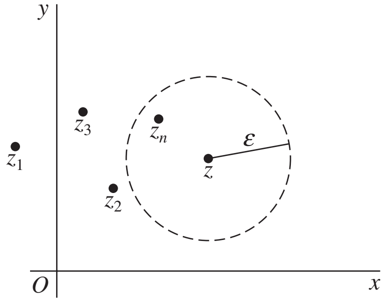

FIGURE 76

Theorem. *Suppose that* $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ *and* $z = x + iy$. *Then*

$$\lim_{n\to\infty} z_n = z \tag{3}$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y. \tag{4}$$

To prove this theorem, we first assume that conditions (4) hold and obtain condition (3) from it. According to conditions (4), there exist, for each positive number ε, positive integers n_1 and n_2 such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_1$$

and

$$|y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_2.$$

Hence if n_0 is the larger of the two integers n_1 and n_2,

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n > n_0.$$

Since

$$|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \le |x_n - x| + |y_n - y|,$$

then,

$$|z_n - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever} \quad n > n_0.$$

Condition (3) thus holds.

Conversely, if we start with condition (3), we know that for each positive number ε, there exists a positive integer n_0 such that

$$|(x_n + iy_n) - (x + iy)| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

But

$$|x_n - x| \le |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

and

$$|y_n - y| \le |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|;$$

and this means that

$$|x_n - x| < \varepsilon \quad \text{and} \quad |y_n - y| < \varepsilon \quad \text{whenever} \quad n > n_0.$$

That is, conditions (4) are satisfied.

Note how the theorem enables us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

whenever we know that both limits on the right exist or that the one on the left exists.

EXAMPLE 1. The sequence

$$z_n = -1 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots)$$

converges to -1 since

$$\lim_{n\to\infty} \left[-1 + i\frac{(-1)^n}{n^2}\right] = \lim_{n\to\infty} (-1) + i \lim_{n\to\infty} \frac{(-1)^n}{n^2} = -1 + i \cdot 0 = -1.$$

Definition (1) can also be used to obtain this result. More precisely,

$$|z_n - (-1)| = \left|i\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2} < \varepsilon \quad \text{whenever} \quad n > \frac{1}{\sqrt{\varepsilon}}.$$

One must be careful when adapting our theorem to polar coordinates, as the following example shows.

EXAMPLE 2. Consider now the same sequence

$$z_n = -1 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots)$$

as in Example 1. If we use the polar coordinates

$$r_n = |z_n| \quad \text{and} \quad \Theta_n = \operatorname{Arg} z_n \qquad (n = 1, 2, \ldots)$$

where $\operatorname{Arg} z_n$ denotes *principal* arguments $(-\pi < \Theta_n \le \pi)$, we find that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \sqrt{1 + \frac{1}{n^4}} = 1$$

but that

$$\lim_{n\to\infty} \Theta_{2n} = \pi \quad \text{and} \quad \lim_{n\to\infty} \Theta_{2n-1} = -\pi \qquad (n = 1, 2, \ldots).$$

Evidently, then, the limit of Θ_n does not exist as n tends to infinity. (See also Exercise 2, Sec. 61.)

61. CONVERGENCE OF SERIES

An infinite ***series***

(1) $$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

of complex numbers ***converges*** to the ***sum*** S if the sequence

(2) $$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \qquad (N = 1, 2, \ldots)$$

of ***partial sums*** converges to S; we then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Note that since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it ***diverges.***

Theorem. *Suppose that* $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ *and* $S = X + iY$. *Then*

(3) $$\sum_{n=1}^{\infty} z_n = S$$

if and only if

(4) $$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

This theorem tells us, of course, that one can write

$$\sum_{n=1}^{\infty}(x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n$$

whenever it is known that the two series on the right converge or that the one on the left does.

To prove the theorem, we first write the partial sums (2) as

(5) $$S_N = X_N + iY_N,$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n.$$

Now statement (3) is true if and only if

(6) $$\lim_{N\to\infty} S_N = S;$$

and, in view of relation (5) and the theorem on sequences in Sec. 60, limit (6) holds if and only if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y. \tag{7}$$

Limits (7) therefore imply statement (3), and conversely. Since X_N and Y_N are the partial sums of the series (4), the theorem here is proved.

This theorem can be useful in showing that a number of familiar properties of series in calculus carry over to series whose terms are complex numbers. To illustrate how this is done, we include here two such properties and present them as corollaries.

Corollary 1. *If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.*

Assuming that series (1) converges, we know from the theorem that if

$$z_n = x_n + iy_n \quad (n = 1, 2, \ldots),$$

then each of the series

$$\sum_{n=1}^{\infty} x_n \quad \text{and} \quad \sum_{n=1}^{\infty} y_n \tag{8}$$

converges. We know, moreover, from calculus that the nth term of a convergent series of real numbers approaches zero as n tends to infinity. Thus, by the theorem in Sec. 60,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + 0 \cdot i = 0;$$

and the proof of Corollary 1 is complete.

It follows from this corollary that the terms of convergent series are ***bounded.*** That is, when series (1) converges, there exists a positive constant M such that $|z_n| \le M$ for each positive integer n. (See Exercise 9.)

For another important property of series of complex numbers that follows from a corresponding property in calculus, series (1) is said to be ***absolutely convergent*** if the series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \qquad (z_n = x_n + iy_n)$$

of real numbers $\sqrt{x_n^2 + y_n^2}$ converges.

Corollary 2. *The absolute convergence of a series of complex numbers implies the convergence of that series.*

To prove Corollary 2, we assume that series (1) converges absolutely. Since

$$|x_n| \le \sqrt{x_n^2 + y_n^2} \quad \text{and} \quad |y_n| \le \sqrt{x_n^2 + y_n^2},$$

we know from the comparison test in calculus that the two series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|$$

must converge. Moreover, since the absolute convergence of a series of real numbers implies the convergence of the series itself, it follows that the series (8) both converge. In view of the theorem in this section, then, series (1) converges. This finishes the proof of Corollary 2.

In establishing the fact that the sum of a series is a given number S, it is often convenient to define the ***remainder*** ρ_N after N terms, using the partial sums (2):

$$\rho_N = S - S_N. \tag{9}$$

Thus $S = S_N + \rho_N$; and, since $|S_N - S| = |\rho_N - 0|$, we see that *a series converges to a number S if and only if the sequence of remainders tends to zero.* We shall make considerable use of this observation in our treatment of ***power series.*** They are series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots,$$

where z_0 and the coefficients a_n are complex constants and z may be any point in a stated region containing z_0. In such series, involving a variable z, we shall denote sums, partial sums, and remainders by $S(z)$, $S_N(z)$, and $\rho_N(z)$, respectively.

EXAMPLE. With the aid of remainders, it is easy to verify that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever} \quad |z| < 1. \tag{10}$$

We need only recall the identity (Exercise 9, Sec. 9)

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

to write the partial sums

$$S_N(z) = \sum_{n=0}^{N-1} z^n = 1 + z + z^2 + \cdots + z^{N-1} \qquad (z \neq 1)$$

as

$$S_N(z) = \frac{1 - z^N}{1 - z}.$$

If

$$S(z) = \frac{1}{1-z},$$

then,

$$\rho_N(z) = S(z) - S_N(z) = \frac{z^N}{1-z} \qquad (z \neq 1).$$

Thus

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|},$$

and it is clear from this that the remainders $\rho_N(z)$ tend to zero when $|z| < 1$ but not when $|z| \geq 1$. Summation formula (10) is, therefore, established.

EXERCISES

1. Use definition (1), Sec. 60, of limits of sequences to show that

$$\lim_{n\to\infty} \left(\frac{1}{n^2} + i\right) = i.$$

2. Let Θ_n $(n = 1, 2, \ldots)$ denote the principal arguments of the numbers

$$z_n = 1 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots),$$

and point out why

$$\lim_{n\to\infty} \Theta_n = 0.$$

(Compare with Example 2, Sec. 60.)

3. Use the inequality (see Sec. 5) $||z_n| - |z|| \leq |z_n - z|$ to show that

$$\text{if} \quad \lim_{n\to\infty} z_n = z, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |z|.$$

4. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 61. Then, with the aid of the theorem in Sec. 61, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

5. Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

6. Show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

7. Let c denote any complex number and show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} c z_n = cS.$$

8. By recalling the corresponding result for series of real numbers and referring to the theorem in Sec. 61, show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S \quad \text{and} \quad \sum_{n=1}^{\infty} w_n = T, \qquad \text{then} \quad \sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

9. Let a sequence z_n $(n = 1, 2, \ldots)$ converge to a number z. Show that there exists a positive number M such that the inequality $|z_n| \leq M$ holds for all n. Do this in each of the following ways.

(a) Note that there is a positive integer n_0 such that

$$|z_n| = |z + (z_n - z)| < |z| + 1$$

whenever $n > n_0$.

(b) Write $z_n = x_n + i y_n$ and recall from the theory of sequences of real numbers that the convergence of x_n and y_n $(n = 1, 2, \ldots)$ implies that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ $(n = 1, 2, \ldots)$ for some positive numbers M_1 and M_2.

62. TAYLOR SERIES

We turn now to ***Taylor's theorem,*** which is one of the most important results of the chapter.

Theorem. *Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0* (Fig. 77). *Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0), \tag{1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

That is, series (1) *converges to $f(z)$ when z lies in the stated open disk.*

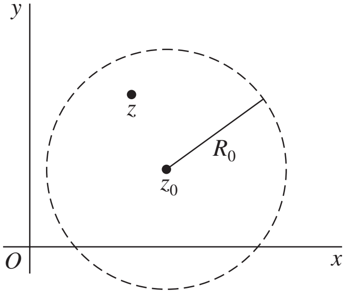

FIGURE 77

This is the expansion of $f(z)$ into a ***Taylor series*** about the point z_0. It is the familiar Taylor series from calculus, adapted to functions of a complex variable. With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

series (1) can, of course, be written

$$(3) \quad f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \quad (|z - z_0| < R_0).$$

Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if f is analytic at z_0, it is analytic throughout some neighborhood $|z - z_0| < \varepsilon$ of that point (Sec. 25); and ε may serve as the value of R_0 in the statement of Taylor's theorem. Also, if f is entire, R_0 can be chosen arbitrarily large; and the condition of validity becomes $|z - z_0| < \infty$. The series then converges to $f(z)$ at each point z in the finite plane.

When it is known that f is analytic everywhere inside a circle centered at z_0, convergence of its Taylor series about z_0 to $f(z)$ for each point z within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 at which f *fails* to be analytic. In Sec. 71, we shall find that this is actually the largest circle centered at z_0 such that the series converges to $f(z)$ for all z interior to it.

In the following section, we shall first prove Taylor's theorem when $z_0 = 0$, in which case f is assumed to be analytic throughout a disk $|z| < R_0$. Series (1) then becomes a ***Maclaurin series***:

$$(4) \qquad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (|z| < R_0).$$

The proof when z_0 is nonzero will follow as an immediate consequence. A reader who wishes to accept the proof of Taylor's theorem can easily skip to the examples in Sec. 64.

63. PROOF OF TAYLOR'S THEOREM

As indicated at the end of Sec. 62, the proof falls naturally into two parts.

The case $z_0 = 0$

To begin the derivation of representation (4) in Sec. 62, we write $|z| = r$ and let C_0 denote the positively oriented circle $|z| = r_0$ where $r < r_0 < R_0$ (see Fig. 78). Since f is analytic inside and on the circle C_0 and since the point z is interior to C_0, the Cauchy integral formula

$$(1) \qquad f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s - z}$$

applies.

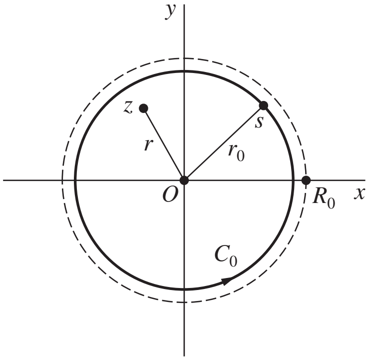

FIGURE 78

Now the factor $1/(s-z)$ in the integrand here can be put in the form

$$\frac{1}{s-z} = \frac{1}{s} \cdot \frac{1}{1-(z/s)}; \tag{2}$$

and we know from the example in Sec. 56 that

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z} \tag{3}$$

when z is any complex number other than unity. Replacing z by z/s in expression (3), then, we can rewrite equation (2) as

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}}\, z^n + z^N \frac{1}{(s-z)s^N}. \tag{4}$$

Multiplying through this equation by $f(s)$ and then integrating each side with respect to s around C_0, we find that

$$\int_{C_0} \frac{f(s)\,ds}{s-z} = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}}\, z^n + z^N \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}.$$

In view of expression (1) and the fact that (Sec. 55)

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}} = \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

this reduces, after we multiply through by $1/(2\pi i)$, to

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!}\, z^n + \rho_N(z), \tag{5}$$

where

$$\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}. \tag{6}$$

Representation (4) in Sec. 62 now follows once it is shown that

$$\lim_{N\to\infty} \rho_N(z) = 0. \tag{7}$$

To accomplish this, we recall that $|z| = r$ and that C_0 has radius r_0, where $r_0 > r$. Then, if s is a point on C_0, we can see that

$$|s - z| \geq ||s| - |z|| = r_0 - r.$$

Consequently, if M denotes the maximum value of $|f(s)|$ on C_0,

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N}\, 2\pi r_0 = \frac{Mr_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N.$$

Inasmuch as $(r/r_0) < 1$, limit (7) clearly holds.

The case $z_0 \neq 0$

In order to verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, we suppose that f is analytic when $|z - z_0| < R_0$ and note that the composite function $f(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < R_0$. This last inequality is, of course, just $|z| < R_0$; and, if we write $g(z) = f(z + z_0)$, the analyticity of g in the disk $|z| < R_0$ ensures the existence of a Maclaurin series representation:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \qquad (|z| < R_0).$$

That is,

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \qquad (|z| < R_0).$$

After replacing z by $z - z_0$ in this equation and its condition of validity, we have the desired Taylor series expansion (1) in Sec. 62.

64. EXAMPLES

In Sec. 72, we shall see that any Taylor series representing a function $f(z)$ about a given point z_0 is unique. More precisely, we will show that if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all points z interior to some circle centered at z_0, then the power series here must be *the* Taylor series for f about z_0, regardless of how those constants arise. This observation often allows us to find the coefficients a_n in Taylor series in more efficient ways than by appealing directly to the formula $a_n = f^{(n)}(z_0)/n!$ in Taylor's theorem.

This section is devoted to finding the following six Maclaurin series expansions, where $z_0 = 0$, and to illustrate how they can be used to find related expansions:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1), \tag{1}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \qquad (|z| < \infty), \tag{2}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad (|z| < \infty), \tag{3}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad (|z| < \infty), \tag{4}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad (|z| < \infty), \tag{5}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad (|z| < \infty). \tag{6}$$

We list these results together in order to have them for ready reference later on. Since the expansions are familiar ones from calculus with z instead of x, the reader should, however, find them easy to remember.

In addition to collecting expansions (1) through (6) together, we now present their derivations as Examples 1 through 6, along with some other series that are immediate consequences. The reader should always keep in mind that

(*a*) the regions of convergence can be determined before the actual series are found;
(*b*) there may be several reasonable ways to find the desired series.

EXAMPLE 1. Representation (1) was, of course, obtained earlier in Sec. 61, where Taylor's theorem was not used. In order to see how Taylor's theorem can be used, we first note that the point $z = 1$ is the only singularity of the function

$$f(z) = \frac{1}{1-z}$$

in the finite plane. So the desired Maclaurin series converges to $f(z)$ when $|z| < 1$.

The derivatives of $f(z)$ are

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \qquad (n = 1, 2, \ldots).$$

Hence if we agree that $f^{(0)}(z) = f(z)$ and $0! = 1$, we find that $f^{(n)}(0) = n!$ when $n = 0, 1, 2, \ldots$; and upon writing

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n,$$

we arrive at the series representation (1).

If we substitute $-z$ for z in equation (1) and its condition of validity, and note that $|z| < 1$ when $|-z| < 1$, we see that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n \qquad (|z| < 1).$$

If, on the other hand, we replace the variable z in equation (1) by $1 - z$, we have the Taylor series representation

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n (z-1)^n \qquad (|z-1| < 1).$$

This condition of validity follows from the one associated with expansion (1) since $|1 - z| < 1$ is the same as $|z - 1| < 1$.

For another application of expansion (1), we now seek a Taylor series representation of the function

$$f(z) = \frac{1}{1-z}$$

about the point $z_0 = i$. Since the distance between z_0 and the singularity $z = 1$ is $|1 - i| = \sqrt{2}$, the condition of validity is $|z - i| < \sqrt{2}$. (See Fig. 79.) To find the series, which involves powers of $z - i$, we first write

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\cdot\frac{1}{1-\left(\dfrac{z-i}{1-i}\right)}.$$

Because

$$\left|\frac{z-i}{1-i}\right| = \frac{|z-i|}{|1-i|} = \frac{|z-i|}{\sqrt{2}} < 1$$

when $|z - i| < \sqrt{2}$, expansion (1) now tells us that

$$\frac{1}{1-\left(\dfrac{z-i}{1-i}\right)} = \sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n \qquad (|z-i| < \sqrt{2});$$

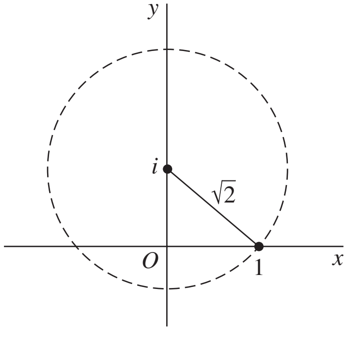

FIGURE 79
$|z - i| < \sqrt{2}$

and we arrive at the Taylor series expansion

$$\frac{1}{1-z} = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2}).$$

EXAMPLE 2. Since the function $f(z) = e^z$ is entire, it has a Maclaurin series representation that is valid for all z. Here $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, \ldots)$; and because $f^{(n)}(0) = 1 (n = 0, 1, 2, \ldots)$, expansion (2) follows. Note that if $z = x + i0$, the expansion becomes

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

The entire function z^3e^{2z} is also represented by a Maclaurin series. The simplest way to show this is to replace z by $2z$ in expression (2) and then multiply through the result by z^3:

$$z^3e^{2z} = \sum_{n=0}^{\infty}\frac{2^n}{n!}z^{n+3} \qquad (|z| < \infty).$$

Finally, if we replace n by $n - 3$ here, we have

$$z^3e^{2z} = \sum_{n=3}^{\infty}\frac{2^{n-3}}{(n-3)!}z^n \qquad (|z| < \infty).$$

EXAMPLE 3. One can use expansion (2) and the definition (Sec. 37)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

to find the Maclaurin series for the entire function $f(z) = \sin z$. To give the details, we refer to expansion (1) and write

$$\sin z = \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{i^nz^n}{n!} \qquad (|z| < \infty).$$

But $1 - (-1)^n = 0$ when n is even, and so we can replace n by $2n + 1$ in this last series:

$$\sin z = \frac{1}{2i}\sum_{n=0}^{\infty}\left[1 - (-1)^{2n+1}\right]\frac{i^{2n+1}z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

Inasmuch as

$$1 - (-1)^{2n+1} = 2 \quad \text{and} \quad i^{2n+1} = (i^2)^n i = (-1)^n i,$$

this reduces to expansion (3).

EXAMPLE 4. Using term by term differentiation, which will be justified in Sec. 71, we differentiate each side of equation (3) and write

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dz} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} z^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty).$$

Expansion (4) is now verified.

EXAMPLE 5. Because $\sinh z = -i \sin(iz)$, as pointed out in Sec. 39, we need only recall expansion (3) for $\sin z$ and write

$$\sinh z = -i \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n+1}}{(2n+1)!} \qquad (|z| < \infty),$$

which becomes

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty).$$

EXAMPLE 6. Since $\cosh z = \cos(iz)$, according to Sec. 39, the Maclaurin series (4) for $\cos z$ reveals that

$$\cosh z = \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n}}{(2n)!} \qquad (|z| < \infty),$$

and we arrive at the Maclaurin series representation

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty).$$

Observe that the Taylor series for $\cosh z$ about the point $z_0 = -2\pi i$, for example, is obtained by replacing the variable z on each side of this last equation by $z + 2\pi i$ and then recalling (Sec. 39) that $\cosh(z + 2\pi i) = \cosh z$ for all z:

$$\cosh z = \sum_{n=0}^{\infty} \frac{(z+2\pi i)^{2n}}{(2n)!} \qquad (|z| < \infty).$$

65. NEGATIVE POWERS OF $(z - z_0)$

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem there. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $(z - z_0)$. Such series are extremely important and are taken up in the next section. They are often obtained by using one or more of the six Maclaurin series listed at the beginning of Sec. 64. In order that the reader be accustomed to series involving negative powers of $(z - z_0)$, we pause here with several examples before exploring their general theory.