$

$$= \frac{1}{10}(-.072y_1^3 + 1.115y_1^2 - 16.4y_1 - 24.625)$$

$$L(y_1) = (.04 \times 2 - .255)y_1^2 + (5 - .8 \times 2)y_1 + (4 \times 2 - 15)$$

$$= -.175y_1^2 + 3.4y_1 - 7$$

$$g_1(x_1) = \max_{y_1 \ge x_1} \{-1(y_1 - x_1) - .175y_1^2 + 3.4y_1 - 7 + \frac{.8}{10}(-.07y_1^3 + 1.115y_1^2 - 16.4y_1 - 24.625)\}$$

$$= \max_{y_1 \ge x_1} \{-.00576y_1^3 - .075y_1^2 + .89y_1 - 8.97 + x_1\}$$

$$\frac{\partial\{\cdot\}}{\partial y_1} = -.01728y_1^2 - .15y_1 + .89 = 0$$

$$y_1^* = 9.02$$

continued...

1 continued

<u>optimal policy</u>:

$$\text{Period 1}\begin{cases}\text{order } 9.02-x_1, & x_1\le 9.02\\ \text{order } 0, & x_1\ge 9.02\end{cases}$$

$$\text{Period 2}\begin{cases}\text{order } 7.5-x_2, & x_2\le 7.5\\ \text{order } 0, & x_2\ge 7.5\end{cases}$$

For the <u>infinite</u> model:

$$\frac{1}{10}\int_0^{y^*} dD=\frac{3+.2(2-1)}{3+.1+.2\times 2}=.915$$

$$y_1^*=9.15>y_2^*>y_1^*$$

2

$$\int_0^{y^*} f(D)\,dD=.08\int_0^{y^*} D\,dD$$

$$=.04\,y^{*2}$$

Thus,

$$.04\,y^{*2}=\frac{p+(1-\alpha)(r-c)}{p+h+(1-\alpha)r}$$

$$=\frac{10+.1\times 2}{10+1+.1\times 10}=.85$$

Thus, $y^*=4.61$

<u>Policy</u>:

order $4.61-x$, if $x\le 4.61$

order 0, if $x\ge 4.61$

3

$$g(x)=\min_{y\ge x}\Big\{c(y-x)+h\int_0^y (y-D)^2 f(D)\,dD+p\int_y^\infty (D-y)^2 f(D)\,dD+\alpha\int_0^\infty g(y-D)\,f(D)\,dD\Big\}$$

$$\frac{\partial\{\cdot\}}{\partial y}=c+2h\int_0^y (y-D)f(D)\,dD-2p\int_y^\infty (D-y)f(D)\,dD+\alpha E\{g'(y-D)\}$$

Continued...

3 continued

Since this is a cost function, $g'(y-D)=-c$.

Now, $\frac{\partial\{\cdot\}}{\partial y}=0$ yields,

$$\Big\{(1-\alpha)c+2hy^*\int_0^{y^*} f(D)\,dD-2h\int_0^{y^*} D f(D)\,dD+2py^*\Big(1-\int_0^{y^*} f(D)\,dD\Big)-2pE\{D\}+2p\int_0^{y} D f(D)\,dD\Big\}=0$$

This simplifies to

$$(h-p)\Big\{y^*\int_0^{y^*} f(D)\,dD-\int_0^{y^*} Df(D)\,dD\Big\}+py^*=\frac{2pE\{D\}-(1-\alpha)c}{2} \quad (1)$$

or

$$y^*\Big\{\frac{1}{h-p}+\int_0^{y^*} f(D)\,dD-\int_0^{y^*} Df(D)\,dD=\frac{2pE\{D\}-(1-\alpha)c}{2(h-p)}$$

y^* can be determined from the last equation. When $h=p$, (1) yields

$$y^*=\frac{2pE\{D\}-(1-\alpha)c}{2p}$$

This result is independent of $f(D)$ except in so far as $E\{D\}$ is concerned.