EXAMPLE 4. Using term by term differentiation, which will be justified in Sec. 71, we differentiate each side of equation (3) and write

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dz} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} z^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$(|z| < \infty).$$

Expansion (4) is now verified.

EXAMPLE 5. Because $\sinh z = -i \sin(iz)$, as pointed out in Sec. 39, we need only recall expansion (3) for $\sin z$ and write

$$\sinh z = -i \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n+1}}{(2n+1)!} \qquad (|z| < \infty),$$

which becomes

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

EXAMPLE 6. Since $\cosh z = \cos(iz)$, according to Sec. 39, the Maclaurin series (4) for $\cos z$ reveals that

$$\cosh z = \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n}}{(2n)!} \qquad (|z| < \infty),$$

and we arrive at the Maclaurin series representation

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty).$$

Observe that the Taylor series for $\cosh z$ about the point $z_0 = -2\pi i$, for example, is obtained by replacing the variable z on each side of this last equation by $z + 2\pi i$ and then recalling (Sec. 39) that $\cosh(z + 2\pi i) = \cosh z$ for all z:

$$\cosh z = \sum_{n=0}^{\infty} \frac{(z + 2\pi i)^{2n}}{(2n)!} \qquad (|z| < \infty).$$

65. NEGATIVE POWERS OF $(z - z_0)$

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem there. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $(z - z_0)$. Such series are extremely important and are taken up in the next section. They are often obtained by using one or more of the six Maclaurin series listed at the beginning of Sec. 64. In order that the reader be accustomed to series involving negative powers of $(z - z_0)$, we pause here with several examples before exploring their general theory.

EXAMPLE 1. Using the familiar Maclaurin series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \qquad (|z| < \infty),$$

we can see that

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots\right) = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \frac{z^2}{4!} - \cdots$$

when $0 < |z| < \infty$.

EXAMPLE 2. From the Maclaurin series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

it follows that when $0 < |z| < \infty$,

$$z^3 \cosh\left(\frac{1}{z}\right) = z^3 \sum_{n=0}^{\infty} \frac{1}{(2n)!z^{2n}} = \sum_{n=0}^{\infty} \frac{1}{(2n)!z^{2n-3}}.$$

We note that $2n - 3 < 0$ when n is 0 or 1 but that $2n - 3 > 0$ when $n \geq 2$. Hence this last series can be rewritten so that

$$z^3 \cosh\left(\frac{1}{z}\right) = z^3 + \frac{z}{2} + \sum_{n=2}^{\infty} \frac{1}{(2n)!z^{2n-3}} \qquad (0 < |z| < \infty).$$

Anticipating a standard form for such an expansion in the next section, we can replace n by $n + 1$ in this series to arrive at

$$z^3 \cos\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}} \qquad (0 < |z| < \infty).$$

EXAMPLE 3. For our next example, let us expand the function

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5} = \frac{1}{z^3} \cdot \frac{2(1 + z^2) - 1}{1 + z^2} = \frac{1}{z^3}\left(2 - \frac{1}{1 + z^2}\right)$$

into a series involving powers of z. We cannot find a Maclaurin series since $f(z)$ is not analytic at $z = 0$. But we do know that

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + z^4 + \cdots \qquad (|z| < 1);$$

and, after replacing z by $-z^2$ on each side here, we have

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + z^8 - \cdots \qquad (|z| < 1).$$

So when $0 < |z| < 1$,

$$f(z) = \frac{1}{z^3}(2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots.$$

We call such terms as $1/z^3$ and $1/z$ *negative* powers of z since they can be written z^{-3} and z^{-1}, respectively. As already noted at the beginning of this section, the theory of expansions involving negative powers of $(z - z_0)$ will be discussed in the next section.

The reader will notice that in the series obtained in Examples 1 and 3 the negative powers appear first but that the positive powers appear first in Example 2. Whether the positive or negative powers come first is usually immaterial in the applications later on. Also, these three examples involve powers of $(z - z_0)$ when $z_0 = 0$. Our final example here does, however, involve a nonzero z_0.

EXAMPLE 4. We propose here to expand the function

$$\frac{e^z}{(z+1)^2}$$

in powers of $(z + 1)$. We start with the Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)$$

and replace z by $(z + 1)$:

$$e^{z+1} = \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!} \qquad (|z+1| < \infty).$$

Dividing through this equation by $e(z + 1)^2$ reveals that

$$\frac{e^z}{(z+1)^2} = \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!e}.$$

So we have

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty} \frac{(z+1)^{n-2}}{n!}\right] \qquad (0 < |z+1| < \infty),$$

which is the same as

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right] \qquad (0 < |z+1| < \infty).$$

EXERCISES*

1. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

*In these and subsequent exercises on series expansions, it is recommended that the reader use, when possible, representations (1) through (6) in Sec. 64.

2. Obtain the Taylor series

$$e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

for the function $f(z) = e^z$ by

(*a*) using $f^{(n)}(1)$ $(n = 0, 1, 2, \ldots)$; (*b*) writing $e^z = e^{z-1}e$.

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+4} = \frac{z}{4} \cdot \frac{1}{1 + (z^4/4)}.$$

Ans. $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{2}).$

4. With the aid of the identity (see Sec. 37)

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

5. Use the identity $\sinh(z + \pi i) = -\sinh z$, verified in Exercise 7(*a*), Sec. 39, and the fact that $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

Ans. $-\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!} \quad (|z - \pi i| < \infty).$

6. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

7. Show that if $f(z) = \sin z$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus give an alternative derivation of the Maclaurin series (3) for $\sin z$ in Sec. 64.

8. Rederive the Maclaurin series (4) in Sec. 64 for the function $f(z) = \cos z$ by

(*a*) using the definition

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

in Sec. 37 and appealing to the Maclaurin series (2) for e^z in Sec. 64;

(*b*) showing that

$$f^{(2n)}(0) = (-1)^n \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

9. Use representation (3), Sec. 64, for $\sin z$ to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

10. Derive the expansions

$$(a)\quad \frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \qquad (0 < |z| < \infty);$$

$$(b)\quad \frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \qquad (0 < |z| < \infty).$$

11. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

66. LAURENT SERIES

We turn now to a statement of ***Laurent's theorem***, which enables us to expand a function $f(z)$ into a series involving positive and negative powers of $(z - z_0)$ when the function fails to be analytic at z_0.

Theorem. *Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain* (Fig. 80). *Then, at each point in the domain, $f(z)$ has the series representation*

$$(1)\qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (R_1 < |z - z_0| < R_2),$$

where

$$(2)\qquad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots)$$

and

$$(3)\qquad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \qquad (n = 1, 2, \ldots).$$

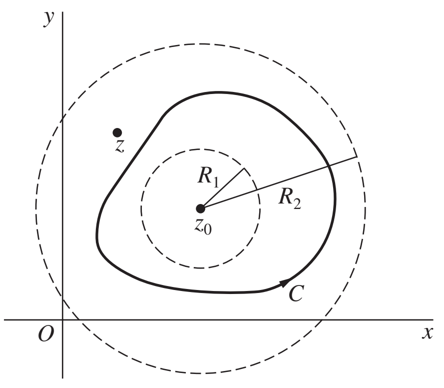

FIGURE 80

Note how replacing n by $-n$ in the second series in representation (1) enables us to write that series as

$$\sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}},$$

where

$$b_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n = -1, -2, \ldots).$$

Thus

$$f(z) = \sum_{n=-\infty}^{-1} b_{-n}(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad (R_1 < |z-z_0| < R_2).$$

If

$$c_n = \begin{cases} b_{-n} & \text{when } n \le -1, \\ a_n & \text{when } n \ge 0, \end{cases}$$

this becomes

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \qquad (R_1 < |z-z_0| < R_2) \tag{4}$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

In either one of the forms (1) and (4), the representation of $f(z)$ is called a ***Laurent series***.

Observe that the integrand in expression (3) can be written $f(z)(z-z_0)^{n-1}$. Thus it is clear that when f is actually analytic throughout the disk $|z-z_0| < R_2$, this integrand is too. Hence all of the coefficients b_n are zero; and, because (Sec. 55)

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

expansion (1) reduces to a Taylor series about z_0.

If, however, f fails to be analytic at z_0 but is otherwise analytic in the disk $|z-z_0| < R_2$, the radius R_1 can be chosen arbitrarily small. Representation (1) is then valid in the punctured disk $0 < |z-z_0| < R_2$. Similarly, if f is analytic at each point in the finite plane exterior to the circle $|z-z_0| = R_1$, the condition of validity is $R_1 < |z-z_0| < \infty$. Note that if f is analytic *everywhere* in the finite plane except at z_0, series (1) is valid at each point of analyticity, or when $0 < |z-z_0| < \infty$.

We shall prove Laurent's theorem first when $z_0 = 0$, which means that the annulus is centered at the origin. The verification of the theorem when z_0 is arbitrary will follow readily; and, as was the case with Taylor's theorem, a reader can skip the entire proof without difficulty.

68. EXAMPLES

The coefficients in a Laurent series are generally found by means other than appealing directly to the integral representations in Laurent's theorem (Sec. 66). This has already been illustrated in Sec. 65, where the series found were actually Laurent series. The reader is encouraged to go back to Sec. 65, as well as to Exercises 10 and 11 of that section, in order to see how in each case the punctured plane or disk in which the series is valid can now be predicted by Laurent's theorem. Also, we shall always assume that the Maclaurin series expansions (1) through (6) in Sec. 64 are well known, since we shall need them so often in finding Laurent series. As was the case with Taylor series, we defer the proof of uniqueness of Laurent series till Sec. 72.

EXAMPLE 1. The function

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z} \cdot \frac{1}{1+z^2}$$

has singularities at the points $z = 0$ and $z = \pm i$. Let us find the Laurent series representation of $f(z)$ that is valid in the punctured disk $0 < |z| < 1$ (see Fig. 82).

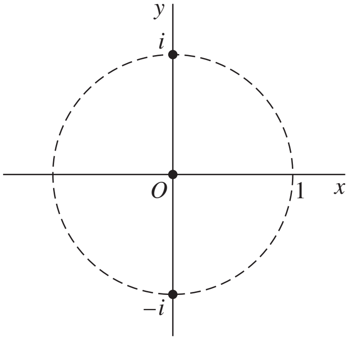

FIGURE 82

Since $|-z^2| < 1$ when $|z| < 1$, we may substitute $-z^2$ for z in the Maclaurin series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1). \tag{1}$$

The result is

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (|z| < 1),$$

and so

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} \qquad (0 < |z| < 1).$$

That is,

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{2n-1} \qquad (0 < |z| < 1).$$

Replacing n by $n + 1$, we arrive at

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1} \qquad (0 < |z| < 1).$$

In standard form, then,

$$f(z) = \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1} + \frac{1}{z} \qquad (0 < |z| < 1). \tag{2}$$

(See also Exercise 3.)

EXAMPLE 2. The function

$$f(z) = \frac{z+1}{z-1},$$

which has the singular point $z = 1$, is analytic in the domains (Fig. 83)

$$D_1 : |z| < 1 \quad \text{and} \quad D_2 : 1 < |z| < \infty.$$

In these domains $f(z)$ has series representations in powers of z. Both series can be found by making appropriate replacements for z in the same expansion (1) that was used in Example 1.

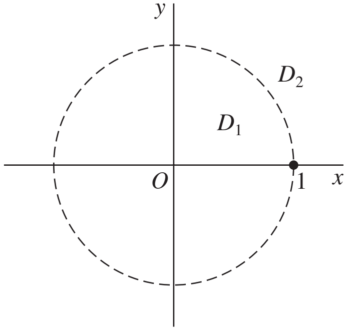

FIGURE 83

We consider first the domain D_1 and note that the series asked for is a Maclaurin series. In order to use series (1), we write

$$f(z) = -(z+1)\frac{1}{1-z} = -z\frac{1}{1-z} - \frac{1}{1-z}.$$

Then

$$f(z) = -z\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

Replacing n by $n-1$ in the first of the two series on the far right here yields the desired Maclaurin series:

$$f(z) = -\sum_{n=1}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -1 - 2\sum_{n=1}^{\infty} z^n \qquad (|z| < 1). \tag{3}$$

The representation of $f(z)$ in the unbounded domain D_2 is a Laurent series, and the fact that $|1/z| < 1$ when z is a point in D_2 suggests that we use series (1) to write

$$f(z) = \frac{1 + \dfrac{1}{z}}{1 - \dfrac{1}{z}} = \left(1 + \frac{1}{z}\right)\frac{1}{1 - \dfrac{1}{z}} = \left(1 + \frac{1}{z}\right)\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$
$$(1 < |z| < \infty).$$

Substituting $n-1$ for n in the last of these series reveals that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{1}{z^n} \qquad (1 < |z| < \infty),$$

and we arrive at the Laurent series

$$f(z) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n} \qquad (1 < |z| < \infty). \tag{4}$$

EXAMPLE 3. Replacing z by $1/z$ in the Maclaurin series expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we have the Laurent series representation

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \qquad (0 < |z| < \infty).$$

Note that no positive powers of z appear here, since the coefficients of the positive powers are zero. Note, too, that the coefficient of $1/z$ is unity; and, according to Laurent's theorem in Sec. 66, that coefficient is the number

$$b_1 = \frac{1}{2\pi i}\int_C e^{1/z}\, dz$$

where C is any positively oriented simple closed contour around the origin. Since $b_1 = 1$, then,

$$\int_C e^{1/z}\,dz = 2\pi i.$$

This method of evaluating certain integrals around simple closed contours will be developed in considerable detail in Chap. 6 and then used extensively in Chap. 7.

EXAMPLE 4. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series, where $z_0 = i$. That is,

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n (z-i)^n \qquad (0 < |z-i| < \infty)$$

where $c_{-2} = 1$ and all of the other coefficients are zero. From expression (5), Sec. 66, for the coefficients in a Laurent series, we know that

$$c_n = \frac{1}{2\pi i}\int_C \frac{dz}{(z-i)^{n+3}} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

where C is, for instance, any positively oriented circle $|z-i| = R$ about the point $z_0 = i$. Thus [compare with Exercise 13, Sec. 46]

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{when } n \neq -2, \\ 2\pi i & \text{when } n = -2. \end{cases}$$

EXERCISES

1. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

Ans. $1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$.

2. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$.

3. Find the Laurent series that represents the function $f(z)$ in Example 1, Sec. 68, when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}}$.

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

$$Ans. \sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \quad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty).$$

5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2},$$

which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains (Fig. 84)

$$D_1 : |z| < 1, \quad D_2 : 1 < |z| < 2, \quad D_3 : 2 < |z| < \infty.$$

Find the series representation in powers of z for $f(z)$ in each of those domains.

$$Ans. \sum_{n=0}^{\infty}(2^{-n-1} - 1)z^n \text{ in } D_1; \ \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n} \text{ in } D_2; \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \text{ in } D_3.$$

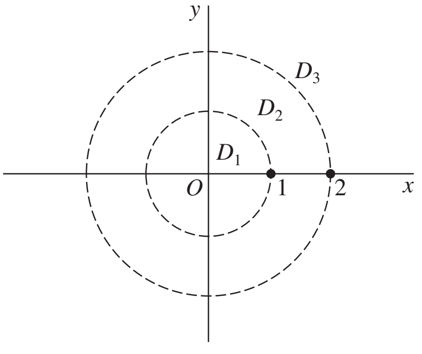

FIGURE 84

6. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

7. (*a*) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(*b*) After writing $z = e^{i\theta}$ in the equation obtained in part (*a*), equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 61.)