CHAPTER

3

ELEMENTARY FUNCTIONS

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

30. THE EXPONENTIAL FUNCTION

The exponential function can be defined by writing

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{1}$$

where Euler's formula (see Sec. 7)

$$e^{iy} = \cos y + i \sin y \tag{2}$$

is used and y is to be taken in radians. We see from this definition that e^z reduces to the usual exponential function in calculus when $y = 0$; and, following the convention used in calculus, we often write $\exp z$ for e^z.

Note that since the *positive* nth root $\sqrt[n]{e}$ of e is assigned to e^x when $x = 1/n$ $(n = 2, 3, \ldots)$, expression (1) tells us that the complex exponential function e^z is also $\sqrt[n]{e}$ when $z = 1/n$ $(n = 2, 3, \ldots)$. This is an exception to the convention (Sec. 10) that would ordinarily require us to interpret $e^{1/n}$ as the *set* of nth roots of e.

Note, too, that when definition (1) is written in the form

$$e^z = \rho e^{i\phi} \qquad \text{where} \qquad \rho = e^x \text{ and } \phi = y,$$

it becomes clear that

(3) $$|e^z| = e^x \qquad \text{and} \qquad \arg(e^z) = y + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Moreover, since e^x is never zero,

(4) $$e^z \neq 0 \qquad \text{for any complex number } z.$$

In addition to property (4), there are a number of other properties that carry over from e^x to e^z, and we mention a few of them here.

According to definition (1), $e^x e^{iy} = e^{x+iy}$; and this is consistent with the additive property $e^{x_1} e^{x_2} = e^{x_1+x_2}$ of the exponential function in calculus. The extension

(5) $$e^{z_1} e^{z_2} = e^{z_1+z_2}$$

to complex analysis is easy to verify. To do this, we write

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Then

$$e^{z_1} e^{z_2} = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}) = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}).$$

But x_1 and x_2 are both real, and we know from Sec. 8 that

$$e^{iy_1} e^{iy_2} = e^{i(y_1+y_2)}.$$

Hence

$$e^{z_1} e^{z_2} = e^{(x_1+x_2)} e^{i(y_1+y_2)};$$

and, since

$$(x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2,$$

the right-hand side of this last equation becomes $e^{z_1+z_2}$. Property (5) is now established.

Observe how property (5) enables us to write $e^{z_1-z_2} e^{z_2} = e^{z_1}$, or

(6) $$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

From this and the fact that $e^0 = 1$, it follows that $1/e^z = e^{-z}$.

There are a number of other important properties of e^z that are expected. According to Example 1 in Sec. 23, for instance,

(7) $$\frac{d}{dz} e^z = e^z$$

everywhere in the z plane. Note that the differentiability of e^z for all z tells us that e^z *is entire* (Sec. 25).

Some properties of e^z are, on the other hand, *not* expected. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z *is periodic,* with a pure imaginary period of $2\pi i$:

$$e^{z+2\pi i} = e^z. \tag{8}$$

For another property of e^z that e^x does not have, we note that while e^x is always positive, e^z can be negative. We recall (Sec. 6), for instance, that $e^{i\pi} = -1$. In fact,

$$e^{i(2n+1)\pi} = e^{i2n\pi + i\pi} = e^{i2n\pi}e^{i\pi} = (1)(-1) = -1 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

There are, moreover, values of z such that e^z is *any* given nonzero complex number. This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

EXAMPLE. In order to find numbers $z = x + iy$ such that

$$e^z = 1 + \sqrt{3}i, \tag{9}$$

we write equation (9) as

$$e^x e^{iy} = 2\, e^{i\pi/3}.$$

Then, in view of the statement in italics at the beginning of Sec. 10, regarding the equality of two nonzero complex numbers in exponential form,

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Because $\ln(e^x) = x$, it follows that

$$x = \ln 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots);$$

and so

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{10}$$

EXERCISES

1. Show that

(*a*) $\exp(2 \pm 3\pi i) = -e^2$; (*b*) $\exp\left(\dfrac{2+\pi i}{4}\right) = \sqrt{\dfrac{e}{2}}\,(1+i)$;

(*c*) $\exp(z + \pi i) = -\exp z$.

2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that the function $f(z) = \exp \overline{z}$ is not analytic anywhere.

4. Show in two ways that the function $f(z) = \exp(z^2)$ is entire. What is its derivative?

Ans. $f'(z) = 2z\exp(z^2)$.

5. Write $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z+i)+\exp(iz^2)| \le e^{2x}+e^{-2xy}.$$

6. Show that $|\exp(z^2)| \le \exp(|z|^2)$.

7. Prove that $|\exp(-2z)| < 1$ if and only if $\operatorname{Re} z > 0$.

8. Find all values of z such that

(a) $e^z = -2$;   (b) $e^z = 1+i$;   (c) $\exp(2z-1) = 1$.

Ans. (a) $z = \ln 2 + (2n+1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \dfrac{1}{2}\ln 2 + \left(2n + \dfrac{1}{4}\right)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$;

(c) $z = \dfrac{1}{2} + n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$. (Compare with Exercise 4, Sec. 29.)

10. (a) Show that if e^z is real, then $\operatorname{Im} z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$.

(b) If e^z is pure imaginary, what restriction is placed on z?

11. Describe the behavior of $e^z = e^x e^{iy}$ as (a) x tends to $-\infty$; (b) y tends to ∞.

12. Write $\operatorname{Re}(e^{1/z})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in some domain D. State why the functions

$$U(x,y) = e^{u(x,y)}\cos v(x,y), \quad V(x,y) = e^{u(x,y)}\sin v(x,y)$$

are harmonic in D.

14. Establish the identity

$$(e^z)^n = e^{nz} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

in the following way.

(a) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.

(b) Verify it for negative integers n by first recalling from Sec. 8 that

$$z^n = (z^{-1})^m \qquad (m = -n = 1, 2, \ldots)$$

when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part (a), together with the property $1/e^z = e^{-z}$ (Sec. 30) of the exponential function.

31. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z \tag{1}$$

for w, where z is any *nonzero* complex number. To do this, we note that when z and w are written $z = re^{i\Theta}$ $(-\pi < \Theta \le \pi)$ and $w = u + iv$, equation (1) becomes

$$e^u e^{iv} = re^{i\Theta}.$$

According to the statement in italics at the beginning of Sec. 10 about the equality of two nonzero complex numbers expressed in exponential form, this tells us that

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi$$

where n is any integer. Since the equation $e^u = r$ is the same as $u = \ln r$, it follows that equation (1) is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus, if we write

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{2}$$

equation (1) tells us that

$$e^{\log z} = z \qquad (z \ne 0), \tag{3}$$

Inasmuch as equation (2) becomes

$$\log x = \ln x + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

when $z = x > 0$ and since equation (3) then reduces to the familiar identity

$$e^{\ln x} = x \qquad (x > 0) \tag{4}$$

in calculus, equation (4) suggests that we use expression (2) as the definition of the (multiple-valued) ***logarithmic function*** of a nonzero complex variable $z = re^{i\theta}$.

It should be emphasized that it is *not* true that the left-hand side of equation (3) with the order of the exponential and logarithmic functions reversed reduces to just z. More precisely, since expression (2) can be written

$$\log z = \ln |z| + i \arg z$$

and since (Sec. 30)

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

when $z = x + iy$, we know that

$$\log(e^z) = \ln |e^z| + i \arg(e^z) = \ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i$$
$$(n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$\log(e^z) = z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

The ***principal value*** of $\log z$ is the value obtained from equation (2) when $n = 0$ there and is denoted by $\operatorname{Log} z$. Thus

$$\operatorname{Log} z = \ln r + i\Theta. \tag{6}$$

Note that Log z is well defined and single-valued when $z \neq 0$ and that

$$\log z = \text{Log}\, z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

It reduces to the usual logarithm in calculus when z is a positive real number. To see this, one need only write $z = x$ $(x > 0)$, in which case equation (6) becomes $\text{Log}\, z = \ln x$.

32. EXAMPLES

In this section we illustrate material in Sec. 31.

EXAMPLE 1. If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\Theta = -2\pi/3$. Hence

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i$$
$$(n = 0, \pm 1, \pm 2, \ldots).$$

EXAMPLE 2. From expression (2) in Sec. 31, we find that

$$\log 1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

As anticipated, $\text{Log}\, 1 = 0$.

The next example reminds us that although we were unable to find logarithms of *negative* real numbers in calculus, it is now possible.

EXAMPLE 3. Observe that

$$\log(-1) = \ln 1 + i(\pi + 2n\pi) = (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and that $\text{Log}\,(-1) = \pi i$.

Special care must be taken in anticipating that familiar properties of $\ln x$ in calculus carry over to be properties of $\log z$ and $\text{Log}\, z$.

EXAMPLE 4. The identity

$$\text{Log}[(1 + i)^2] = 2\,\text{Log}(1 + i) \tag{1}$$

is valid since

$$\text{Log}[(1 + i)^2] = \text{Log}\,(2i) = \ln 2 + i\frac{\pi}{2}$$

and

$$2\,\text{Log}(1 + i) = 2\left(\ln\sqrt{2} + i\frac{\pi}{4}\right) = \ln 2 + i\frac{\pi}{2}.$$

On the other hand,

(2) $$\text{Log}[(-1+i)^2] \neq 2\,\text{Log}(-1+i)$$

because

$$\text{Log}[(-1+i)^2] = \text{Log}(-2i) = \ln 2 - i\frac{\pi}{2}$$

and

$$2\,\text{Log}(-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + i\frac{3\pi}{2}.$$

While statement (1) might be expected, we see that statement (2) would not be true as an equality.

EXAMPLE 5. It is shown in Exercise 5, Sec. 33, that

(3) $$\log(i^{1/2}) = \frac{1}{2}\log i$$

in the sense that the set of values on the left is the same as the set of values on the right. But

(4) $$\log(i^2) \neq 2\log i$$

because

$$\ln(i^2) = \log(-1) = (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

according to Example 3, and since

$$2\log i = 2\left[\ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right] = (4n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Upon comparing statements (3) and (4), we find that familiar properties of logarithms in calculus are sometimes but not always true in complex analysis.

33. BRANCHES AND DERIVATIVES OF LOGARITHMS

If $z = re^{i\theta}$ is a nonzero complex number, the argument θ has any one of the values $\theta = \Theta + 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\Theta = \text{Arg}\, z$. Hence the definition

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

of the multiple-valued logarithmic function in Sec. 31 can be written

(1) $$\log z = \ln r + i\theta.$$

If we let α denote any real number and restrict the value of θ in expression (1) so that $\alpha < \theta < \alpha + 2\pi$, the function

(2) $$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi),$$

with components

(3) $$u(r,\theta) = \ln r \quad \text{and} \quad v(r,\theta) = \theta,$$

is *single-valued* and continuous in the stated domain (Fig. 35). Note that if the function (2) were to be defined on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v are near α and also points such that the values of v are near $\alpha + 2\pi$.

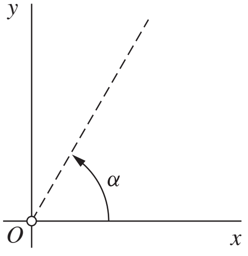

FIGURE 35

The function (2) is not only continuous but also analytic throughout the domain $r > 0$, $\alpha < \theta < \alpha + 2\pi$ since the first-order partial derivatives of u and v are continuous there and satisfy the polar form (Sec. 24)

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations. Furthermore, according to Sec. 24,

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}};$$

that is,

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi). \tag{4}$$

In particular,

$$\frac{d}{dz}\,\mathrm{Log}\, z = \frac{1}{z} \qquad (|z| > 0, -\pi < \mathrm{Arg}\, z < \pi). \tag{5}$$

A ***branch*** of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f. The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f. Observe that for each fixed α, the single-valued function (2) is a branch of the multiple-valued function (1). The function

$$\mathrm{Log}\, z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi) \tag{6}$$

is called the ***principal branch.***

A ***branch cut*** is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points (Sec. 25) of F, and any point that is common to all branch cuts of f is called a ***branch point.*** The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The branch cut for the principal branch (6) consists of the origin and the ray $\Theta = \pi$. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

We saw in Example 5, Sec. 32, that the set of values of $\log(i^2)$ is not the set of values of $2\log i$. The following example does show, however, that equality can occur when a specific branch of the logarithm is used. In that case, of course, there is only one value of $\log(i^2)$ that is to be taken, and the same is true of $2\log i$.

EXAMPLE. In order to show that

$$\log(i^2) = 2\log i \tag{7}$$

when the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right)$$

is used, write

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i$$

and then observe that

$$2\log i = 2\left(\ln 1 + i\frac{\pi}{2}\right) = \pi i.$$

It is interesting to contrast equality (7) with the result $\log(i^2) \neq 2\log i$ in Exercise 4, where a different branch of $\log z$ is used.

In Sec. 34, we shall consider other identities involving logarithms, sometimes with qualifications as to how they are to be interpreted. A reader who wishes to pass to Sec. 35 can simply refer to results in Sec. 34 when needed.

EXERCISES

1. Show that

(*a*) $\operatorname{Log}(-ei) = 1 - \dfrac{\pi}{2}i$; (*b*) $\operatorname{Log}(1-i) = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i$.

2. Show that

(*a*) $\log e = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*b*) $\log i = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*c*) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Show that $\operatorname{Log}(i^3) \neq 3\operatorname{Log} i$.

4. Show that $\log(i^2) \neq 2\log i$ when the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$$

is used. (Compare this with the example in Sec. 33.)

5. (*a*) Show that the two square roots of i are

$$e^{i\pi/4} \quad \text{and} \quad e^{i5\pi/4}$$

Then show that

$$\log(e^{i\pi/4}) = \left(2n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(e^{i5\pi/4}) = \left[(2n+1) + \frac{1}{4}\right]\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Conclude that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(*b*) Show that

$$\log(i^{1/2}) = \frac{1}{2}\log i,$$

as stated in Example 5, Sec. 32, by finding the values on the right-hand side of this equation and then comparing them with the final result in part (*a*).

6. Given that the branch $\log z = \ln r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity (Sec. 31)

$$e^{\log z} = z \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi)$$

and using the chain rule.

7. Show that a branch (Sec. 33)

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function can be written

$$\log z = \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

in rectangular coordinates. Then, using the theorem in Sec. 23, show that the given branch is analytic in its domain of definition and that

$$\frac{d}{dz}\log z = \frac{1}{z}$$

there.

8. Find all roots of the equation $\log z = i\pi/2$.

Ans. $z = i$.

9. Suppose that the point $z = x + iy$ lies in the horizontal strip $\alpha < y < \alpha + 2\pi$. Show that when the branch $\log z = \ln r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is used, $\log(e^z) = z$. [Compare with equation (5), Sec. 31.]

10. Show that

(*a*) the function $f(z) = \text{Log}(z - i)$ is analytic everywhere except on the portion $x \le 0$ of the line $y = 1$;

(*b*) the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

11. Show in two ways that the function $\ln(x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

12. Show that

$$\text{Re}\,[\log(z - 1)] = \frac{1}{2}\ln[(x - 1)^2 + y^2] \qquad (z \ne 1).$$

Why must this function satisfy Laplace's equation when $z \ne 1$?

34. SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2}$$

was in Sec. 9. That is, if values of two of the three logarithms are specified, then there is a value of the third such that equation (1) holds.

The verification of statement (1) can be based on statement (2) in the following way. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|.$$

So it follows from this and equation (2) that

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2). \tag{3}$$

Finally, because of the way in which equations (1) and (2) are to be interpreted, equation (3) is the same as equation (1).

EXAMPLE 1. To illustrate statement (1), write $z_1 = z_2 = -1$ and recall from Examples 2 and 3 in Sec. 32 that

$$\log 1 = 2n\pi i \quad \text{and} \quad \log(-1) = (2n + 1)\pi i,$$

where $n = 0, \pm 1, \pm 2, \ldots$. Noting that $z_1 z_2 = 1$ and using the values

$$\log(z_1 z_2) = 0 \quad \text{and} \quad \log z_1 = \pi i,$$

we find that equation (1) is satisfied when the value $\log z_2 = -\pi i$ is chosen.

If, on the other hand, principal values are used when $z_1 = z_2 = -1$,

$$\text{Log}(z_1 z_2) = 0 \quad \text{and} \quad \text{Log}\, z_1 + \text{Log}\, z_2 = 2\pi i.$$

Thus statement (1) is not always true when principal values are used in all three terms. In our next example, however, principal values can be used everywhere in equation (1) when certain restrictions are placed on the nonzero numbers z_1 and z_2.

EXAMPLE 2. Let z_1 and z_2 denote nonzero complex numbers lying to the right of the imaginary axis, so that

$$\text{Re}\, z_1 > 0 \quad \text{and} \quad \text{Re}\, z_2 > 0.$$

Thus

$$z_1 = r_1 \exp(i\Theta_1) \quad \text{and} \quad z_2 = r_2 \exp(i\Theta_2),$$

where

$$-\frac{\pi}{2} < \Theta_1 < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}.$$

Now it is important to notice that $-\pi < \Theta_1 + \Theta_2 < \pi$ since this means that

$$\text{Arg}\,(z_1 z_2) = \Theta_1 + \Theta_2.$$

Consequently,

$$\begin{aligned} \text{Log}(z_1 z_2) &= \ln |z_1 z_2| + i\,\text{Arg}\,(z_1 z_2) \\ &= \ln(r_1 r_2) + i(\Theta_1 + \Theta_2) \\ &= (\ln r_1 + i\Theta_1) + (\ln r_2 + i\Theta_2). \end{aligned}$$

That is,

$$\text{Log}(z_1 z) = \text{Log}\, z_1 + \text{Log}\, z_2.$$

(Compare this result with the one in Exercise 6, Sec. 9.)

Verification of the statement

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2, \tag{4}$$

which is to be interpreted in the same way as statement (1), is left to the exercises.

We include here two other properties of $\log z$ that will be of special interest in Sec. 35. If z is a nonzero complex number, then

$$z^n = e^{n \log z} \qquad (n = 0 \pm 1, \pm 2, \ldots) \tag{5}$$

for any value of $\log z$ that is taken. When $n = 1$, this reduces, of course, to relation (3), Sec. 31. Equation (5) is readily verified by writing $z = re^{i\theta}$ and noting that each side becomes $r^n e^{in\theta}$.

It is also true that when $z \neq 0$,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right) \qquad (n = 1, 2, \ldots). \tag{6}$$

That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write $z = r\exp(i\Theta)$, where Θ is the principal value of $\arg z$. Then, in view of definition (2), Sec. 31, of $\log z$,

$$\exp\left(\frac{1}{n}\log z\right) = \exp\left[\frac{1}{n}\ln r + \frac{i(\Theta + 2k\pi)}{n}\right]$$

where $k = 0, \pm 1, \pm 2, \ldots$. Thus

$$\exp\left(\frac{1}{n}\log z\right) = \sqrt[n]{r}\exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

Because $\exp(i2k\pi/n)$ has distinct values only when $k = 0, 1, \ldots, n-1$, the right-hand side of equation (7) has only n values. That right-hand side is, in fact, an expression for the nth roots of z (Sec. 10), and so it can be written $z^{1/n}$. This establishes property (6), which is actually valid when n is a negative integer too (see Exercise 4).

EXERCISES

1. Show that for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare with Example 2 in Sec. 34.)

2. Verify expression (4), Sec. 34, for $\log(z_1/z_2)$ by

(*a*) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 9);

(*b*) showing that $\log(1/z) = -\log z$ $(z \neq 0)$, in the sense that $\log(1/z)$ and $-\log z$ have the same set of values, and then referring to expression (1), Sec. 34, for $\log(z_1 z_2)$.

3. By choosing specific nonzero values of z_1 and z_2, show that expression (4), Sec. 34, for $\log(z_1/z_2)$ is not always valid when *log* is replaced by *Log*.

4. Show that property (6), Sec. 34, also holds when n is a negative integer. Do this by writing $z^{1/n} = (z^{1/m})^{-1}$ $(m = -n)$, where n has any one of the negative values $n = -1, -2, \ldots$ (see Exercise 9, Sec. 11), and using the fact that the property is already known to be valid for positive integers.

5. Let z denote any nonzero complex number, written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \ldots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation

$$\log(z^{1/n}) = \frac{1}{n}\ln r + i\,\frac{\Theta + 2(pn + k)\pi}{n},$$