9. *Nested Squares*. A square $\sigma_0 : a_0 \le x \le b_0, c_0 \le y \le d_0$ is divided into four equal squares by line segments parallel to the coordinate axes. One of those four smaller squares $\sigma_1 : a_1 \le x \le b_1, c_1 \le y \le d_1$ is selected according to some rule. It, in turn, is divided into four equal squares one of which, called σ_2, is selected, etc. (see Sec. 49). Prove that there is a point (x_0, y_0) which belongs to each of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the result in Exercise 8 to each of the sequences of closed intervals $a_n \le x \le b_n$ and $c_n \le y \le d_n$ $(n = 0, 1, 2, \ldots)$.

54. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}. \tag{1}$$

Expression (1) is called the ***Cauchy integral formula***. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f *on* C.

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 68). Since the quotient $f(z)/(z - z_0)$ is analytic between and on the contours C_ρ and C, it follows from the principle of deformation of paths (Sec. 53) that

$$\int_C \frac{f(z)\,dz}{z - z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z - z_0}.$$

This enables us to write

$$\int_C \frac{f(z)\,dz}{z - z_0} - f(z_0) \int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz. \tag{2}$$

But [see Exercise 13, Sec. 46]

$$\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i,$$

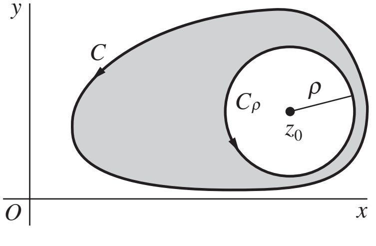

FIGURE 68

and so equation (2) becomes

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{3}$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that corresponding to each positive number ε, however small, there is a positive number δ such that

$$|f(z)-f(z_0)| < \varepsilon \quad \text{whenever} \quad |z-z_0| < \delta. \tag{4}$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z-z_0| = \rho < \delta$ when z is on C_ρ, it follows that the *first* of inequalities (4) holds when z is such a point; and the theorem in Sec. 47, giving upper bounds for the moduli of contour integrals, tells us that

$$\left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| < \frac{\varepsilon}{\rho}\, 2\pi\rho = 2\pi\varepsilon.$$

In view of equation (3), then,

$$\left| \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it follows that

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = 0.$$

Hence equation (1) is valid, and the theorem is proved.

When the Cauchy integral formula is written as

$$\int_C \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0), \tag{5}$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 1$ about the origin. Since the function

$$f(z) = \frac{\cos z}{z^2+9}$$

is analytic inside and on C and since the origin $z_0 = 0$ is interior to C, equation (5) tells us that

$$\int_C \frac{\cos z}{z(z^2+9)}\,dz = \int_C \frac{(\cos z)/(z^2+9)}{z-0}\,dz = 2\pi i f(0) = \frac{2\pi i}{9}.$$

55. AN EXTENSION OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula in the theorem in Sec. 50 can be extended so as to provide an integral representation for derivatives $f^{(n)}(z_0)$ of f at z_0.

Theorem. *Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

this theorem includes the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}. \tag{2}$$

Verification of expression (1) will be taken up in Sec. 56.

When written in the form

$$\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \ldots), \tag{3}$$

expression (1) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 50 when $n = 0$.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (3) shows that

$$\int_C \frac{dz}{z-z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = 0 \qquad (n = 1, 2, \ldots).$$

(Compare with Exercise 13, Sec. 46.)

Expression (1) can also be useful in slightly different notation. Namely, if s denotes points on C and if z is a point interior to C, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}} \quad (n = 0, 1, 2, \ldots), \tag{4}$$

where $f^{(0)}(z) = f(z)$ and, of course, $0! = 1$. Our next example illustrates the use of expression (4) in the form

$$\int_C \frac{f(s)\,ds}{(s-z)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z) \quad (n = 0, 1, 2, \ldots), \tag{5}$$

which includes the special case

$$\int_C \frac{f(s)\,ds}{s-z} = 2\pi i\, f(z). \tag{6}$$

EXAMPLE 3. If n is a nonnegative integer and $f(z) = (z^2-1)^n$, expression (4), becomes

$$\frac{d^n}{dz^n}(z^2-1)^n = \frac{n!}{2\pi i}\int_C \frac{(s^2-1)^n ds}{(s-z)^{n+1}} \quad (n = 0, 1, 2, \ldots), \tag{7}$$

where C is any simple closed contour surrounding z. In view of equation (7), one can write the Legendre polynomial*

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n \quad (n = 0, 1, 2, \ldots) \tag{8}$$

as

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n ds}{(s-z)^{n+1}} \quad (n = 0, 1, 2, \ldots). \tag{9}$$

Because

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1},$$

expression (9) reveals that

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s+1)^n ds}{s-1} \quad (n = 0, 1, 2, \ \ldots);$$

and by writing $f(s) = (s+1)^n$ and $z = 1$ in equation (6), we arrive at the values

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\, 2\pi i\ (1+1)^n = 1 \quad (n = 0, 1, 2, \ldots).$$

The values $P_n(-1) = (-1)^n (n = 0, 1, 2, \ldots)$ can be found (Exercise 8, Sec. 57) in a similar way.

*See Exercise 10, Sec. 20, and the footnote with it.

maximum value of $|f(z)|$ *on* C_R*, then*

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \quad (n = 1, 2, \ldots). \tag{2}$$

Inequality (2) is called ***Cauchy's inequality*** and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots),$$

in the theorem in Sec. 55 when n is a positive integer. We need only apply the theorem in Sec. 47, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \quad (n = 1, 2, \ldots),$$

where M_R is as in the statement of Theorem 3. This inequality is, of course, the same as inequality (2).

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(*a*) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; (*b*) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$; (*c*) $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(*d*) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$; (*e*) $\displaystyle\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (*a*) 2π; (*b*) $\pi i/4$; (*c*) $-\pi i/2$; (*d*) 0; (*e*) $i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(*a*) $g(z) = \dfrac{1}{z^2+4}$; (*b*) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans. (*a*) $\pi/2$; (*b*) $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,ds \qquad (|z| \ne 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\,ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following the procedure used in Sec. 56, prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s - z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}\,(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

8. Show that $P_n(-1) = (-1)^n\,(n = 0, 1, 2, \ldots)$, where $P_n(z)$ are the Legendre polynomials in Example 3, Sec. 55.

Suggestion: Note that

$$\frac{(s^2 - 1)^n}{(s + 1)^{n+1}} = \frac{(s - 1)^n}{s + 1}.$$

9. Follow the steps below to verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s - z)^3}$$

in Sec. 56.

(*a*) Use expression (2) in Sec. 56 for $f'(z)$ to show that

$$\frac{f'(z + \Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s - z)^3} = \frac{1}{2\pi i}\int_C \frac{3(s - z)\Delta z - 2(\Delta z)^2}{(s - z - \Delta z)^2(s - z)^3} f(s)\,ds.$$

(*b*) Let D and d denote the largest and smallest distances, respectively, from z to points on C. Also, let M be the maximum value of $|f(s)|$ on C and L the length of C. With the aid of the triangle inequality and by referring to the derivation of expression (2) in Sec. 56 for $f'(z)$, show that when $0 < |\Delta z| < d$, the value of the integral on the right-hand side in part *(a)* is bounded from above by

$$\frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d - |\Delta z|)^2 d^3} L.$$

(*c*) Use the results in parts (*a*) and (*b*) to obtain the desired expression for $f''(z)$.

10. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

58. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

Cauchy's inequality in Theorem 3 of Sec. 57 can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as ***Liouville's theorem***, states this result in a slightly different way.

Theorem 1. *If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

To start the proof, we assume that f is as stated and note that since f is entire, Theorem 3 in Sec. 57 can be applied with any choice of z_0 and R. In particular, Cauchy's inequality (2) in that theorem tells us that when $n = 1$,

$$|f'(z_0)| \leq \frac{M_R}{R}. \tag{1}$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that $|f(z)| \leq M$ for all z; and, because the constant M_R in inequality (1) is always less than or equal to M, it follows that

$$\left|f'(z_0)\right| \leq \frac{M}{R}, \tag{2}$$

where R can be arbitrarily large. Now the number M in inequality (2) is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function, according to the theorem in Sec. 25.

The following theorem is called the ***fundamental theorem of algebra*** and follows readily from Liouville's theorem.

Theorem 2. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that $P(z)$ is *not* zero for any value of z. Then the quotient $1/P(z)$ is clearly entire. It is also bounded in the complex plane.

To see that it is bounded, we first recall statement (6) in Sec. 5. Namely, there is a positive number R such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n} \quad \text{whenever } |z| > R.$$

So $1/P(z)$ is bounded in the region *exterior to* the disk $|z| \le R$. But $1/P(z)$ is continuous on that closed disk, and this means that $1/P(z)$ is bounded there too (Sec. 18). Hence $1/P(z)$ is bounded in the entire plane.

It now follows from Liouville's theorem that $1/P(z)$, and consequently $P(z)$, is constant. But $P(z)$ is not constant, and we have reached a contradiction.*

The fundamental theorem tells us that any polynomial $P(z)$ of degree n ($n \ge 1$) can be expressed as a product of linear factors:

$$P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n), \tag{3}$$

where c and z_k ($k = 1, 2, \ldots, n$) are complex constants. More precisely, the theorem ensures that $P(z)$ has a zero z_1. Then, according to Exercise 8, Sec. 59,

$$P(z) = (z - z_1)Q_1(z),$$

where $Q_1(z)$ is a polynomial of degree $n - 1$. The same argument, applied to $Q_1(z)$, reveals that there is a number z_2 such that

$$P(z) = (z - z_1)(z - z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we arrive at expression (3). Some of the constants z_k in expression (3) may, of course, appear more than once, but it is clear that $P(z)$ can have no more than n *distinct* zeros.

59. MAXIMUM MODULUS PRINCIPLE

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

Lemma. *Suppose that $|f(z)| \le |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.*

To prove this, we assume that f satisfies the stated conditions and let z_1 be any point other than z_0 in the given neighborhood. We then let ρ be the distance between z_1 and z_0. If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, centered at z_0 and

*For an interesting proof of the fundamental theorem of algebra using the Cauchy–Goursat theorem, see R. P. Boas, Jr., *Amer. Math. Monthly*, Vol. 71, No. 2, p. 180, 1964.

passing through z_1 (Fig. 72), the Cauchy integral formula tells us that

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)\,dz}{z - z_0}; \tag{1}$$

and the parametric representation

$$z = z_0 + \rho e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

for C_ρ enables us to write equation (1) as

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta. \tag{2}$$

We note from expression (2) that when a function is analytic within and on a given circle, its value at the center is the arithmetic mean of its values on the circle. This result is called ***Gauss's mean value theorem***.

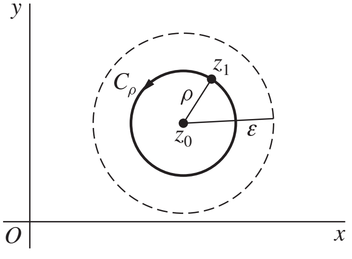

FIGURE 72

From equation (2), we obtain the inequality

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta. \tag{3}$$

On the other hand, since

$$|f(z_0 + \rho e^{i\theta})| \le |f(z_0)| \qquad (0 \le \theta \le 2\pi), \tag{4}$$

we find that

$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta \le \int_0^{2\pi} |f(z_0)|\,d\theta = 2\pi |f(z_0)|.$$

Thus

$$|f(z_0)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta. \tag{5}$$

It is now evident from inequalities (3) and (5) that

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta,$$

or

$$\int_0^{2\pi} [|f(z_0)| - |f(z_0 + \rho e^{i\theta})|]\,d\theta = 0.$$

The integrand in this last integral is continuous in the variable θ; and, in view of condition (4), it is greater than or equal to zero on the entire interval $0 \le \theta \le 2\pi$. Because the value of the integral is zero, then, the integrand must be identically equal to zero. That is,

$$|f(z_0 + \rho e^{i\theta})| = |f(z_0)| \qquad (0 \le \theta \le 2\pi). \tag{6}$$

This shows that $|f(z)| = |f(z_0)|$ *for all points* z *on the circle* $|z - z_0| = \rho$.

Finally, since z_1 is any point in the deleted neighborhood $0 < |z - z_0| < \varepsilon$, we see that the equation $|f(z)| = |f(z_0)|$ is, in fact, satisfied by all points z lying on any circle $|z - z_0| = \rho$, where $0 < \rho < \varepsilon$. Consequently, $|f(z)| = |f(z_0)|$ everywhere in the neighborhood $|z - z_0| < \varepsilon$. But we know from Example 4. Sec. 26, that when the modulus of an analytic function is constant in a domain, the function itself is constant there. Thus $f(z) = f(z_0)$ for each point z in the neighborhood, and the proof of the lemma is complete.

This lemma can be used to prove the following theorem, which is known as the ***maximum modulus principle.***

Theorem. *If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in it.*

Given that f is analytic in D, we shall prove the theorem by assuming that $|f(z)|$ *does* have a maximum value at some point z_0 in D and then showing that $f(z)$ must be constant throughout D.

The general approach here is similar to that taken in the proof of the lemma in Sec. 28. We draw a polygonal line L lying in D and extending from z_0 to any other point P in D. Also, d represents the shortest distance from points on L to the boundary of D. When D is the entire plane, d may have any positive value. Next, we observe that there is a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L such that z_n coincides with the point P and

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

In forming a finite sequence of neighborhoods (Fig. 73)

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n$$

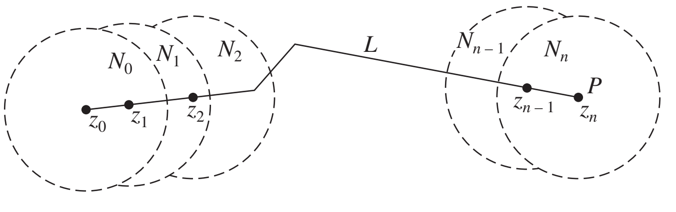

FIGURE 73