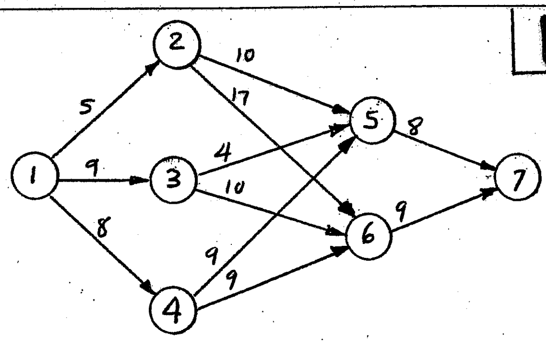
CHAPTER 10

Deterministic Dynamic Programming

1



Stage 1:

To city	shortest distance	from city
2	5	1
3	9	1
4	8	1

Stage 2:

To city	shortest distance	from city
5	$\min\{5+10, \underline{9+4}, 8+9\} = 13$	3
6	$\min\{5+17, 9+10, \underline{8+9}\} = 17$	4

Stage 3:

To city	shortest distance	from city
7	$\min\{\underline{13+8}, 17+9\} = 21$	5

Optimum solution: shortest distance = 21 miles

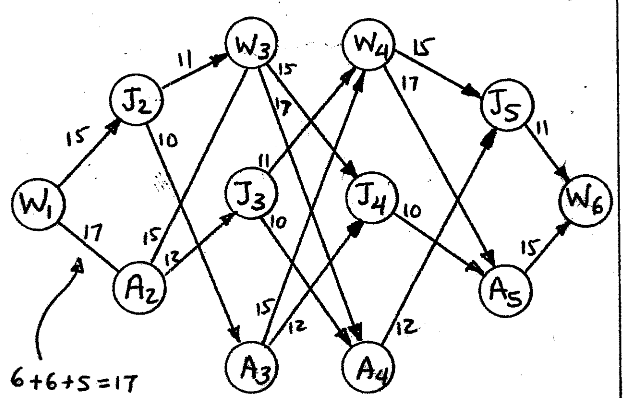
Route: $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$

2

Define node N_i as:

$N \equiv W, J,$ and A for Washington, Jefferson, and Adams

i = day on which N is visited



Stage 1:

To	Longest distance	From
J_2	15	W_1
A_2	17	W_1

Stage 2:

To	Longest distance	From
W_3	$\max\{15+11, \underline{17+15}\} = 32$	A_2
J_3	$17+12 = 29$	A_2
A_3	$15+10 = 25$	J_2

Stage 3:

To	Longest distance	From
W_4	$\max\{\underline{29+11}, \underline{25+15}\} = 40$	J_3 or A_3
J_4	$\max\{\underline{32+15}, 25+12\} = 47$	W_3
A_4	$\max\{\underline{32+17}, 29+10\} = 49$	W_3

Stage 4:

To	Longest distance	From
J_5	$\max\{40+15, \underline{49+12}\} = 61$	A_4
A_5	$\max\{\underline{40+17}, \underline{47+10}\} = 57$	W_4 or J_4

Stage 5:

To	Longest distance	From
W_6	$\max\{\underline{61+11}, \underline{57+15}\} = 72$	J_5 or A_5

Solution: 72 miles or 14.4 miles/day

To determine the optimum routes, start from stage 5.

$W_6 \rightarrow J_5 \rightarrow A_4 \rightarrow W_3 \rightarrow A_2 \rightarrow W_1$

$W_6 \rightarrow A_5 \rightarrow J_4 \rightarrow W_3 \rightarrow A_2 \rightarrow W_1$

$W_6 \rightarrow A_5 \rightarrow W_4 \rightarrow A_3 \rightarrow J_2 \rightarrow W_1$

$W_6 \rightarrow A_5 \rightarrow W_4 \rightarrow J_3 \rightarrow A_2 \rightarrow W_1$

The routes can be summarized as:

Day	1	2	3	4	5
Route 1	W	A	W	A	J
Route 2	W	A	W	J	A
Route 3	W	J	A	W	A
Route 4	W	A	J	W	A

All routes visit J once and each of W and A twice

continued...

1

$$f_i(x_i) = \min_{\substack{\text{feasible}\\(x_i,\,x_{i+1})\\\text{routes}}} \{d(x_i, x_{i+1}) + f_{i+1}(x_{i+1})\},\ i=1,2$$

Stage 3:

$$f_3(x_3) = \min_{\substack{\text{feasible}\\(x_3,\,x_4)}} \{d(x_3, x_4)\}$$

x_3	$d(x_3, x_4)$	Optimum sol	
	$x_4 = 7$	$f_3(x_3)$	x_4^*
5	8	8	7
6	9	9	7

Stage 2:

$$f_2(x_2) = \min_{\substack{\text{feasible}\\(x_2,\,x_3)}} \{d(x_2, x_3) + f_3(x_3)\}$$

x_2	$d(x_2, x_3) + f_3(x_3)$		Opt. Sol.	
	$x_3 = 5$	$x_3 = 6$	$f_2(x_2)$	x_3^*
2	10+8 = 18	17+9 = 26	18	5
3	4+8 = 12	10+9 = 19	12	5
4	9+8 = 17	9+9 = 18	17	5

Stage 1:

$$f_1(x_1) = \min_{\substack{\text{feasible}\\(x_1,\,x_2)}} \{d(x_1, x_2) + f_2(x_2)\}$$

x_1	$d(x_1, x_2) + f_2(x_2)$			Opt. Sol.	
	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_1(x_1)$	x_2^*
1	5+18 = 23	9+12 = 21	8+17 = 25	21	3

Solution: distance = 21
route = 1 − 3 − 5 − 7

2

$$f_i(x_i) = \max_{\substack{\text{feasible}\\(x_i,\,x_{i+1})\\\text{routes}}} \{d(x_i, x_{i+1}) + f_{i+1}(x_{i+1})\}$$

$i = 1, 2, 3, 4$

Stage 5: $f_5 = \max_{\substack{\text{feasible}\\(x_5,\,x_6)}} \{d(x_5, x_6)\}$

x_5	$d(x_5, x_6)$	Opt. Sol.	
	$x_6 = W_6$	$f_5(x_5)$	x_6^*
J_5	11	11	W_6
A_5	15	15	W_6

continued...

Stage 4:

x_4	$d(x_4, x_5) + f_5(x_5)$		Opt. Sol.	
	$x_5 = J_5$	$x_5 = A_5$	$f_4(x_4)$	x_5^*
W_4	15+11 = 26	17+15 = 32	32	A_5
J_4	—	10+15 = 25	25	A_5
A_4	12+11 = 23	—	23	J_5

Stage 3:

x_3	$d(x_3, x_4) + f_4(x_4)$			Opt. Sol.	
	$x_4 = W_4$	$x_4 = J_4$	$x_4 = A_4$	$f_3(x_3)$	x_4^*
W_3	—	15+25 = 40	17+23 = 40	40	J_4, A_4
J_3	11+32 = 43	—	10+23 = 33	43	W_4
A_3	15+32 = 47	17+25 = 42	—	47	W_4

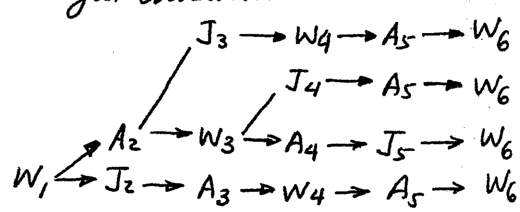
Stage 2:

x_2	$d(x_2, x_3) + f_3(x_3)$			Opt. Sol.	
	$x_3 = W_3$	$x_3 = J_3$	$x_3 = A_3$	$f_2(x_2)$	x_3^*
J_2	11+40 = 51	—	10+47 = 57	57	A_3
A_2	15+40 = 55	12+43 = 55	—	55	W_3, J_3

Stage 1:

x_1	$d(x_1, x_2) + f_2(x_2)$		Opt. Sol.	
	$x_2 = J_2$	$x_2 = A_2$	$f_1(x_1)$	x_2^*
W_1	15+57 = 72	17+55 = 72	72	A_2, J_2

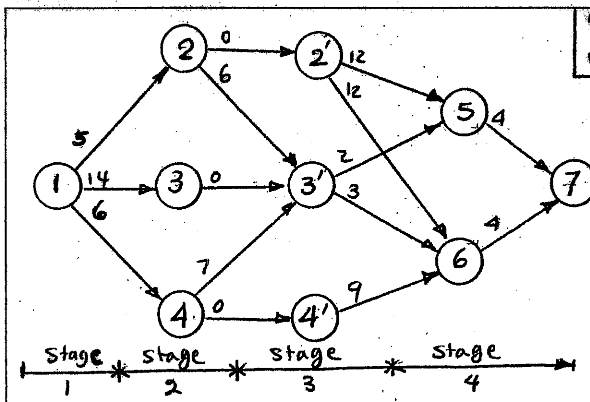
Solution:
Longest distance = 72 miles



Routes:

	Day 1	2	3	4	5
Route 1:	W	A	J	W	A
Route 2:	W	A	W	J	A
Route 4:	W	A	W	A	J
Routes:	W	J	A	W	A

3



$$f_i(x_i) = \min_{\substack{\text{feasible}\\ (x_i,\, x_{i+1})\\ \text{routes}}} \left\{ d(x_i, x_{i+1}) + f_{i+1}(x_{i+1}) \right\}$$

$i = 1, 2, 3, 4$

Stage 4:

x_4	$d(x_4, x_5)$: $x_5 = 7$	Opt. Sol. $f_4(x_4)$	x_5^*
5	4	4	7
6	4	4	7

Stage 3:

x_3	$d(x_3, x_4) + f_4(x_4)$: $x_4 = 5$	$x_4 = 6$	Opt. Sol. f_4	x_4^*
2'	12 + 4 = 16	12 + 4 = 16	16	5, 6
3'	2 + 4 = 6	3 + 4 = 7	6	5
4'	—	9 + 4 = 13	13	6

Stage 2:

x_2	$d(x_2, x_3) + f_3(x_3)$: $x_3 = 2'$	$x_3 = 3'$	$x_3 = 4'$	Opt. Sol. f_2	x_3^*
2	0 + 16 = 16	6 + 6 = 12	—	12	3'
3	—	0 + 6 = 6	—	6	3'
4	—	7 + 6 = 13	0 + 13 = 13	13	3', 4'

Stage 1:

x_1	$d(x_1, x_2) + f_2(x_2)$: $x_2 = 2$	$x_2 = 3$	$x_2 = 4$	Opt. Sol. $f_1(x_1)$	x_2^*
1	5 + 12 = 17	14 + 6 = 20	6 + 13 = 19	17	2

Solution:

Distance = 17

Route: 1 - 2 - 3' - 5 - 7

Since 3 is the same as 3', the optimal route is

1 - 2 - 3 - 5 - 7.

continued...

1

$(x_1=3) \rightarrow m_1=0 \rightarrow (x_2=3) \rightarrow m_2=1 \rightarrow (x_3=3-3=0) \rightarrow m_3=0.$

Solution:

$(m_1, m_2, m_3) = (0, 3, 0)$

Revenue $= 47$

2

(a)

Stage 3: $\max m_3 = [\frac{6}{2}] = 3$

	$40m_3$				Opt. Sol.	
x_3	$m_3=0$	$m_3=1$	$m_3=2$	$m_3=3$	f_3	m_3^*
0	0	–	–	–	0	0
1	0	–	–	–	0	0
2	0	40	–	–	40	1
3	0	40	–	–	40	1
4	0	40	80	–	80	2
5	0	40	80	–	80	2
6	0	40	80	120	120	3

Stage 2: $\max m_2 = [\frac{6}{1}] = 6$

	$20m_2 + f_3(x_2 - m_2)$							Opt. Sol.	
x_2	$m_2=0$	1	2	3	4	5	6	f_2	m_2^*
0	0	–	–	–	–	–	–	0	0
1	0	20	–	–	–	–	–	20	1
2	40	20	40	–	–	–	–	40	0,2
3	40	60	40	60	–	–	–	60	1,3
4	80	60	80	60	80	–	–	80	2,4,0
5	80	100	80	100	80	100	–	100	1,3,5
6	120	100	120	100	120	100	120	120	0,2,4,6

Stage 1: $\max m_1 = [\frac{6}{4}] = 1$

	$70m_1 + f_2(x_1 - 4m_1)$		Opt. Sol.	
x_1	$m_1=0$	$m_1=1$	f_1	m_1^*
6	$0+120=120$	$70+40=110$	120	0

Optimum solutions:

$(m_1, m_2, m_3) = (0, 0, 3)$
$= (0, 2, 2)$
$= (0, 4, 1)$
$= (0, 6, 0)$

Value $= 120$

continued...

(b) Stage 3: $\max m_3 = [\frac{4}{3}] = 1$

	$80m_3$		Opt. Sol.	
x_3	$m_3=0$	$m_3=1$	f_3	m_3^*
0	0	–	0	0
1	0	–	0	0
2	0	–	0	0
3	0	80	80	1
4	0	80	80	1

Stage 2: $\max m_2 = [4/2] = 2$

	$60m_2 + f_3(x_2 - 2m_2)$			Opt. Sol.	
x_2	$m_2=0$	$m_2=1$	$m_2=2$	f_2	m_2^*
0	0	–	–	0	0
1	0	–	–	0	0
2	0	60	–	60	1
3	80	60	–	80	0
4	80	60	120	120	2

Stage 1: $\max m_1 = [4/1] = 4$

	$30m_1 + f_2(x_1 - m_1)$					Opt. Sol.	
x_1	$m_1=0$	1	2	3	4	f_1	m_1^*
4	120	90	120	90	120	120	0,2,4

Alternative optima:

$$\left.\begin{aligned}(m_1, m_2, m_3) &= (0, 2, 0)\\ &= (2, 1, 0)\\ &= (4, 0, 0)\end{aligned}\right\} \text{value} = 120$$

3

Stage 3: $w_3 = 1,\ r_3 = 14,\ K_3 = -4$

Dynamic Programming (Backward) Knapsack Model (with Setup Cost)

Input Data and Stage Calculations

Number of stages, N=	3	Res. Cap. W=	4			
Current stage= 3	w3=	1	r3=	14	K3=	-4
Are m3-values correct?	yes	yes	yes	yes	yes	
m3=	0	1	2	3	4	Stage Optimum Solution
r3*m3=	0	10	24	38	52	
w3*m3=	0	1	2	3	4	f3, m3
x3= 0	0	111111	111111	111111	111111	0, 0
x3= 1	0	10	111111	111111	111111	10, 1
x3= 2	0	10	24	111111	111111	24, 2
x3= 3	0	10	24	38	111111	38, 3
x3= 4	0	10	24	38	52	52, 4

Optimum Solution Summary

x	f	m
Stage 3		
0	0	0
1	10	1
2	24	2
3	38	3
4	52	4

Stage 2: $w_2 = 3,\ r_2 = 47,\ K_2 = -15$

Dynamic Programming (Backward) Knapsack Model (with Setup Cost)

Input Data and Stage Calculations

Number of stages, N=	3	Res. Cap. W=	4			
Current stage= 2	w2=	3	r2=	47	K2=	-15
Are m2-values correct?	yes	yes				
m2=	0	1				Stage Optimum Solution
r2*m2=	0	32				
Stage3 f3 / w2*m2=	0	3				f2, m2
0 / x2= 0	0	111111				0, 0
10 / x2= 1	10	111111				10, 0
24 / x2= 2	24	111111				24, 0
38 / x2= 3	38	32				38, 0
52 / x2= 4	52	42				52, 0

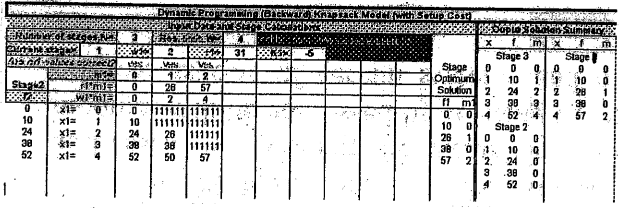
Optimum Solution Summary

x	f	m
Stage 3		
0	0	0
1	10	1
2	24	2
3	38	3
4	52	4
Stage 2		
0	0	0
1	10	0
2	24	0
3	38	0
4	52	0

continued...

Set 10.3a

Stage 1: $w_1 = 2,\ r_1 = 31,\ K_1 = -5$



Optimum solution:

$_1 = 4 \rightarrow (m_1 = 2) \rightarrow x_2 = (4 - 2\times 2 = 0) \rightarrow$
$(m_2 = 0) \rightarrow x_3 = 0 \rightarrow m_3 = 0$

value = 57

4

x_1 = number of food items
x_2 = number of first-aid items
x_3 = number of cloth pieces

Maximize $z = 3x_1 + 4x_2 + 5x_3$

subject to

$$x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_3 \le 3$$

$$x_1 \ge 1,\ 1 \le x_2 \le 2,\ x_3 \ge 1$$

Define the state y_i as the volume assigned to items $i, i+1, \ldots,$ and n

Recursive equations:

$$f_3(y_3) = \max_{x_3 = 1, \ldots, [\frac{y_3}{2}]} \{5x_3\}$$

$$f_2(y_2) = \max_{x_2 = 1, \ldots, \min[\frac{y_2}{4}, 2]} \left\{4x_2 + f_3\left(y_2 - \frac{x_2}{4}\right)\right\}$$

$$f_1(y_1) = \max_{x_1 = 1, \ldots, y_1} \{3x_1 + f_2(y_1 - x_1)\}$$

Stage 3: (Note: $[a, b) \equiv a \le y < b$)

y_3	$5x_3$: $x_3=1$	2	3	4	5	6	Opt. Sol. f_3	x_3^*
[.5, 1)	5	—	—	—	—	—	5	1
[1, 1.5)	5	10	—	—	—	—	10	2
[1.5, 2)	5	10	15	—	—	—	15	3
[2, 2.5)	5	10	15	20	—	—	20	4
[2.5, 3)	5	10	15	20	25	—	25	5
3	5	10	15	20	25	30	30	6

continued...

Stage 2:

y_2	$4x_2 + f_3(y_2 - x_2/4)$: $x_2 = 1$	$x_2 = 2$	Opt. Sol. f_2	x_2^*
.25	—	—	—	—
.50	—	—	—	—
.75	4+5 = 9	—	9	1
1.00	4+5 = 9	8+5 = 13	13	2
1.25	4+10 = 14	8+5 = 13	14	1
1.50	4+10 = 14	8+10 = 18	18	2
1.75	4+15 = 19	8+10 = 18	19	1
2.00	4+15 = 19	8+15 = 23	23	2
2.25	4+20 = 24	8+15 = 23	24	1
2.50	4+20 = 24	8+20 = 28	28	2
2.75	4+25 = 29	8+20 = 28	29	1
3.00	4+25 = 29	8+25 = 33	33	2

Stage 1:

y_1	$3x_1 + f_2(y_1 - x_1)$: $x_1 = 1$	$x_1 = 2$	Opt. Sol. f_1	x_1^*
3	3 + 23 = 26	6+13 = 19	26	1

Solution:

$(y_1 = 3) \rightarrow x_1 = 1 \rightarrow (y_2 = 3 - 1 = 2) \rightarrow x_2 = 2 \rightarrow$
$(y_1 = 2 - .5 = 1.5) \rightarrow x_3 = 3$

Revenue = 26

$(x_1, x_2, x_3) = (1, 2, 3)$

5

x_i = number of courses allocated to departments $i, i+1, \ldots,$ and n.

$m_i = 1, 2, \ldots, 7$, $i = 1, 2, 3, 4$

$x_4 = 1, 2, \ldots, 7$ $\quad x_2 = 3, 4, \ldots, 9$

$x_3 = 2, 3, \ldots, 8$ $\quad x_1 = 4, 5, \ldots, 10$

$$f_i(x_i) = \max_{m_i} \{v(m_i) + f_{i+1}(x_i - m_i)\}$$

where $v(m_i)$ = value of m_i courses

continued...

Stage 4:

x_4	$v(m_4)$: $m_4=1$	2	3	4	5	6	7	Opt. Sol. f_4	m_4^*
1	10							10	1
2		20						20	2
3			30					30	3
4				40				40	4
5					50			50	5
6						60		60	6
7							70	70	7

Stage 3:

x_3	$v(m_3)+f_4(x_3-m_3)$: $m_3=1$	2	3	4	5	6	7	Opt. Sol. f_3	m_3^*
2	50	—	—	—	—	—	—	50	1
3	60	70	—	—	—	—	—	70	2
4	70	80	90	—	—	—	—	90	3
5	80	90	100	110	—	—	—	110	4
6	90	100	110	120	110	—	—	120	4
7	100	110	120	130	120	110	—	130	4
8	110	120	130	140	130	120	110	140	4

Stage 2:

x_2	$v(m_2)+f_3(x_2-m_2)$: $m_2=1$	2	3	4	5	6	7	Opt. Sol. f_2	m_2^*
3	70	—	—	—	—	—	—	70	1
4	90	120	—	—	—	—	—	120	2
5	110	140	140	—	—	—	—	140	2,3
6	130	160	160	150	—	—	—	160	2,3
7	140	180	180	170	150	—	—	180	2,3
8	150	190	200	190	170	150	—	200	3
9	160	200	210	210	190	170	150	210	3,4

stage 1:

x_1	$v(m_1)+f_2(x_1-m_1)$: $m_1=1$	2	3	4	5	6	7	Opt. Sol. f_1	m_1^*
10	235	250	240	240	240	220	170	250	3

Solution: $m_1=2$, $m_2=3$, $m_3=4$, $m_4=1$

Total number of points = 250

6

x_1 = number of (2') rows of tomato

x_2 = number of (3') rows of bean

x_3 = number of (2') rows of corn

Maximize $z = 10x_1 + 3x_2 + 7x_3$

Subject to

$$2x_1 + 3x_2 + 2x_3 \le 10$$

$$0 \le x_1 \le 2, \quad x_2 \ge 1, \quad x_3 \ge 0$$

continued...

Define the states as:

y_3 = number of width-feet assigned to corn

y_2 = number of width-feet assigned to corn and bean

y_1 = number of width-feet assigned to corn, bean, and tomato

$y_1 = 10, \quad y_2 = 2, 3, \dots, 10, \quad y_3 = 0, 1, \dots, 7$

Stage 3: $f_3(y_3) = \max_{2x_3 \le y_3} \{7x_3\}$

y_3	$7x_3$: $x_3=0$	1	2	3	4	5	Opt. Sol. f_3	x_3^*
0	0	—	—	—	—	—	0	0
1	0	—	—	—	—	—	0	0
2	0	7	—	—	—	—	7	1
3	0	7	—	—	—	—	7	1
4	0	7	14	—	—	—	14	2
5	0	7	14	—	—	—	14	2
6	0	7	14	21	—	—	21	3
7	0	7	14	21	—	—	21	3

Stage 2: $f_2(y_2) = \max_{\substack{3x_2 \le y_2 \\ x_2 \ge 1}} \{3x_2 + f_3(y_2 - 3x_2)\}$

y_2	$3x_2 + f_3(y_2-3x_2)$: $x_2=1$	$x_2=2$	$x_2=3$	Opt. Sol. f_2	x_2^*
3	3 + 0 = 3	—	—	3	1
4	3 + 0 = 3	—	—	3	1
5	3 + 7 = 10	—	—	10	1
6	3 + 7 = 10	6 + 0 = 6	—	10	1
7	3 + 14 = 17	6 + 0 = 6	—	17	1
8	3 + 14 = 17	6 + 7 = 13	—	17	1
9	3 + 21 = 24	6 + 7 = 13	9 + 0 = 9	24	1
10	3 + 21 = 24	6 + 14 = 20	9 + 0 = 9	24	1

Stage 1: $f_1(y_1) = \max_{\substack{2x_1 \le y_1 \\ x_1 \le 2}} \{10x_1 + f_2(y_1 - 2x_1)\}$

y_1	$10x_1 + f_2(y_1-2x_1)$: $x_1=0$	$x_1=1$	$x_1=2$	Opt. Sol. f_1	x_1^*
10	0+24=24	10+17=27	20+10=30	30	2

continued...

Solution:

$(y_1=10) \rightarrow x_1=2 \rightarrow (y_2=10-4=6) \rightarrow x_2=1$
$\rightarrow (y_3=6-3=3) \rightarrow x_1=1$

Plant 2 rows of tomatoes, 1 row of beans, and 1 row of corn.

7

$x_j = 1$ if application j is selected, and 0 if otherwise.

maximize $z = 78x_1+64x_2+68x_3+62x_4+85x_5$

subject to

$$7x_1+4x_2+6x_3+5x_4+8x_5 \le 23$$
$$x_j = (0,1), \quad j=1,2,\dots,5$$

Stage 5: $f_5(y_5) = \max_{8x_5 \le y_5} \{85x_5\}$

y_5	$85x_5$: $x_5=0$	$x_5=1$	Opt. Sol. f_5	x_5^*
0	0	—	0	0
1	0	—	0	0
⋮	⋮	⋮	⋮	⋮
7	0	—	0	0
8	0	85	85	1
9	0	85	85	1
⋮	⋮	⋮	⋮	⋮
23	0	85	85	1

Stage 4:

$$f_4(y_4) = \max_{5x_4 \le y_4} \{62x_4 + f_5(y_4-5x_4)\}$$

y_4	$62x_4+f_5(y_4-5x_4)$: $x_4=0$	$x_4=1$	Opt. Sol. f_4	x_4^*
0	0+0 = 0	—	0	0
1	0+0 = 0	—	0	0
⋮	⋮	⋮	⋮	⋮
5	0+0 = 0	62+0 = 62	62	1
6	0+0 = 0	62+0 = 62	62	1
7	0+0 = 0	62+0 = 62	62	1
8	0+85 = 85	62+0 = 62	85	0
⋮	⋮	⋮	⋮	⋮
12	0+85 = 85	62+0 = 62	85	0
13	0+85 = 85	62+85 = 147	147	1
14	0+85 = 85	62+85 = 147	147	1
⋮	⋮	⋮	⋮	⋮
23	0+85 = 85	62+85 = 147	147	1

Stage 3: $f_3(y_3) = \max_{6x_3 \le y_3} \{68x_3 + f_4(y_3-6x_3)\}$

y_3	$68x_3+f_4(y_3-6x_3)$: $x_3=0$	$x_3=1$	Opt. Sol. f_3	x_3^*
0	0 + 0 = 0	—	0	0
1	0 + 0 = 0	—	0	0
2	0 + 0 = 0	—	0	0
3	0 + 0 = 0	—	0	0
4	0 + 0 = 0	—	0	0
5	0 + 62 = 62	—	62	0
6	0 + 62 = 62	68 + 0 = 68	68	1
7	0 + 62 = 62	68 + 0 = 68	68	1
8	0 + 85 = 85	68 + 0 = 68	85	0
9	0 + 85 = 85	68 + 0 = 68	85	0
10	0 + 85 = 85	68 + 0 = 68	85	0
11	0 + 85 = 85	68 + 62 = 130	130	1
12	0 + 85 = 85	68 + 62 = 130	130	1
13	0 + 147 = 147	68 + 62 = 130	147	0
14	0 + 147 = 147	68 + 85 = 153	153	1
15	0 + 147 = 147	68 + 85 = 153	153	1
16	0 + 147 = 147	68 + 85 = 153	153	1
17	0 + 147 = 147	68 + 85 = 153	153	1
18	0 + 147 = 147	68 + 85 = 153	153	1
19	0 + 147 = 147	68 + 147 = 215	215	1
20	0 + 147 = 147	68 + 147 = 215	215	1
21	0 + 147 = 147	68 + 147 = 215	215	1
22	0 + 147 = 147	68 + 147 = 215	215	1
23	0 + 147 = 147	68 + 147 = 215	215	1

continued...

continued...

Stage 2:

$$f_2(y_2) = \max_{4x_2 \le y_2} \{64x_2 + f_3(y_2 - 4x_2)\}$$

	$64x_2 + f_3(y_2 - 4x_2)$		Opt. Sol.	
y_2	$x_2 = 0$	$x_2 = 1$	f_2	x_2^*
0	0 + 0 = 0	—	0	0
1	0 + 0 = 0	—	0	0
2	0 + 0 = 0	—	0	0
3	0 + 0 = 0	—	0	0
4	0 + 0 = 0	64 + 0 = 64	64	1
5	0 + 62 = 62	64 + 0 = 64	64	1
6	0 + 68 = 68	64 + 0 = 64	68	0
7	0 + 68 = 68	64 + 0 = 64	68	0
8	0 + 85 = 85	64 + 0 = 64	85	0
9	0 + 85 = 85	64 + 62 = 126	126	1
10	0 + 85 = 85	64 + 68 = 132	132	1
11	0 + 130 = 130	64 + 68 = 132	132	1
12	0 + 130 = 130	64 + 85 = 149	149	1
13	0 + 147 = 147	64 + 85 = 149	149	1
14	0 + 153 = 153	64 + 85 = 149	153	0
15	0 + 153 = 153	64 + 130 = 194	194	1
16	0 + 153 = 153	64 + 130 = 194	194	1
17	0 + 153 = 153	64 + 147 = 211	211	1
18	0 + 153 = 153	64 + 153 = 217	217	1
19	0 + 215 = 215	64 + 153 = 217	217	1
20	0 + 215 = 215	64 + 153 = 217	217	1
21	0 + 215 = 215	64 + 153 = 217	217	1
22	0 + 215 = 215	64 + 153 = 217	217	1
23	0 + 215 = 215	64 + 215 = 279	279	1

Stage 1:

$$f_1(y_1) = \max_{7x_1 \le y_1} \{78x_1 + f_2(y_1 - 7x_1)\}$$

	$78x_1 + f_2(y_1 - 7x_1)$		Opt. Sol.	
y_1	$x_1 = 0$	$x_1 = 1$	f_1	x_1^*
23	0 + 279 = 279	78 + 194 = 272	279	0

Solution: $(y_1 = 23) \rightarrow x_1 = 0 \rightarrow (y_2 = 23) \rightarrow x_2 = 1 \rightarrow (y_3 = 23 - 4 = 19) \rightarrow x_3 = 1 \rightarrow (y_4 = 19 - 6 = 13) \rightarrow x_4 = 1 \rightarrow (y_5 = 13 - 5 = 8) \rightarrow x_5 = 1$

All but the first application are accepted.

$x_j = 1$ if precinct j is selected, and 0 if otherwise.

Maximize $z = 31x_1 + 26x_2 + 35x_3 + 28x_4 + 24x_5$

subject to

$$3.5x_1 + 2.5x_2 + 4x_3 + 3x_4 + 2x_5 \le 10$$

$$x_j = (0, 1), \quad j = 1, 2, \ldots, 5$$

Stage 5: $f_5(y_5) = \max_{\substack{2x_5 \le y_5 \\ x_5 = (0,1)}} \{24x_5\}$

	$24x_5$		Opt. Sol.	
y_5	$x_5 = 0$	$x_5 = 1$	f_5	x_5^*
0	0	—	0	0
.5	0	—	0	0
1.	0	—	0	0
1.5	0	—	0	0
2.	0	24	24	1
2.5	0	24	24	1
↓	↓	↓	↓	↓
10	0	24	24	1

Stage 4:

$$f_4(y_4) = \max_{\substack{3x_4 \le y_4 \\ x_4 = 0,1}} \{28x_4 + f_5(y_4 - 3x_4)\}$$

	$28x_4 + f_5(y_4 - 3x_4)$		Opt. Sol.	
y_4	$x_4 = 0$	$x_4 = 1$	f_4	x_4^*
0	0 + 0 = 0	—	0	0
.5	0 + 0 = 0	—	0	0
1.	0 + 0 = 0	—	0	0
1.5	0 + 0 = 0	—	0	0
2.	0 + 24 = 24	—	24	0
2.5	↓	—	24	0
3.		28 + 0 = 28	28	1
3.5		28 + 0 = 28	28	1
4.		28 + 0 = 28	28	1
4.5		28 + 0 = 28	28	1
5.		28 + 24 = 52	52	1
↓		↓	↓	↓
10	0 + 24 = 24	28 + 24 = 52	52	1

continued...

Set 10.3a

Stage 3:

$$f_3(y_3) = \max_{\substack{4x_3 \le y_3 \\ x_3 = 0,1}} \{35x_3 + f_4(y_3 - 4x_3)\}$$

y_3	$35x_3 + f_4(y_3 - 4x_3)$: $x_3 = 0$	$x_3 = 1$	Opt. Sol.: f_3	x_3^*
0	0+0 = 0	—	0	0
.5	0+0 = 0	—	0	0
1.	0+0 = 0	—	0	0
1.5	0+0 = 0	—	0	0
2.	0+24 = 24	—	24	0
2.5	0+24 = 24	—	24	0
3.	0+28 = 28	—	28	0
3.5	0+28 = 28	—	28	0
4.	0+28 = 28	35+0 = 35	35	0
4.5	0+28 = 28	35+0 = 35	35	0
5.	0+52 = 52	35+0 = 35	52	0
5.5		35+0 = 35	52	0
6.		35+24 = 59	59	1
6.5		35+24 = 59	59	1
7.		35+28 = 63	63	1
7.5		35+28 = 63	63	1
8.		35+28 = 63	63	1
8.5		35+28 = 63	63	1
9.		35+52 = 87	87	1
9.5		35+52 = 87	87	1
10.	0+52 = 52	35+52 = 87	87	1

Stage 2:

$$f_2(y_2) = \max_{\substack{2.5x_2 \le y_2 \\ x_2 = 0,1}} \{26x_2 + f_3(y_2 - 2.5x_2)\}$$

continued...

y_2	$26x_2 + f_3(y_2 - 2.5x_2)$: $x_2 = 0$	$x_2 = 1$	Opt. Sol.: f_2	x_2^*
0	0+0 = 0	—	0	0
.5	0+0 = 0	—	0	0
1.	0+0 = 0	—	0	0
1.5	0+0 = 0	—	0	0
2.	0+24 = 24	—	24	0
2.5	0+24 = 24	26+0 = 26	26	1
3.	0+28 = 28	26+0 = 26	28	0
3.5	0+28 = 28	26+0 = 26	28	0
4.	0+35 = 35	26+0 = 26	35	0
4.5	0+35 = 35	26+24 = 50	50	1
5.	0+35 = 35	26+24 = 50	50	1
5.5	0+35 = 35	26+28 = 54	54	1
6.	0+59 = 59	26+28 = 54	59	0
6.5	0+59 = 59	26+35 = 61	61	1
7.	0+63 = 63	26+35 = 61	63	0
7.5	0+63 = 63	26+35 = 61	63	0
8.	0+63 = 63	26+35 = 61	63	0
8.5	0+63 = 63	26+59 = 85	85	1
9.	0+87 = 87	26+59 = 85	87	0
9.5	0+87 = 87	26+63 = 89	89	1
10.	0+87 = 87	26+63 = 89	89	1

Stage 1:

$$f_1(y_1) = \max_{\substack{3.5x_1 \le y_1 \\ x_1 = 0,1}} \{31x_1 + f_2(y_1 - 3.5x_1)\}$$

y_1	$31x_1 + f_2(y_1 - 3.5x_1)$: $x_1 = 0$	$x_1 = 1$	f_1	x_1^*
10	0+89 = 89	31+61 = 92	92	1

Solution:

$(y_1 = 10) \rightarrow x_1 = 1 \rightarrow (y_2 = 10 - 3.5 = 6.5)$
$\rightarrow x_2 = 1 \rightarrow (y_3 = 6.5 - 2.5 = 4) \rightarrow$
$x_3 = 1 \rightarrow (y_4 = 4 - 4 = 0) \rightarrow x_4 = 0 \rightarrow$
$(y_5 = 0) \rightarrow x_5 = 0.$

Allocate funds to precincts 1, 2, and 3. Total population reached is 3100 + 2600 + 3500 = 9200.

9

k_j = number of parallel units in component j, $j = 1, 2, 3$

The problem can be written as

maximize $r = r_1(k_1) \cdot r_2(k_2) \cdot r_3(k_3)$

subject to

$c_1(k_1) + c_2(k_2) + c_3(k_3) \le 10$

where

$r_j(k_j)$ = reliability of component j given k_j parallel units

$C_j(k_j)$ = cost of component j given k_j parallel units

Define state as

y_j = capital assigned to components $j, j+1, \ldots, 3$

Stage 3: $f_3(y_3) = \max_{k_3 = 1, 2, 3} \{R_3(k_3)\}$

	$R_3(k_3)$			Optimal Solution	
	$k_3 = 1$	$k_3 = 2$	$k_3 = 3$		
y_3	$R = .5, c = 2$	$R = .7, c = 4$	$R = .9, c = 5$	$f_3(y_3)$	k_3^*
2	.5	—	—	.5	1
3	.5	—	—	.5	1
4	.5	.7	—	.7	2
5	.5	.7	.9	.9	3
6	.5	.7	.9	.9	3

Stage 2: $f_2(y_2) = \max_{k_2 = 1, 2, 3} \{R_2(k_2) \cdot f_3[y_2 - c_2(k_2)]\}$

	$R_2(k_2) \cdot f_3[y_2 - c_2(k_2)]$			Optimal Solution	
	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$		
y_2	$R = .7, c = 3$	$R = .8, c = 5$	$R = .9, c = 6$	$f_2(y_2)$	k_2^*
5	$.7 \times .5 = .35$	—	—	.35	1
6	$.7 \times .5 = .35$	—	—	.35	1
7	$.7 \times .7 = .49$	$.8 \times .5 = .40$	—	.49	1
8	$.7 \times .9 = .63$	$.8 \times .5 = .40$	$.9 \times .5 = .45$	.63	1
9	$.7 \times .9 = .63$	$.8 \times .7 = .56$	$.9 \times .5 = .45$	.63	1

Stage 1: $f_1(y_1) = \max_{k_1 = 1, 2, 3} \{R_1(k_1) \cdot f_2[y_1 - c_1(k_1)]\}$

	$R_1(k_1) \cdot f_2[y_1 - c_1(k_1)]$			Optimal Solution	
y_1	$R = .6, c = 1$	$R = .8, c = 2$	$R = .9, c = 3$	$f_1(y_1)$	k_1^*
6	$.6 \times .35 = .210$	—	—	.210	1
7	$.6 \times .35 = .210$	$.8 \times .35 = .280$	—	.280	2
8	$.6 \times .49 = .294$	$.8 \times .35 = .280$	$.9 \times .35 = .315$	.315	3
9	$.6 \times .63 = .378$	$.8 \times .49 = .392$	$.9 \times .35 = .315$	.392	2
10	$.6 \times .63 = .378$	$.8 \times .63 = .504$	$.9 \times .49 = .441$	.504	2

Solution:

$(K_1^*, K_2^*, K_3^*) = (2, 1, 3)$

Composite $r = .504$

continued...

10

State y_j = portion of the quantity c allocated to variables $j, j+1, \ldots,$ and n.

Stage n: $f_n(y_n) = \max_{x_n \le y_n} \{x_n\}$

State	Opt. Sol.	
	f_n	x_n^*
y_n	y_n	y_n

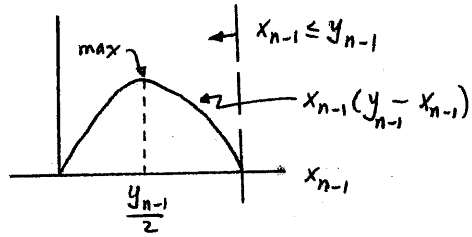
Stage $n-1$: $f_{n-1}(y_{n-1}) = \max_{x_{n-1} \le y_{n-1}} \{x_{n-1} f_n(y_{n-1} - x_{n-1})\}$

Given $f_n(y_n) = y_n$, then

$f_n(y_{n-1} - x_{n-1}) = y_{n-1} - x_{n-1}$

Thus,

$f_{n-1}(y_{n-1}) = \max_{x_{n-1} \le y_{n-1}} \{x_{n-1}(y_{n-1} - x_{n-1})\}$



State	Opt. Sol.	
	f_{n-1}	x_{n-1}^*
y_{n-1}	$(y_{n-1}/2)^2$	$(y_{n-1}/2)$

Stage j

$f_j(y_j) = \max_{x_j \le y_j} \{x_j f_{j+1}(y_j - x_j)\}$

State	Opt. Sol.	
	f_j	x_j^*
y_j	$\left(\frac{y_j}{n-j+1}\right)^{n-j+1}$	$\frac{y_j}{n-j+1}$

Solution: $(y_1 = q) \rightarrow x_1 = \frac{c}{n} \rightarrow (y_2 = \frac{n-1}{n}c) \rightarrow \cdots \rightarrow y_j = \frac{n-j+1}{n}c \rightarrow x_j = \frac{c}{n}$

$x_1 = x_2 = \cdots = x_n = \frac{c}{n}$, $z = \left(\frac{c}{n}\right)^n$

Set 10.3a

11

$$f_n(y_n) = \min_{x_n = y_n} \{x_n^2\}$$

$$f_i(y_i) = \min_{x_i > 0} \left\{x_i^2 + f_{i+1}\left(\frac{y_i}{x_i}\right)\right\}$$

<u>Stage n:</u>

$$f_n(y_n) = y_n^2, \qquad x_n^* = y_n$$

<u>Stage $n-1$:</u>

$$f_{n-1}(y_{n-1}) = \min_{x_{n-1}>0} \left\{x_{n-1}^2 + \left(\frac{y_{n-1}}{x_{n-1}}\right)^2\right\}$$

$$\frac{\partial\{\cdot\}}{\partial x_{n-1}} = 2x_{n-1} - 2\frac{y_{n-1}^2}{x_{n-1}^3} = 0$$

or $x_{n-1}^* = \sqrt{y_{n-1}}$, $f_{n-1}(y_{n-1}) = 2y_{n-1}$

<u>Stage $n-2$:</u>

$$f_{n-2}(y_{n-2}) = \min_{x_{n-2}>0} \left\{x_{n-2}^2 + 2\left(\frac{y_{n-2}}{x_{n-2}}\right)\right\}$$

$$\frac{\partial\{\cdot\}}{\partial x_{n-2}} = 2x_{n-2} - 2\frac{y_{n-2}}{x_{n-2}} = 0$$

or $x_{n-2}^* = (y_{n-2})^{1/3}$, $f_{n-2}(y_{n-2}) = 3y_{n-2}^{2/3}$

<u>Stage i:</u>

Induction yields

$$x_i^* = y_i^{\frac{1}{n-i+1}}, \quad f_i(y_i) = (n-i+1)\, y_i^{\frac{2}{n-i+1}}$$

<u>Stage 1:</u>

$$x_1^* = c^{1/n}, \quad f_1(y_1) = n\, y_1^{2/n}$$

Thus, $y_2 = \frac{y_1}{x_1} = c^{\frac{n-1}{n}} \Rightarrow x_2^* = c^{1/n}$

In general, $y_i = \sqrt[n]{c}$

12

For proper decomposition, let

$x_1 = y_1$, $x_2 = y_4$, $x_3 = y_2$ and $x_4 = y_3$

The problem is then written as

Maximize $z = (x_1+2)^2 + (x_2-5)^2 + x_3 x_4$

Subject to $x_1 + x_2 + x_3 + x_4 \le 5$

$x_1, x_2, x_3, x_4 \ge 0$ and integer

Rearrangement of variables allows mixing multiplicative and additive decomposition

continued...

z_j = amount of the resource allocated to variables $j, j+1, \dots, 4$.

<u>Stage 4:</u> $f_4(z_4) = \max_{x_4 \le y_4} \{x_4\}$

z_4	x_4: $x_4 = 0$	1	2	3	4	5	Opt. Sol. f_4	x_4^*
0	0	–	–	–	–	–	0	0
1	0	1	–	–	–	–	1	1
2	0	1	2	–	–	–	2	2
3	0	1	2	3	–	–	3	3
4	0	1	2	3	4	–	4	4
5	0	1	2	3	4	5	5	5

<u>Stage 3:</u> $f_3(z_3) = \max_{x_3 \le z_3} \{x_3 f_4(z_3 - x_3)\}$

z_3	$x_3 f_4(z_3 - x_3)$: $x_3 = 0$	1	2	3	4	5	Opt. Sol. f_3	x_3^*
0	0x0=0	–	–	–	–	–	0	0
1	0x1=0	1x0=0	–	–	–	–	0	1
2	0x2=0	1x1=1	2x0=0	–	–	–	1	1
3	0x3=0	1x2=2	2x1=2	3x0=0	–	–	2	1,2
4	0x4=0	1x3=3	2x2=4	3x1=3	4x0=0	–	4	2
5	0x5=0	1x4=4	2x3=6	3x2=6	4x1=4	5x0=0	6	2,3

<u>Stage 2:</u> $f_2(z_2) = \max_{x_2 \le z_2} \{(x_2-5)^2 + f_3(z_2 - x_2)\}$

z_2	$(x_2-5)^2 + f_3(z_2-x_2)$: $x_2 = 0$	1	2	3	4	5	Opt. Sol. f_2	x_2^*
0	25+0=25	–	–	–	–	–	25	0
1	25+0=25	16+0=16	–	–	–	–	25	0
2	25+1=26	16+0=16	9+0=9	–	–	–	26	0
3	25+2=27	16+1=17	9+0=9	4+0=4	–	–	27	0
4	25+4=29	16+2=18	9+1=10	4+0=4	1+0=0	–	29	0
5	25+6=31	16+4=20	9+2=11	4+1=5	1+0=0	0+0=0	31	0

<u>Stage 1:</u> $f_1(z_1) = \max_{x_1 \le z_1} \{(x_1+2)^2 + f_2(z_1 - x_1)\}$

z_1	$(x_1+2)^2 + f_2(z_1-x_1)$: $x_1 = 0$	1	2	3	4	5	Opt. Sol. f_1	x_1^*
5	4+31	9+29	16+27	25+26	36+25	49+25	74	5

$(z_1 = 5) \rightarrow x_1 = 5 \rightarrow (z_2 = 0) \rightarrow x_2 = 0 \rightarrow (z_3 = 0) \rightarrow x_3 = 0 \rightarrow (z_4 = 0) \rightarrow x_4 = 0$

<u>Optimum:</u> $(y_1, y_2, y_3, y_4) = (5, 0, 0, 0)$

$z = 74$

13

Define state as

y_i = amount of the resource allocated to variable $i, i+1, \ldots,$ and n

$$g_n(y_n) = \min_{x_3 = y_3} \{f_3(y_3)\}$$

$$g_i(y_i) = \min_{0 \le x_i \le y_i} \{\max[f_i(x_i), g_{i+1}(y_i - x_i)]\}$$

Stage 3: $g_3(y_3) = \min_{x_3 = y_3} \{x_3 - 2\}$

State	$g_3(y_3)$	x_3^*
y_3	$y_3 - 2$	y_3

Stage 2: $\min_{0 \le x_2 \le y_2} \{\max[(5x_2+3), (y_2 - x_2 - 2)]\}$

State	$g_2(y_2)$	x_2^*
$y_2 < 5$	0	3
$y_2 \ge 5$	$\frac{x_2 - 5}{6}$	$\frac{5}{6}x_2 - \frac{7}{6}$

Stage 1: $g_1(y_1) = \min_{x_1 \le y_1} \{\max[x_1 + 5, g_2(y_1 - x_1)]\}$

State	$g_1(y_1)$	x_1^*
$y_1 \le \frac{37}{5}$	0	5
$y_1 > \frac{37}{5}$	$\frac{5y_1 - 37}{11}$	$\frac{5y_1 + 18}{11}$

$(y_1 = 10) \to x_1 = \frac{50-37}{11} = \frac{13}{11} \to$

$(y_2 = \frac{97}{11}) \to x_2 = \frac{97/11 - 5}{6} = \frac{7}{11} \to$

$(y_3 = \frac{90}{11}) \to x_3 = \frac{90}{11}$

$$g_1(10) = \frac{5 \times 10 + 18}{11} = \frac{68}{11}$$

Set 10.3b

1

(a) Stage 5: $b_5 = 8$

x_4	$x_5 = 8$	Opt. Sol. f_5	x_5^*
6	$0 + 4 + 2(2) = 8$	8	8
7	$0 + 4 + 2(1) = 6$	6	8
8	$0 + 0 = 0$	0	8

Stage 4: $b_4 = 6$

x_3	$x_4 = 6$	$x_4 = 7$	$x_4 = 8$	Opt. Sol. f_4	x_4^*
3	$0+(4+6)+8$	$3+(4+8)+6$	$6+(4+10)+0$	18	6
4	$0+(4+4)+8$	$3+(4+6)+6$	$6+(4+8)+0$	16	6
5	$0+(4+2)+8$	$3+(4+4)+6$	$6+(4+6)+0$	14	6
6	$0+0+8$	$3+(4+2)+6$	$6+(4+4)+0$	8	6
7	$0+0+8$	$3+0+6$	$6+(4+2)+0$	8	6
8	$0+0+8$	$3+0+6$	$6+0+0$	6	8

Stage 3: $b_3 = 3$

x_2	$x_3 = 3$	4	5	6	7	8	Opt. Sol. f_3	x_3^*
5	$0+0+18$	$3+0+16$	$6+0+14$	$9+4+2+8$	$12+4+4+8$	$15+4+6+6$	18	3
6	$0+0+18$	$3+0+16$	$6+0+14$	$9+0+8$	$12+4+2+8$	$5+4+4+6$	17	6
7	$0+0+18$	$3+0+16$	$6+0+14$	$9+0+8$	$12+0+8$	$15+4+2+6$	17	6
8	$0+0+18$	$3+0+16$	$6+0+14$	$9+0+8$	$12+0+8$	$15+0+6$	17	6

Stage 2: $b_2 = 5$

x_1	$x_2 = 5$	6	7	8	Opt. Sol. f_2	x_2^*
6	$0+0+18$	$3+0+17$	$6+4+2+17$	$9+4+4+17$	18	5
7	$0+0+18$	$3+0+17$	$6+0+17$	$9+4+2+17$	18	5
8	$0+0+18$	$3+0+17$	$6+0+17$	$9+0+17$	18	5

Stage 1: $b_1 = 6$

x_0	$x_1 = 6$	7	8	Opt. Sol. f_1	x_1^*
0	$0+(4+12)+18$	$3+(4+14)+18$	$6+(4+16)+18$	34	6

Week i	b_i	x_i	
1	6	6	Hire 6
2	5	5	Fire 1
3	3	3	Fire 2
4	6	6	Hire 3
5	8	8	Hire 2

continued...

(b) Stage 5: $b_5 = 2$

x_4	$x_5 = 2$	Opt. Sol. f_5	x_5^*
	$0+0$	0	2

Stage 4: $b_4 = 8$

x_3	$x_4 = 8$	Opt. Sol. f_4	x_4^*
7	$0+(4+2)+1$	6	8
8	$0+0+0$	0	8

Stage 3: $b_3 = 7$

x_2	$x_3 = 7$	$x_3 = 8$	Opt. Sol. f_3	x_3^*
4	$0+4+6+6$	$3+4+8+0$	15	8
5	$0+4+4+6$	$3+4+6+0$	13	8
6	$0+4+2+6$	$3+4+4+0$	11	8
7	$0+0+6$	$3+4+2+0$	6	7
8	$0+0+6$	$3+0+0$	6	7

Stage 2: $b_2 = 4$

x_1	$x_2 = 4$	5	6	7	8	Opt. Sol. f_2	x_2^*
8	$0+0+15$	$3+0+13$	$6+0+11$	$9+0+6$	$12+0+6$	15	4,7

Stage 1: $b_1 = 8$

x_0	$x_1 = 8$	Opt. Sol. f_1	x_1^*
0	$0 + (4 + 2 \times 8) + 15$	35	8

Optimum solution:

Week i	b_i	x_i	
1	8	8	Hire 8
2	4	7	Fire 1
3	7	7	—
4	8	8	Hire 1
5	2	2	Fire 6

Alternative optimum:

Week i	b_i	x_i	
1	8	8	Hire 8
2	4	4	Fire 4
3	7	8	Hire 4
4	8	8	—
5	2	2	Fire 6

2

Let

$C_3(x_{i-1}-x_i) = 100(x_{i-1}-x_i)$

be the severance cost of $x_{i-1}-x_i$ laborers, $x_{i-1} > x_i$.

$$f_i(x_i) = \min_{x_i \ge b_i} \{C_1(x_i-b_i) + C_2(x_i-x_{i-1}) + C_3(x_{i-1}-x_i) + f_{i+1}(x_i)\}$$

$i = 1, 2, \ldots, n$

Stage 5 ($b_5 = 6$):

x_4	$C_1(x_5-6)+C_2(x_5-x_4)+C_3(x_4-x_5)$: $x_5 = 6$	Optimum solution: $f_5(x_4)$	x_5^*
4	3(0) + 4 + 2(2) + 0 = 8	8	6
5	3(0) + 4 + 2(1) + 0 = 6	6	6
6	3(0) + 0 + 0 = 0	0	6

Stage 4 ($b_4 = 4$):

x_3	$C_1(x_4-4)+C_2(x_4-x_3)+C_3(x_4-x_5)+f_5(x_4)$: $x_4 = 4$	$x_4 = 5$	$x_4 = 6$	Optimum solution: $f_4(x_3)$	x_4^*
8	3(0) + 0 + 4 + 8 = 12	3(1) + 0 + 3 + 6 = 12	3(2) + 0 + 2 + 0 = 8	8	6

Stage 3 ($b_3 = 8$):

x_2	$C_1(x_3-8)+C_2(x_3-x_2)+C_3(x_2-x_3)+f_4(x_3)$: $x_3 = 8$	Optimum solution: $f_3(x_2)$	x_3^*
7	0 + 4 + 2(1) + 0 + 8 = 14	14	8
8	0 + 0 + 0 + 8 = 8	8	8

Stage 2 ($b_2 = 7$):

x_1	$C_1(x_2-7)+C_2(x_2-x_1)+C_3(x_1-x_2)+f_3(x_2)$: $x_2 = 7$	$x_2 = 8$	Optimum solution: $f_2(x_1)$	x_2^*
5	0 + 4 + 2(2) + 0 + 14 = 22	3(1) + 4 + 2(3) + 0 + 8 = 21	21	8
6	0 + 4 + 2(1) + 0 + 14 = 20	3(1) + 4 + 2(2) + 0 + 8 = 19	19	8
7	0 + 0 + 0 + 14 = 14	3(1) + 4 + 2(1) + 0 + 8 = 17	14	7
8	0 + 0 + 1 + 14 = 15	3(1) + 0 + 0 + 8 = 11	11	8

Stage 1 ($b_1 = 5$):

x_0	$C_1(x_1-5)+C_2(x_1-x_0)+C_3(x_0-x_1)+f_2(x_1)$: $x_1 = 5$	$x_1 = 6$	$x_1 = 7$	$x_1 = 8$	Optimum solution: $f_1(x_0)$	x_1^*
0	0 + 4 + 2(5) + 0 + 21 = 35	3(1) + 4 + 2(6) + 0 + 19 = 38	3(2) + 4 + 2(7) + 0 + 14 = 38	3(2) + 4 + 2(8) + 0 + 11 = 37	35	5

The optimum solution is determined as

$x_0 = 0 \rightarrow x_1^* = 5 \rightarrow x_2^* = 8 \rightarrow x_3^* = 8 \rightarrow x_4^* = 6 \rightarrow x_5 = 6$

The solution can be translated to the following plan:

Week i	Minimum Labor Force b_i	Actual Labor Force x_i	Decision
1	5	5	Hire 5 workers
2	7	8	Hire 3 workers
3	8	8	No change
4	4	6	Fire 2 workers
5	6	6	No change

3

Let

x_i = number of cars rented in wk i

$C_i(x_i)$ = renal cost in week i

$$= \begin{cases} 220x_i, & \text{if } x_i \le x_{i-1} \\ 500 + 220x_i, & \text{if } x_i > x_{i-1} \end{cases}$$

$$f_i(x_{i-1}) = \min_{x_i \ge b_i} \{C_i(x_i) + f_{i+1}(x_i)\}$$

$i = 1, 2, 3, 4$

continued...

Stage 4: $b_4 = 8$

x_3	$x_4 = 8$	Opt. Sol. f_4	x_4^*
7	500 + 220×8 = 2260	2260	8
8	220×8 = 1760	1760	8

Stage 3: $b_3 = 7$

x_2	$x_3 = 7$	$x_3 = 8$	Opt. Sol. f_3	x_3^*
4	500 + 220(7) + 2260 = 4300	500 + 220(8) + 1760 = 4020	4020	8
5	500 + 220(7) + 2260 = 4300	500 + 220(8) + 1760 = 4020	4020	8
6	500 + 220(7) + 2260 = 4300	500 + 220(8) + 1760 = 4020	4020	8
7	220×7 + 2260 = 3800	500 + 220(8) + 1760 = 4020	3800	7
8	220×7 + 2260 = 3800	220×8 + 1760 = 3520	3520	8

Stage 2: $b_2 = 4$

x_1	$x_2 = 4$	5	6	7	8	Opt. Sol. f_2	x_2^*
7	220(4) + 4020 = 4900	220(5) + 4020 = 5120	220(6) + 4020 = 5340	220(7) + 3800 = 5340	500 + 220(8) + 3520 = 5780	4900	4
8	220(4) + 4020 = 4900	220(5) + 4020 = 5120	220(6) + 4020 = 5340	220(7) + 3800 = 5340	220(8) + 3520 = 5280	4900	4

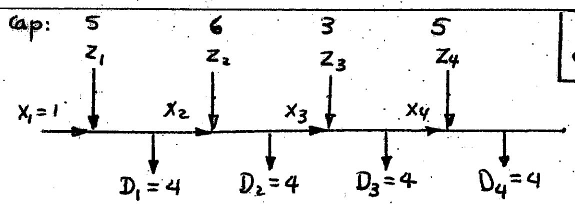
Stage 1: $b_1 = 7$

x_0	$x_1 = 7$	$x_1 = 8$	Opt. Sol. f_1	x_1^*
0	500 + 220(7) + 4900 = 6940	500 + 220(8) + 4900 = 7160	6940	4

Solution:

Week i	b_i	x_i	
1	7	7	Rent 7 cars
2	4	4	Return 3
3	7	8	Rent 4
4	8	8	—

4



c/unit	\$30	33	35	42
h/unit	\$2	3	4	—

z_i = amount produced in period i

$x_1 = 1$, $\quad 0 \le z_1 \le 5$

$0 \le x_2 \le 2$, $\quad 0 \le z_2 \le 6$

$0 \le x_3 \le 4$, $\quad 0 \le z_3 \le 3$

$0 \le x_4 \le 3$, $\quad 0 \le z_4 \le$

Stage 4: $f_4(x_4) = \min\limits_{\substack{z_4 \ge 0 \\ z_4 + x_4 = 4}} \{42 z_4\}$

x_4	$z_4=0$	1	2	3	4	Opt. Sol. f_4	z_4^*
0	—	—	—	—	42x4	168	4
1	—	—	—	42x3	—	126	3
2	—	—	42x2	—	—	84	2
3	—	42x1	—	—	—	42	1
4	0	—	—	—	—	0	0

Stage 3: $f_3(x_3) = \min\limits_{\substack{z_3 \ge 0 \\ z_3 + x_3 \ge 4}} \{35 z_3 + 4(x_3 + z_3 - 4) + f_4(x_3 + z_3 - 4)\}$

x_3	$z_3=0$	1	2	3	4	f_3	z_3^*
0	—	—	—	—	140+0 +168 =308	308	4
1	—	—	—	105+0 +168 =273	140+4 +126 =270	270	4
2	—	—	70+0 +168 =238	105+4 +126 =235	140+8 +84 =232	232	4
3	—	35+0 +168 =203	70+4 +126 =200	105+8 +84 =193	140+12 +42 =194	193	3
4	0+0 +168 =168	35+4 +126 =165	70+8 +84 =162	105+12 +42 =159	140+16 +0 =156	156	4

continued...

Stage 2:

$f_2(x_2) = \min\limits_{\substack{z_2 \ge 0 \\ z_2 + x_2 \ge 4}} \{33 z_2 + 3(x_2 + z_2 - 4) + f_3(x_2 + z_2 - 4)\}$

x_2	$z_2=0$	1	2	3	4	5	6	Opt. Sol. f_2	z_2
0	—	—	—	—	132 +308 =440	165 +3 270 =438	198 +6 232 =436	436	6
1	—	—	—	99 +308 =407	132 +3 +270 =405	165 +6 +232 =403	198 +9 +193 =400	400	6
2	—	—	66 +308 =374	99 +3 +270 =372	132 +6 +232 =370	165 +9 +193 =367	198 +12 +156 =366	366	6

Stage 1:

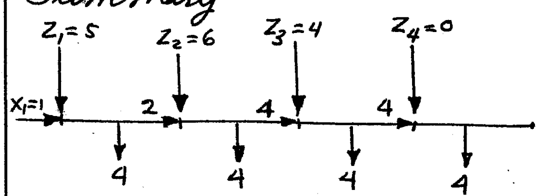
$f_1(x_1) = \min\limits_{\substack{z_1 \ge 0 \\ z_1 + x_1 \ge 4}} \{30 z_1 + 2(x_1 + z_1 - 4) + f_2(x_1 + z_1 - 4)\}$

x_1	$z_1=0$	1	2	3	4	5	Opt. Sol. f_1	z_1^*
1	—	—	—	90 +436 =526	120 + 2 +400 =522	150 +4 +366 =520	520	5

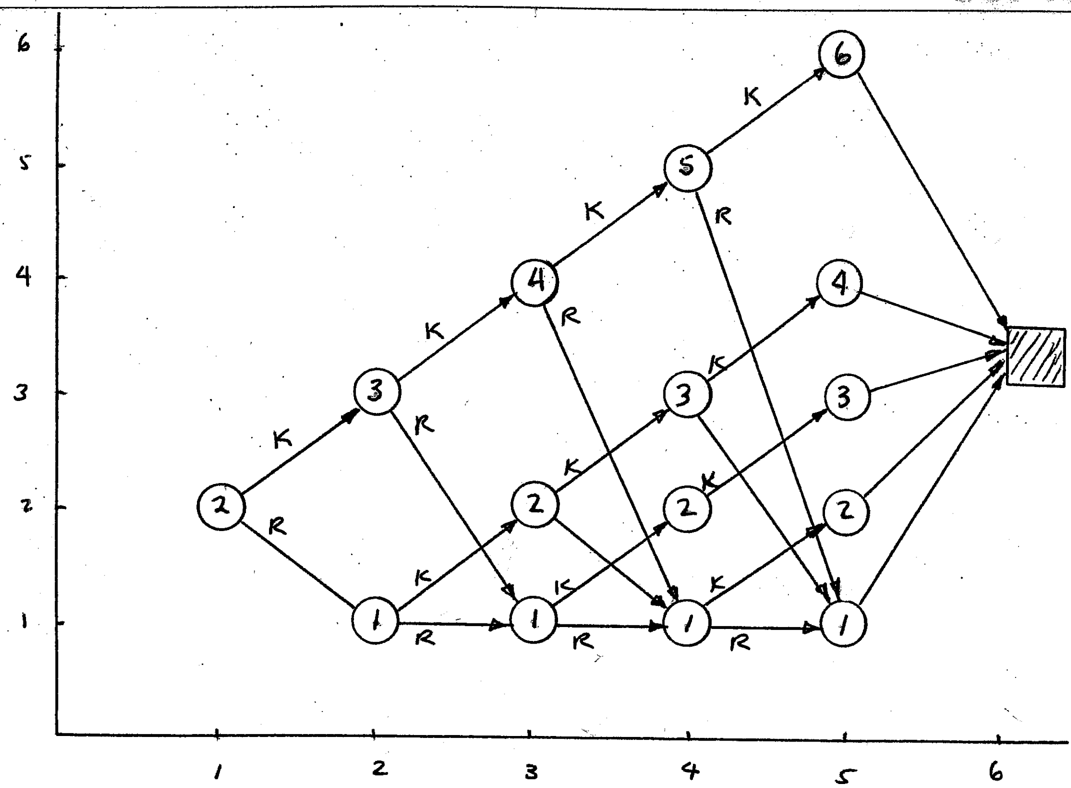
Solution: Cost = \$520

$(x_1 = 1) \rightarrow z_1 = 5 \rightarrow (x_2 = 1+5-4 = 2) \rightarrow$
$z_2 = 6 \rightarrow (x_3 = 2+6-4 = 4) \rightarrow z_3 = 4 \rightarrow$
$(x_4 = 4+4-4 = 4) \rightarrow z_4 = 0$

Summary



(a)



Stage 4:

t	K	R	Opt. Sol. f_4	Dec.
1	19+60 − .6 = 78.4	20 + 80 + 80 − 100.2 = 79.8	79.8	R
2	18.5 + 50 − 1.2 = 67.3	20 + 60 + 80 − 100.2 = 59.8	67.3	K
3	17.2 + 30 − 1.5 = 45.7	20 + 50 + 80 − 100.2 = 49.8	49.8	R
5	14 + 10 − 1.8 = 22.2	20 + 10 + 80 − 100.2 = 9.8	22.2	K

Stage 3:

t	K	R	Opt. Sol. f_3	Dec.
1	19 − .6 + 67.3 = 85.7	20+80 − 100.2 + 79.8 = 79.6	85.7	K
2	18.5 − 1.2 + 49.8 = 67.1	20+ 60 − 100.2 + 79.8 = 59.6	67.1	K
4	15.5 − 1.7 + 22.2 = 36.	20+30 − 100.2 + 79.8 = 29.6	36	K

Stage 2:

t	K	R	Opt. Sol. f_2	Dec.
1	19 − .6 + 67.1 = 85.5	20+80 − 100.2 + 85.7 = 85.5	85.5	K, R
3	17.2 − 1.5 + 36 = 51.7	20+50 − 100.2 + 85.7 = 55.5	55.5	R

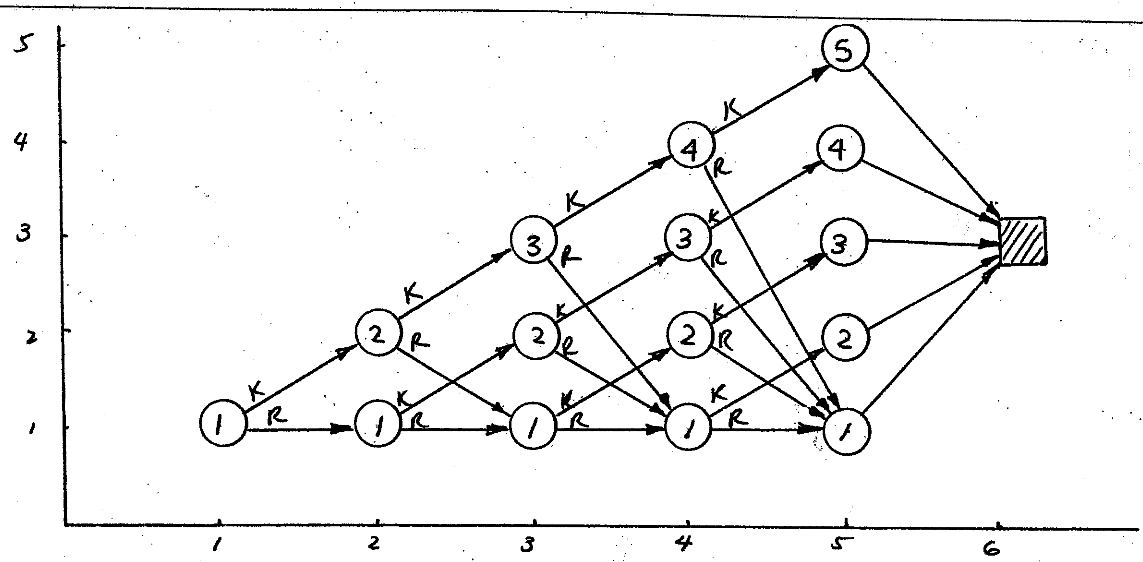
Stage 1:

t	K	R	Opt. Sol. f_1	Dec.
2	18.5 − 1.2 + 55.5 = 72.8	20+60 − 100.2 + 85.5 = 65.3	72.8	K

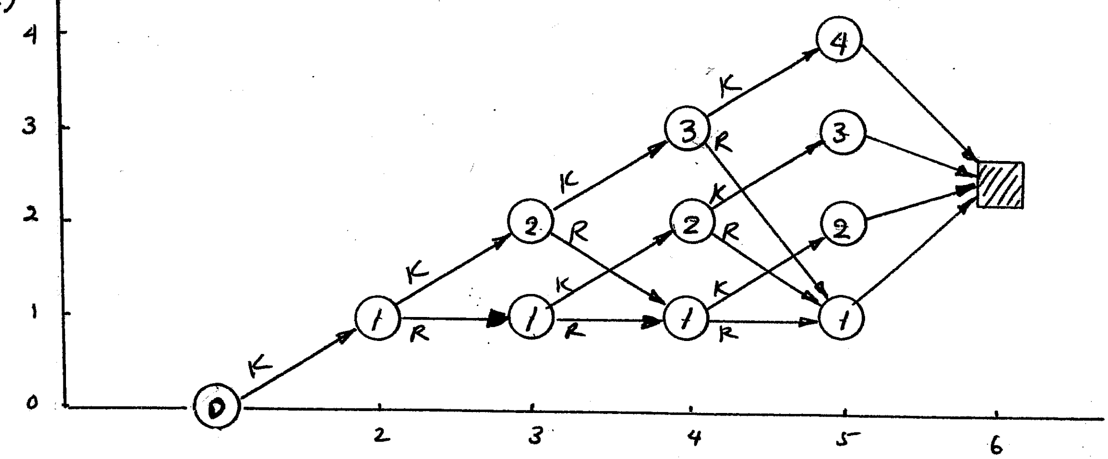
Solution: K → R → K → K, revenue = $72,800

continued...

(b)



(c)



2

Since income from mowing is constant, it need not be taken into account.

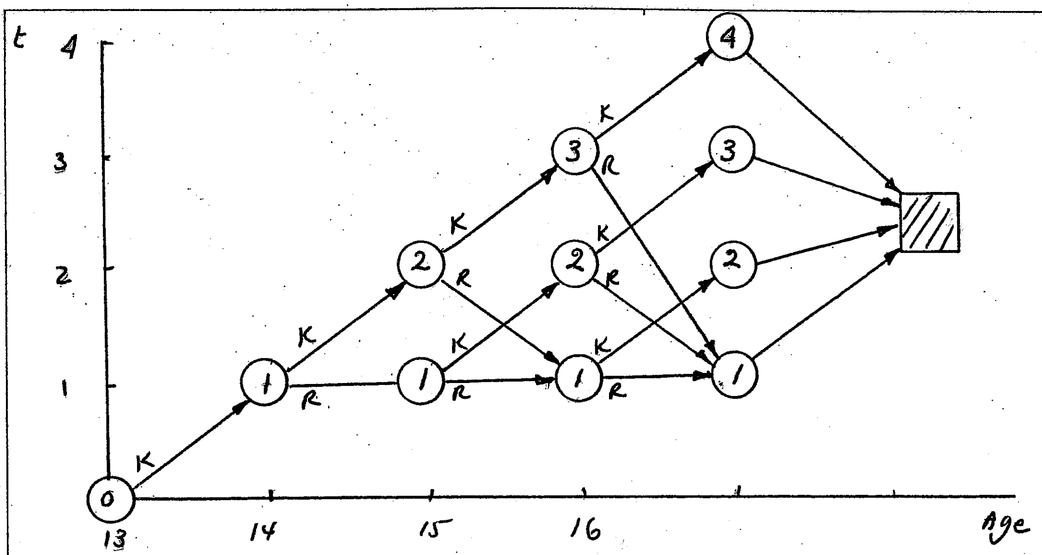
f_4 . $f_4(t) = \min \begin{cases} C(t) - S(t), & K \\ I(t) + C(1) - S(t), & R \end{cases}$

$f_i(t) = \min \begin{cases} C(t) + f_{i+1}(t+1), & K \\ I(t) + C(1) - S(t) + f_{i+1}(1), & R \end{cases}$

where,

$C(t)$ = operating cost per year for a t-year old mower
$I(t)$ = cost of a new mower after t years
$S(t)$ = salvage value of a t-year old mower
$f_i(t)$ = minimum cost for periods $i, i+1, \ldots,$ and 4 given t-year mower.

continued...



Stage 4:

t	K	R	Opt. Sol. f_4	Dec.
1	144 − 130 = 14	260 + 120 − 150 − 150 = 80	14	K
2	168 − 110 = 58	260 + 120 − 135 − 150 = 95	58	K
3	192 − 90 = 102	260 + 120 − 120 − 150 = 110	102	K

Stage 3:

t	K	R	Opt. Sol. f_3	Dec.
1	144 + 58 = 202	240 + 120 − 150 + 14 = 224	202	K
2	168 + 102 = 270	240 + 120 − 135 + 14 = 239	239	R

Stage 2:

t	K	R	Opt. Sol. f_2	Dec
1	144 + 239 = 338	220 + 120 − 150 + 202 = 392	338	K

Stage 1: The only option available at the start is K. Cost = 120 + 338 = 458

Solution: K → K → R → K, total cost = \$458

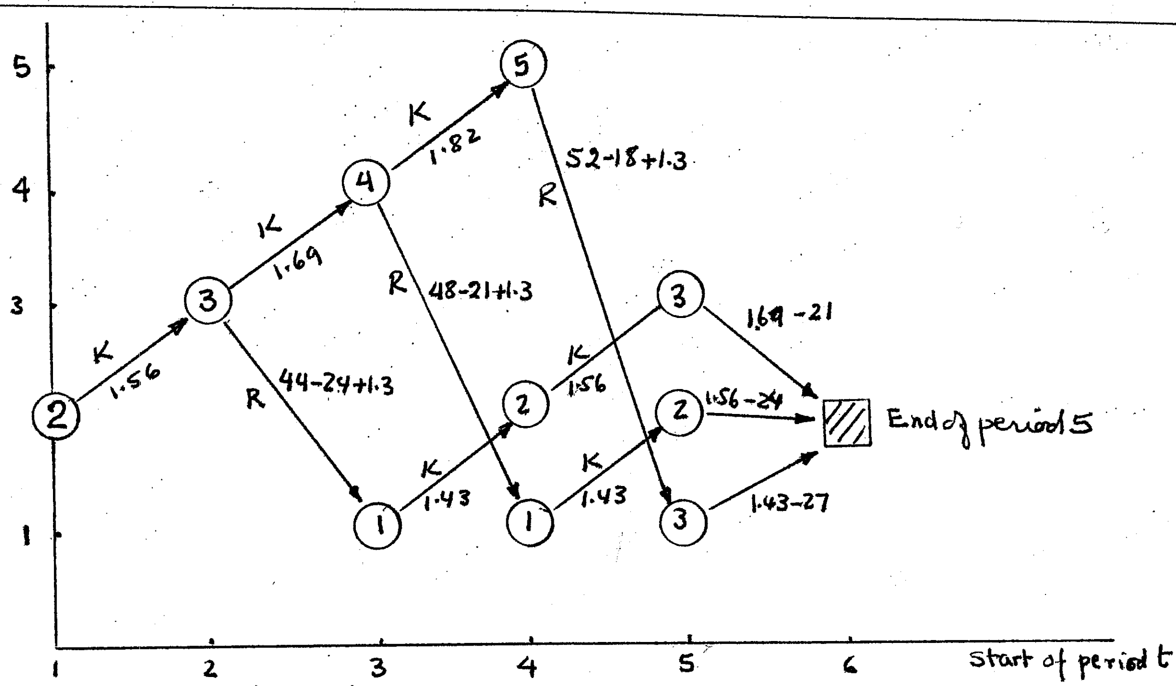
3

$$f_i(t) = \min \begin{cases} c(t) + f_{i+1}(t+1), & K \\ I(t) + c(1) - s(t) + f_{i+1}(1), & R \end{cases} \quad (2 \le t \le 5)$$

$$f_5(t) = \min \begin{cases} c(t) - s(t), & K \\ I(t) + c(1) - S(t), & R \end{cases}$$

continued...

Set 10.3c



Stage 5: (Start of year 5)

t	K	R	Optimum f_5	Dec
1	$1.43 - 27 = -25.57$	—	-25.57	K
2	$1.56 - 24 = -22.44$	—	-22.44	K
3	$1.69 - 21 = -19.31$	—	-19.31	K

Stage 4 (Start of year 4):

t	K	R	Optimum f_4	Dec.
1	$1.43 + (-22.94) = -21.51$	—	-21.51	K
2	$1.56 + (-19.31) = -17.75$	—	-17.75	K
5	—	$52 - 18 + 1.3 + (-25.57) = 9.73$	9.73	R

Stage 3 (Start of year 3):

t	K	R	Optimum f_3	Dec.
1	$1.43 + (-17.75) = -16.32$	—	-16.32	K
4	$1.82 + (9.73) = 11.55$	$48 - 21 + 1.3 + (-21.51) = 6.79$	6.79	R

Stage 2 (Start of year 2):

t	K	R	Optimum f_2	Dec
3	$1.69 + 6.79 = 8.48$	$44 - 24 + 1.3 - 16.32 = 4.98$	4.98	R

Stage 1 (Start of year 1): Keep is the only option. Cost $= 1.56 + 4.98 = 6.54$

Solution:

$K \rightarrow R \rightarrow K \rightarrow K \rightarrow K$. Cost = \$6540

4

(a)

$$f_N(T_N) = \max_{T_N \le N} \begin{cases} N^2 - T_N^2 + N - (T_N+1), & K \\ (N^2-0) + N - (0+1) - C + N - T_N, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \le N} \begin{cases} (N^2 - T_i^2) + f_{i+1}(T_i+1), & K \\ (N^2-0) + (N - T_i) - C + f_{i+1}(1), & R \end{cases}$$

For $N = 3$, $C = 10$,

$$f_3(T_3) = \max_{T_3 \le 3} \begin{cases} 11 - T_3 - T_3^2, & K \\ 4 - T_3, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \le 3} \begin{cases} 9 - T_i^2 + f_{i+1}(T_i+1), & K \\ 2 - T_i + f_{i+1}(1), & R \end{cases} \quad i = 1, 2$$

(b)

Stage 3

T_3	K	R	Optimum f_3	Dec^n
1	9	3	9	K
2	5	2	5	K
3	-1	1	1	R

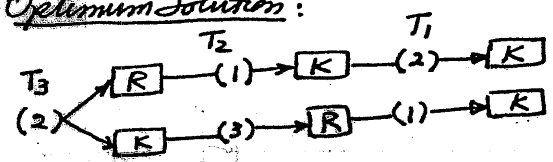
Stage 2:

T_2	K	R	Optimum f_2	Dec^n
1	8+5 = 13	1+9 = 10	13	K
2	5+1 = 6	0+9 = 9	9	R
3	—	-1+9 = 8	8	R

Stage 1:

T_1	K	R	Optimum f_1	Dec^n
1	8+9 = 17	1+13 = 14	17	K
2	5+8 = 13	0+13 = 13	13	K,R
3	—	-1+13 = 12	12	R

Optimum Solution:



Return = 13, (K,K,R) or (K,R,K)

5

$$f_4(T_4) = \max_{T_4 \le 4} \begin{cases} \frac{4}{1+T_4} + 4 - (T_4+1), & K \\ \frac{4}{1+0} + 4 - (0+1) + 6 + (4 - T_4), & R \end{cases}$$

$$= \max_{T_4 \le 4} \begin{cases} \frac{4}{1+T_4} - T_4 + 3, & K \\ 5 - T_4, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \le 4} \begin{cases} \frac{4}{1+T_i} + f_{i+1}(T_i+1), & K \\ 2 - T_i + f_{i+1}(1), & R \end{cases}$$

Stage 4

T_4	K	R	Opt. Sol. f_4	Dec.
1	4.00	4	4	K,R
2	2.33	3	3	R
3	1.00	2	2	R
4	-0.20	1	1	R

Stage 3

T_3	K	R	Opt. Sol. f_3	Dec.
1	2 + 3 = 5	1 + 4 = 5	5	K,R
2	1.33 + 2 = 3.33	0 + 4 = 4	4	R
3	1.00 + 1 = 2.00	-1 + 4 = 3	3	R
4	0.80 + (-) = —	-2 + 4 = 2	2	R

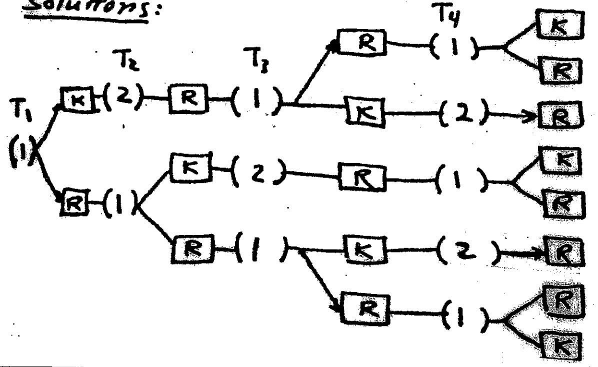
Stage 2:

T_2	K	R	Opt. Sol. f_2	Dec.
1	2 + 4 = 6	1 + 5 = 6	6	K,R
2	1.33 + 3 = 4.33	0 + 5 = 5	5	R
3	1.00 + 2 = 3	-1 + 5 = 4	4	R
4	.80 + (-) = —	-2 + 5 = 3	3	R

Stage 1:

T_1	K	R	Opt. Sol. f_1	Dec
1	2 + 5 = 7	1 + 6 = 7	7	K,R
2	1.33 + 4 = 5.33	0 + 6 = 6	6	R
3	1.00 + 3 = 4	-1 + 6 = 5	5	R
4	.8 + (-) = —	-2 + 6 = 4	4	R

Solutions:



continued...

1

$P_1 = 5, \quad P_2 = 4, \quad P_3 = 3, \quad P_4 = 2$

$\alpha_1 = (1 + .085)$

$= 1.085$

$\alpha_2 = (1 + .08)$

$= 1.08$

$$q_{ij} = \begin{array}{c|c|c} & 1 & 2 \\ 1 & .018 & .023 \\ 2 & .017 & .022 \\ 3 & .021 & .026 \\ 4 & .025 & .030 \end{array}$$

Stage 4: $f_4(x_4) = \max\limits_{0 \le I_4 \le x_4} \{S_4\}$

$S_4 = (1.085 + .025 - 1.08 - .03) I_4 + (1.08 + .03) x_4$

$= 1.11 x_4$

	Opt. Sol.	
State	$f_4(x_4)$	I_4^*
x_4	$1.11\, x_4$	$0 \le I_4 \le x_4$

Stage 3: $f_3(x_3) = \max\limits_{0 \le I_3 \le x_3} \{S_3 + f_4(x_4)\}$

$S_3 = (1.085^2 - 1.08^2) I_3 + 1.08^2 x_3$

$= .010825\, I_3 + 1.1664\, x_3$

$x_4 = P_4 + (q_{31} - q_{32}) I_3 + q_{32} x_3$

$= 2000 + (.021 - .026) I_3 + .026 x_3$

$= 2000 - .005 I_3 + .026 x_3$

$f_3(x_3) = \max\limits_{0 \le I_3 \le x_3} \{.010825 I_3 + 1.1664 x_3 + 1.11(2000 - .005 I_3 + .026 x_3)\}$

$= \max\limits_{0 \le I_3 \le x_3} \{2220 + .005275 I_3 + 1.19526 x_3\}$

	Opt. Sol.	
State	$f_3(x_3)$	I_3^*
x_3	$2220 + 1.200535 x_3$	x_3

Stage 2: $f_2(x_2) = \max\limits_{0 \le I_2 \le x_2} \{S_2 + f_3(x_3)\}$

$S_2 = (1.085^3 - 1.08^3) I_2 + 1.08^3 x_2$

$= .0175771 I_2 + 1.259712 x_2$

$x_3 = 3000 + (.017 - .022) I_2 + .022 x_2$

$= 3000 - .05 I_2 + .022 x_2$

continued...

1 continued

$f_2(x_2) = \max\limits_{0 \le I_2 \le x_2} \{.0175771 I_2 + 1.259712 x_2 + 2220 + 1.200535(3000 - .05 I_2 + .022 x_2)\}$

$= \max\limits_{0 \le I_2 \le x_2} \{5821.61 - .0424496 I_2 + 1.2861238 x_2\}$

	Opt. Sol.	
State	$f_2(x_2)$	I_2^*
	$5821.61 + 1.2861238 x_2$	0

Stage 1: $f_1(x_1) = \max\limits_{0 \le I_1 \le x_1} \{S_1 + f_2(x_2)\}$

$S_1 = (1.085^4 - 1.08^4) I_1 + 1.08^4 x_1$

$= .0253697 I_1 + 1.360489 x_1$

$x_2 = 4000 - .005 I_1 + .023 x_1$

$f_1(x_1) = \max\limits_{0 \le I_1 \le x_1} \{.0253697 I_1 + 1.360489 x_1 + 5821.61 + 1.2861238(4000 - .005 I_1 + .023 x_1)\}$

$= \max\limits_{0 \le I_1 \le x_1} \{10{,}966.11 + .018939 I_1 + 1.3900698 x_1\}$

	Opt. Sol.	
State	$f_1(x_1)$	I_1^*
$x_1 = 5000$	$10{,}966.11 + 1.4090088 x_1$	5000

$x_2 = 4000 - .005 \times 5000 + .023 \times 5000 = \$4090$

$x_3 = 3000 - .005 \times 0 + .022 \times 4090 \cong \$3090$

$x_4 = 2000 - .005 \times 3090 + .026 \times 3090 = \$2065$

Solution:

$I_1 = x_1 = 5000$	:	Invest \$5000 in FB
$I_2 = 0$	:	Invest \$4090 in SB
$I_3 = 3090$	:	Invest \$3090 in FB
$0 \le I_4 \le \$2065$	:	Invest \$2065 in FB, SB, or both.

2

x_i = cumulative amount available at the end of period i before a decision is made.

$$f_i(x_i) = \max_{y_i \le x_i} \{ g(y_i) + f_{i+1}(\alpha(x_i - y_i)) \}$$

$$f_n(x_n) = \max_{y_n = x_n} \{ g(y_n) \}$$

where,

$\alpha = 1.09$, $g(y) = \sqrt{y}$, $x_1 = 10{,}000\alpha$

Stage n:

$$f_n(x_n) = \sqrt{x_n}, \quad y_n^* = x_n$$

Stage $n-1$:

$$f_{n-1}(x_{n-1}) = \max_{y_{n-1} \le x_{n-1}} \left\{ \sqrt{y_{n-1}} + \sqrt{\alpha(x_{n-1} - y_{n-1})} \right\}$$

$$\frac{\partial\{\cdot\}}{\partial y_{n-1}} = \frac{1}{2\sqrt{y_{n-1}}} - \frac{\alpha}{2\sqrt{\alpha(x_{n-1} - y_{n-1})}} = 0$$

$$y_{n-1}^* = \frac{x_{n-1}}{1+\alpha}$$

Because $\frac{\partial^2\{\cdot\}}{\partial y_{n-1}^2} < 0$, y_{n-1}^* is a maximum point.

$$f_{n-1}(x_{n-1}) = \sqrt{(1+\alpha)x_{n-1}}$$

Stage $n-2$:

$$f_{n-2}(x_{n-2}) = \max_{y_{n-2} \le x_{n-2}} \left\{ \sqrt{y_{n-2}} + \sqrt{\alpha(1+\alpha)(x_{n-2} - y_{n-2})} \right\}$$

$$y_{n-2}^* = \frac{x_{n-2}}{1+\alpha+\alpha^2}$$

$$f_{n-2}(x_{n-2}) = \sqrt{(1+\alpha+\alpha^2)x_{n-2}}$$

Stage i:

By induction, we can show that

$$y_i^* = \frac{x_i}{(1+\alpha+\cdots+\alpha^{n-i})}$$

continued...

$$f_i(x_i) = \sqrt{(1+\alpha+\cdots+\alpha^{n-i})x_i}$$

Hence,

$$x_1 = \alpha C, \quad C = \$10{,}000$$

$$y_1^* = \frac{\alpha C}{(1+\alpha+\cdots+\alpha^{n-1})} = \frac{C(1-\alpha)}{(1-\alpha^n)}$$

$$f_1(x_1) = \sqrt{(1+\alpha+\cdots+\alpha^{n-1})x_1}$$

Given $x_1 = \alpha C$,

$$f_1(C) = \sqrt{\alpha(1+\alpha+\cdots+\alpha^{n-1})C} = \sqrt{\frac{\alpha(1-\alpha^n)}{(1-\alpha)}C}$$

$$x_2 = \alpha(x_1 - y_1) = \alpha^2 C\left(1 - \frac{1}{1+\alpha+\cdots+\alpha^{n-1}}\right) = \alpha^3 C\left(\frac{1-\alpha^{n-1}}{1-\alpha^n}\right)$$

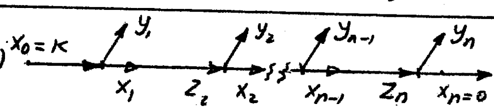
$$y_2^* = \alpha^3 C \frac{(1-\alpha)}{1-\alpha^n}$$

In general, we have

$$y_i^* = \alpha^{i+1} C \left(\frac{1-\alpha}{1-\alpha^{n-i+1}}\right)$$

3

(a)



$$f_n(z_n) = \max_{y_n = z_n \le 2^n K} \{ p_n y_n \}$$

$$f_i(z_i) = \max_{y_i \le z_i \le 2K} \{ p_i y_i + f_{i+1}(2[z_i - y_i]) \}$$

$$i = 1, 2, \ldots, n-1$$

continued...

Set 10.3d

(b) Stage (year) 3:

z_3	$120y_3$									Optimum	
	$y_3=0$	1	2	3	4	5	6	7	8	f_3	y_3^*
0	0									0	0
1		120								120	1
2			240							240	2
3				360						360	3
4					480					480	4
5						600				600	5
6							720			720	6
7								840		840	7
8									960	960	8

Stage (year) 2:

z_2	$130y_2 + f_3(2[z_2-y_2])$						
	$y_2=0$	1	2	3	4	f_2	y_2^*
0	0 + 0 = 0	—				0	0
1	0 + 240 = 240	130 + 0 = 130	—			240	0
2	0 + 480 = 480	130 + 240 = 370	260 + 0 = 260	—		480	0
3	0 + 720 = 720	130 + 480 = 610	260 + 240 = 500	390 + 0 = 390	—	720	0
4	0 + 960 = 960	130 + 720 = 850	260 + 480 = 740	390 + 240 = 630	520 + 0 = 520	960	0

Stage (year) 1:

z_1	$100y_1 + f_2(2[z_1-y_1])$			Optimum	
	$y_1=0$	1	2	f_1	y_1^*
0	—	—	—	—	—
1	—	—	—	—	—
2	0 + 960 = 960	100 + 480 = 580	200 + 0 = 200	960	0

Solution:

$z_1 = 2 \rightarrow y_1 = 0 \rightarrow z_2 = 4 \rightarrow y_2 = 0 \rightarrow z_3 = 8 \rightarrow y_3 = 8$

Revenue = \$960

1

(a)

$$f_2(v_2,w_2)=\max_{\substack{0\le 7x_2\le v_2\\ 0\le 2x_2\le w_2}}\{14x_2\}$$

$$=14\min\left\{\frac{v_2}{7},\frac{w_2}{2}\right\}$$

$$x_2^*=\min\left\{\frac{v_2}{7},\frac{w_2}{2}\right\}$$

$$f_1(v_1,w_1)=\max_{\substack{0\le 2x_1\le v_1\\ 0\le 7x_1\le w_1}}\{4x_1+f_2(v_1-2x_1,w_1-7x_1)\}$$

$$=\max\left(4x_1+14\min\left\{\frac{v_1-2x_1}{7},\frac{w_1-7x_1}{2}\right\}\right)$$

For $v_1=w_1=21$, $0\le x_1\le 3$,

$$f_1(21,21)=\max\begin{cases}42, & 0\le x_1\le 7/3\\ 147-45x_1, & 7/3\le x_1\le 3\end{cases}$$

$$=42 \quad \text{for } 0\le x_1^*\le 7/3$$

Next, $v_2=v_1-2x_1=21-2x_1^*$

$w_2=w_2-7x_1=21-7x_1^*$

$$x_2^*=\min\left\{\frac{21-2x_1^*}{7},\frac{21-7x_1^*}{2}\right\}$$

$$=3-\frac{2}{7}x_1^*,\quad 0\le x_1^*\le 7/3$$

Problem has infinite alternative solutions.

(b)

$$f_2(v_2,w_2)=\max_{\substack{0\le x_2\le v_2\\ 0\le 2x_2\le w_2\\ x_2 \text{ integer}}}\{7x_2\}$$

$$=7\min\left\{[v_2],\left[\frac{w_2}{2}\right]\right\}$$

where $[a]$ = largest integer $\le a$.

$$f_1(v_1,w_1)=\max_{\substack{0\le 2x_1\le v_1\\ 0\le 5x_1\le w_1}}\{8x_1+f_2(v_1-2x_1,w_1-5x_1)\}$$

$$=\max\left\{8x_1+7\min\left([8-2x_1],\left[\frac{15-5x_1}{2}\right]\right)\right\}$$

$$x_1\le\min\left\{\left[\frac{v_1}{2}\right],\left[\frac{w_1}{5}\right]\right\}=\min\left\{\left[\frac{8}{2}\right],\left[\frac{15}{5}\right]\right\}=3$$

$$f_1(v_1,w_1)=\max_{x_1=0,1,2,3}\left\{8x_1+7\min\left([8-2x_1],\left[\frac{15-5x_1}{2}\right]\right)\right\}$$

continued...

$$=\max_{x_1=0,1,2,3}\left\{8x_1+7\left[\frac{15-5x_1}{2}\right]\right\}$$

$$=49 \text{ at } x_1^*=0$$

$v_2=v_1-2x_1=v_1=8$

$w_2=w_1-5x_1=w_1=15$

$$x_2^*=\min\left\{[8],\left[\frac{15}{2}\right]\right\}=7$$

Optimum: $(x_1,x_2)=(0,7)$, $z=49$

(c)

Forward formulation:

$$f_1(v_1,w_1)=\max_{\substack{0\le x_1\le v_1\\ 0\le x_1\le w_1}}(7x_1^2+6x_1)$$

$$=\min\{7v_1^2+6v_1,\ 7w_1^2+6w_1\}$$

where $x_1^*=\min\{v_1,w_1\}$

$$f_2(v_2,w_2)=\max_{0\le x_2\le 5}\left\{5x_2^2+\min\left[7(v_2-x_2)^2+6(v_2-x_2),\ 7(w_2-x_2)^2+6(w_2-x_2)\right]\right\}$$

Now, $v_2=10$:

$0\le v_1=10-2x_2 \Rightarrow 0\le x_2\le 5$

$0\le v_1-x_1 \Rightarrow 0\le x_1\le v_1$

$w_2=9$:

$0\le w_1=9+3x_2 \Rightarrow x_2\ge 0$

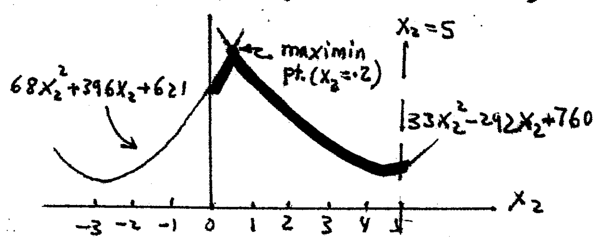
$0\le w_1-x_1 \Rightarrow 0\le x_1\le w_1$

With $v_2=10$ and $w_2=9$, we get

$$f_2(v_2,w_2)$$

$$=\max_{0\le x_2\le 5}\left\{5x_2^2+\min\left[28x_2^2-292x_2+760,\ 63x_2^2+396x_2+621\right]\right\}$$

$$=\max_{0\le x_2\le 5}\left\{\min\left[33x_2^2-292x_2+760,\ 68x_2^2+396x_2+621\right]\right\}$$



continued...

Optimal solution:

$v_2 = 10,\ w_2 = 9 \Rightarrow x_2^* = .2$

$\left.\begin{array}{l} v_1 = 10 - 2\times .2 = 9.6 \\ w_1 = 9 + 3\times .2 = 9.6 \end{array}\right\} \Rightarrow x_1^* = 9.6$

Optimal objective value $= 702.92$

2

Maximize $z = r_1x_1 + r_2x_2 + \cdots + r_nx_n$

Subject to

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n \le W$$
$$v_1x_1 + v_2x_2 + \cdots + v_nx_n \le V$$
$$x_j \ge 0 \text{ and integer}$$

where

x_j = number of units of item j

D.P. backward formulation:

Let

a_j = weight allocated to items $j, j+1, \ldots,$ and n

b_j = volume allocated to items $j, j+1, \ldots,$ and n

$f_j(a_j, b_j)$ = optimum revenue for items $j, j+1,$ and n, given a_j and b_j

$$f_n(a_n, b_n) = \max_{\substack{0 \le w_nx_n \le a_n \\ 0 \le v_nx_n \le b_n}} \{r_nx_n\}$$

$$f_j(a_j, b_j) = \max_{\substack{0 \le w_jx_j \le a_j \\ 0 \le v_jx_j \le b_j}} \{r_jx_j + f_{j+1}(a_j - w_jx_j, b_j - v_jx_j)\}$$

Order of computations

$f_n \rightarrow f_{n-1} \rightarrow \cdots \rightarrow f_1$

$a_1 = W$

$b_1 = V$