must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|}. \tag{7}$$

Thus, since inequality (7) is the same as $|z - z_0| > |z_1 - z_0|$, series (6) must converge absolutely to a continuous function in the domain *exterior to* the circle $|z - z_0| = R_1$, where $R_1 = |z_1 - z_0|$. Also, we know that if a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid in an annulus $R_1 < |z - z_0| < R_2$, then *both* of the series on the right converge uniformly in any closed annulus which is concentric to and interior to that region of validity.

71. INTEGRATION AND DIFFERENTIATION OF POWER SERIES

We have just seen that a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

represents a continuous function at each point interior to its circle of convergence. In this section, we prove that the sum $S(z)$ is actually analytic within that circle. Our proof depends on the following theorem, which is of interest in itself.

Theorem 1. *Let C denote any contour interior to the circle of convergence of the power series* (1)*, and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is,*

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\,dz. \tag{2}$$

To prove this theorem, we note that since both $g(z)$ and the sum $S(z)$ of the power series are continuous on C, the integral over C of the product

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n\, g(z)(z - z_0)^n + g(z)\rho_N(z),$$

where $\rho_N(z)$ is the remainder of the given series after N terms, exists. The terms of the finite sum here are also continuous on the contour C, and so their integrals over

C exist. Consequently, the integral of the quantity $g(z)\rho_N(z)$ must exist; and we may write

$$(3) \qquad \int_C g(z)S(z)\,dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz.$$

Now let M be the maximum value of $|g(z)|$ on C, and let L denote the length of C. In view of the uniform convergence of the given power series (Sec. 69), we know that for each positive number ε there exists a positive integer N_ε such that, for all points z on C,

$$|\rho_N(z)| < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon.$$

Since N_ε is independent of z, we find that

$$\left|\int_C g(z)\rho_N(z)\,dz\right| < M\varepsilon L \quad \text{whenever} \quad N > N_\varepsilon;$$

that is,

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)\,dz = 0.$$

It follows, therefore, from equation (3) that

$$\int_C g(z)S(z)\,dz = \lim_{N\to\infty} \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz.$$

This is the same as equation (2), and Theorem 1 is proved.

If $|g(z)| = 1$ for each value of z in the open disk bounded by the circle of convergence of power series (1), the fact that $(z-z_0)^n$ is entire when $n = 0, 1, 2, \ldots$ ensures that

$$\int_C g(z)(z-z_0)^n\,dz = \int_C (z-z_0)^n\,dz = 0 \qquad (n = 0, 1, 2, \ldots)$$

for every *closed* contour C lying in that domain. According to equation (2), then,

$$\int_C S(z)\,dz = 0$$

for every such contour; and, by Morera's theorem (Sec. 57), the function $S(z)$ is analytic throughout the domain. We state this result as a corollary.

Corollary. *The sum $S(z)$ of power series* (1) *is analytic at each point z interior to the circle of convergence of that series.*

This corollary is often helpful in establishing the analyticity of functions and in evaluating limits.

EXAMPLE 1. To illustrate, let us show that the function defined by means of the equations

$$f(z) = \begin{cases} (\sin z)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. Since the Maclaurin series representation

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{4}$$

is valid for every value of z, the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \tag{5}$$

obtained by dividing each side of equation (4) by z, converges to $f(z)$ when $z \neq 0$. Also, series (5) clearly converges to $f(z)$ when $z = 0$. Hence $f(z)$ is represented by the convergent series (5) for all z; and f is, therefore, an entire function.

Note that since $(\sin z)/z = f(z)$ when $z \neq 0$ and f is continuous at $z = 0$,

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} f(z) = f(0) = 1.$$

This is a result known beforehand because the limit here is the definition of the derivative of $\sin z$ at $z = 0$. That is,

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\sin z - \sin 0}{z - 0} = \cos 0 = 1.$$

We observed in Sec. 62 that the Taylor series for a function f about a point z_0 converges to $f(z)$ at each point z interior to the circle centered at z_0 and passing through the nearest point z_1 where f fails to be analytic. In view of our corollary to Theorem 1, we now know that *there is no larger circle* about z_0 such that at each point z interior to it the Taylor series converges to $f(z)$. For if there were such a circle, f would be analytic at z_1; but f is not analytic at z_1.

We now present a companion to Theorem 1.

Theorem 2. *The power series* (1) *can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{6}$$

To prove this, let z denote any point interior to the circle of convergence of series (1). Then let C be some positively oriented simple closed contour surrounding z and interior to that circle. Also, define the function

$$g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s - z)^2} \tag{7}$$

at each point s on C. Since $g(s)$ is continuous on C, Theorem 1 tells us that

$$\int_C g(s)S(s)\,ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s-z_0)^n\,ds. \tag{8}$$

Now $S(z)$ is analytic inside and on C, and this enables us to write

$$\int_C g(s)S(s)\,ds = \frac{1}{2\pi i}\int_C \frac{S(s)\,ds}{(s-z)^2} = S'(z)$$

with the aid of the integral representation for derivatives in Sec. 55. Furthermore,

$$\int_C g(s)(s-z_0)^n\,ds = \frac{1}{2\pi i}\int_C \frac{(s-z_0)^n}{(s-z)^2}\,ds = \frac{d}{dz}(z-z_0)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus equation (8) reduces to

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n,$$

which is the same as equation (6). This completes the proof.

EXAMPLE 2. In Example 1, Sec. 64, we saw that

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n (z-1)^n \qquad (|z-1| < 1).$$

Differentiation of each side of this equation reveals that

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty}(-1)^n n(z-1)^{n-1} \qquad (|z-1| < 1),$$

or

$$\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n (n+1)(z-1)^n \qquad (|z-1| < 1).$$

72. UNIQUENESS OF SERIES REPRESENTATIONS

The uniqueness of Taylor and Laurent series representations, anticipated in Secs. 64 and 68, respectively, follows readily from Theorem 1 in Sec. 71. We consider first the uniqueness of Taylor series representations.

Theorem 1. *If a series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{1}$$

converges to $f(z)$ at all points interior to some circle $|z-z_0| = R$, then it is the Taylor series expansion for f in powers of $z-z_0$.

converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

The method of proof here is similar to the one used in proving Theorem 1. The hypothesis of this theorem tells us that there is an annular domain about z_0 such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for each point z in it. Let $g(z)$ be as defined by equation (4), but now allow n to be a negative integer too. Also, let C be any circle around the annulus, centered at z_0 and taken in the positive sense. Then, using the index of summation m and adapting Theorem 1 in Sec. 71 to series involving both nonnegative *and* negative powers of $z - z_0$ (Exercise 10), write

$$\int_C g(z) f(z)\, dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz,$$

or

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz. \tag{9}$$

Since equations (6) are also valid when the integers m and n are allowed to be negative, equation (9) reduces to

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = c_n \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which is expression (5), Sec. 66, for the coefficients c_n in the Laurent series for f in the annulus.

EXERCISES

1. By differentiating the Maclaurin series representation

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

obtain the expansions

$$\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)\, z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1 - z)^3} = \sum_{n=0}^{\infty} (n + 1)(n + 2)\, z^n \qquad (|z| < 1).$$

2. By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)\,z^n \qquad (|z| < 1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty).$$

(Compare with Example 2, Sec. 71.)

3. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2}\cdot\frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

4. Show that the function defined by means of the equations

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0, \\ 1/2 & \text{when } z = 0 \end{cases}$$

is entire. (See Example 1, Sec. 71.)

5. Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & \text{when } z \neq \pm\pi/2, \\[2ex] -\dfrac{1}{\pi} & \text{when } z = \pm\pi/2, \end{cases}$$

then f is an entire function.

6. In the w plane, integrate the Taylor series expansion (see Example 1, Sec. 64)

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1| < 1)$$

along a contour interior to its circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1| < 1).$$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{\operatorname{Log} z}{z-1} \qquad \text{when } z \neq 1$$

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty,\ -\pi < \operatorname{Arg} z < \pi.$$

then

$$\int_C g(z)S(z)\,dz = \sum_{n=-\infty}^{\infty} c_n \int_C g(z)(z-z_0)^n\,dz\,.$$

11. Show that the function

$$f_2(z) = \frac{1}{z^2+1} \qquad (z \neq \pm i)$$

is the analytic continuation (Sec. 28) of the function

$$f_1(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (|z| < 1)$$

into the domain consisting of all points in the z plane except $z = \pm i$.

12. Show that the function $f_2(z) = 1/z^2$ $(z \neq 0)$ is the analytic continuation (Sec. 28) of the function

$$f_1(z) = \sum_{n=0}^{\infty} (n+1)(z+1)^n \qquad (|z+1| < 1)$$

into the domain consisting of all points in the z plane except $z = 0$.

73. MULTIPLICATION AND DIVISION OF POWER SERIES

Suppose that each of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n \tag{1}$$

converges within some circle $|z - z_0| = R$. Their sums $f(z)$ and $g(z)$, respectively, are then analytic functions in the disk $|z - z_0| < R$ (Sec. 71), and the product of those sums has a Taylor series expansion which is valid there:

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n \qquad (|z-z_0| < R). \tag{2}$$

According to Theorem 1 in Sec. 72, the series (1) are themselves Taylor series. Hence the first three coefficients in series (2) are given by the equations

$$c_0 = f(z_0)g(z_0) = a_0 b_0,$$

$$c_1 = \frac{f(z_0)g'(z_0) + f'(z_0)g(z_0)}{1!} = a_0 b_1 + a_1 b_0,$$

and

$$c_2 = \frac{f(z_0)g''(z_0) + 2f'(z_0)g'(z_0) + f''(z_0)g(z_0)}{2!} = a_0 b_2 + a_1 b_1 + a_2 b_0.$$

The general expression for any coefficient c_n is easily obtained by referring to ***Leibniz's rule*** (Exercise 7)

$$(3)\qquad [f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) \qquad (n = 1, 2, \ldots),$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n),$$

for the nth derivative of the product of two differentiable functions. As usual, $f^{(0)}(z) = f(z)$ and $0! = 1$. Evidently,

$$c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k};$$

and so expansion (2) can be written

$$\begin{aligned}(4)\qquad f(z)g(z) = a_0 b_0 &+ (a_0 b_1 + a_1 b_0)(z - z_0) \\ &+ (a_0 b_2 + a_1 b_1 + a_2 b_0)(z - z_0)^2 + \cdots \\ &+ \left(\sum_{k=0}^{n} a_k b_{n-k} \right)(z - z_0)^n + \cdots \qquad (|z - z_0| < R).\end{aligned}$$

Series (4) is the same as the series obtained by formally multiplying the two series (1) term by term and collecting the resulting terms in like powers of $z - z_0$; it is called the ***Cauchy product*** of the two given series.

EXAMPLE 1. The function

$$f(z) = \frac{\sinh z}{1 + z}$$

has a singular point at $z = -1$, and so its Maclaurin series representation is valid in the open disk $|z| < 1$. The first four nonzero terms are easily found by writing

$$(\sinh z)\left(\frac{1}{1+z}\right) = \left(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)(1 - z + z^2 - z^3 + \cdots)$$

and multiplying these two series term by term. To be precise, we may multiply each term in the first series by 1, then each term in the first series by $-z$, etc. The following systematic approach is suggested, where like powers of z are assembled vertically so that their coefficients can be readily added:

$$\begin{array}{llllllll}
z & & +\frac{1}{6}z^3 & & +\frac{1}{120}z^5 & +\cdots & & \\
 & -z^2 & & -\frac{1}{6}z^4 & & & -\frac{1}{120}z^6 & -\cdots \\
 & & z^3 & & +\frac{1}{6}z^5 & & & +\cdots \\
 & & & -z^4 & & & +\frac{1}{6}z^6 & -\cdots \\
 & & & & \vdots & & &
\end{array}$$

The desired result, involving four nonzero terms, is found to be

$$\frac{\sinh z}{1+z} = z - z^2 + \frac{7}{6}z^3 - \frac{7}{6}z^4 + \cdots \qquad (|z| < 1). \tag{5}$$

Continuing to let $f(z)$ and $g(z)$ denote the sums of series (1), suppose that $g(z) \neq 0$ when $|z - z_0| < R$. Since the quotient $f(z)/g(z)$ is analytic throughout the disk $|z - z_0| < R$, it has a Taylor series representation

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z - z_0)^n \qquad (|z - z_0| < R), \tag{6}$$

where the coefficients d_n can be found by differentiating $f(z)/g(z)$ successively and evaluating the derivatives at $z = z_0$. The results are the same as those found by formally carrying out the division of the first of series (1) by the second. Since it is usually only the first few terms that are needed in practice, this method is not difficult.

EXAMPLE 2. As pointed out in Sec. 39, the zeros of the entire function $\sinh z$ are $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$. So the reciprocal

$$\frac{1}{\sinh z} = \frac{1}{z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots},$$

which can be written

$$\frac{1}{\sinh z} = \frac{1}{z}\left(\frac{1}{1 + \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots}\right), \tag{7}$$

has a Laurent series representation in the punctured disk $0 < |z| < \pi$. A power series representation of the function in parentheses here can be found by dividing the series in the denominator into unity as follows:

$$
\begin{array}{r|l}
 & 1 - \dfrac{1}{3!}z^2 + \left[\dfrac{1}{(3!)^2} - \dfrac{1}{5!}\right] z^4 + \cdots \\ \hline
1 + \dfrac{1}{3!}z^2 + \dfrac{1}{5!}z^4 + \cdots & 1 \\
 & 1 + \dfrac{1}{3!}z^2 \qquad + \dfrac{1}{5!}z^4 + \cdots \\ \cline{2-2}
 & \quad - \dfrac{1}{3!}z^2 \qquad - \dfrac{1}{5!}z^4 + \cdots \\
 & \quad - \dfrac{1}{3!}z^2 \qquad - \dfrac{1}{(3!)^2}z^4 - \cdots \\ \cline{2-2}
 & \qquad\qquad \left[\dfrac{1}{(3!)^2} - \dfrac{1}{5!}\right] z^4 + \cdots \\
 & \qquad\qquad \left[\dfrac{1}{(3!)^2} - \dfrac{1}{5!}\right] z^4 + \cdots \\ \cline{2-2}
 & \qquad\qquad\qquad \vdots
\end{array}
$$

This shows that

$$\frac{1}{1 + \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots} = 1 - \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^4 + \cdots \qquad (|z| < \pi),$$

or

$$\frac{1}{1 + \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots} = 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots \qquad (|z| < \pi). \tag{8}$$

In view of equation (7), then,

$$\frac{1}{\sinh z} = \frac{1}{z} - \frac{1}{6}z + \frac{7}{360}z^3 + \cdots \qquad (0 < |z| < \pi). \tag{9}$$

Although we have given only the first three nonzero terms of this Laurent series, any number of terms can, of course, be found by continuing the division.

EXERCISES

1. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1).$$

2. By multiplying two Maclaurin series term by term, show that

(*a*) $e^z \sin z = z + z^2 + \dfrac{1}{3}z^3 + \cdots \qquad (|z| < \infty);$

(*b*) $$\frac{e^2}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \cdots \qquad (|z| < 1).$$

3. By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots \qquad (0 < |z| < \pi).$$

4. Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \qquad (0 < |z| < 2\pi).$$

5. Note how the expansion

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi)$$

is an immediate consequence of the Laurent series (8) in Sec. 73. Then use the method illustrated in Example 4, Sec. 68, to show that

$$\int_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3}$$

when C is the positively oriented unit circle $|z| = 1$.

6. Follow these steps, which illustrate an alternative to straightforward division, to obtain equation (8) in Example 2, Sec. 73.

(*a*) Write

$$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots,$$

where the coefficients in the power series on the right are to be determined by multiplying the two series in the equation

$$1 = \left(1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right)(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots).$$

Perform this multiplication to show that

$$(d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!}d_0\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \cdots = 0$$

when $|z| < \pi$.

(*b*) By setting the coefficients in the last series in part (*a*) equal to zero, find the values of d_0, d_1, d_2, d_3, and d_4. With these values, the first equation in part (*a*) becomes equation (8), Sec. 73.

7. Use mathematical induction to establish Leibniz's rule (Sec. 73)

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \qquad (n = 1, 2, \ldots)$$

for the nth derivative of the product of two differentiable functions $f(z)$ and $g(z)$.