has a derivative at $z = 0$. In fact, $f'(0) = 0 + i0 = 0$. We saw in Example 2, Sec. 22, that this function *cannot* have a derivative at any nonzero point since the Cauchy–Riemann equations are not satisfied at such points. (See also Example 3, Sec. 19.)

EXAMPLE 3. When using the theorem in this section to find a derivative at a point z_0, one must be careful not to use the expression for $f'(z)$ in the statement of the theorem before the *existence* of $f'(z)$ at z_0 is established.

Consider, for instance, the function

$$f(z) = x^3 + i(1 - y)^3.$$

Here

$$u(x, y) = x^3 \quad \text{and} \quad v(x, y) = (1 - y)^3,$$

and it would be a mistake to say that $f'(z)$ exists everywhere and that

$$f'(z) = u_x + iv_x = 3x^2. \tag{6}$$

To see this, we observe that the first Cauchy–Riemann equation $u_x = v_y$ can hold only if

$$x^2 + (1 - y)^2 = 0 \tag{7}$$

and that the second equation $u_y = -v_x$ is always satisfied. Condition (7) thus tells us that $f'(z)$ can exist only when $x = 0$ and $y = 1$. In view of equation (6), then, our theorem tells us that $f'(z)$ exists *only* when $z = i$, in which case $f'(i) = 0$.

24. POLAR COORDINATES

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

to restate the theorem in Sec. 23 in polar coordinates.

Depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta} \qquad (z \neq 0)$$

when $w = f(z)$, the real and imaginary components of $w = u + iv$ are expressed in terms of either the variables x and y or r and θ. Suppose that the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighborhood of a given nonzero point z_0 and are continuous at z_0. The first-order partial derivatives of u and v with respect to r and θ also have those properties, and the chain rule for differentiating real-valued functions of two real variables can be used to write them in terms of the ones with respect to x and y. More precisely, since

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$

one can write

(2) $$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r \sin\theta + u_y r \cos\theta.$$

Likewise,

(3) $$v_r = v_x \cos\theta + v_y \sin\theta, \quad v_\theta = -v_x r \sin\theta + v_y r \cos\theta.$$

If the partial derivatives of u and v with respect to x and y also satisfy the Cauchy–Riemann equations

(4) $$u_x = v_y, \quad u_y = -v_x$$

at z_0, equations (3) become

(5) $$v_r = -u_y \cos\theta + u_x \sin\theta, \quad v_\theta = u_y r \sin\theta + u_x r \cos\theta$$

at that point. It is then clear from equations (2) and (5) that

(6) $$r u_r = v_\theta, \quad u_\theta = -r v_r$$

at z_0.

If, on the other hand, equations (6) are known to hold at z_0, it is straightforward to show (Exercise 7) that equations (4) must hold there. Equations (6) are, therefore, an alternative form of the Cauchy–Riemann equations (4).

In view of equations (6) and the expression for $f'(z_0)$ that is found in Exercise 8, we are now able to restate the theorem in Sec. 23 using r and θ.

Theorem. *Let the function*

$$f(z) = u(r, \theta) + i v(r, \theta)$$

be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that

(*a*) *the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;*

(*b*) *those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form*

$$r u_r = v_\theta, \quad u_\theta = -r v_r$$

of the Cauchy–Riemann equations at (r_0, θ_0).

Then $f'(z_0)$ exists, its value being

$$f'(z_0) = e^{-i\theta}(u_r + i v_r),$$

where the right-hand side is to be evaluated at (r_0, θ_0).

EXAMPLE 1. If

$$f(z) = \frac{1}{z^2} = \frac{1}{(re^{i\theta})^2} = \frac{1}{r^2} e^{-i2\theta} = \frac{1}{r^2}(\cos 2\theta - i \sin 2\theta),$$

where $z \neq 0$, the component functions are

$$u = \frac{\cos 2\theta}{r^2} \quad \text{and} \quad v = -\frac{\sin 2\theta}{r^2}.$$

Since

$$r u_r = -\frac{2\cos 2\theta}{r^2} = v_\theta, \qquad u_\theta = -\frac{2\sin 2\theta}{r^2} = -r\,v_r$$

and since the other conditions in the theorem are satisfied at every nonzero point $z = re^{i\theta}$, the derivative of f exists when $z \neq 0$. Moreover, according to the theorem,

$$f'(z) = e^{-i\theta}\left(-\frac{2\cos 2\theta}{r^3} + i\,\frac{2\sin 2\theta}{r^3}\right) = -2e^{-i\theta}\frac{e^{-i2\theta}}{r^3} = -\frac{2}{(re^{i\theta})^3} = -\frac{2}{z^3}.$$

EXAMPLE 2. The theorem can be used to show that any branch

$$f(z) = \sqrt{r}\,e^{i\theta/2} \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

of the square root function $z^{1/2}$ has a derivative everywhere in its domain of definition. Here

$$u(r,\ \theta) = \sqrt{r}\,\cos\frac{\theta}{2} \quad \text{and} \quad v(r,\ \theta) = \sqrt{r}\,\sin\frac{\theta}{2}.$$

Inasmuch as

$$r u_r = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} = -r v_r$$

and since the remaining conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. The theorem also tells us that

$$f'(z) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + i\,\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right);$$

and this reduces to

$$f'(z) = \frac{1}{2\sqrt{r}}\,e^{-i\theta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2\sqrt{r}\,e^{i\theta/2}} = \frac{1}{2f(z)}.$$

EXERCISES

1. Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \bar{z}$; (b) $f(z) = z - \bar{z}$;

(c) $f(z) = 2x + ixy^2$; (d) $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 23 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$; (b) $f(z) = e^{-x}e^{-iy}$;

(c) $f(z) = z^3$; (d) $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. (b) $f''(z) = f(z)$; (d) $f''(z) = -f(z)$.

3. From results obtained in Secs. 21 and 23, determine where $f'(z)$ exists and find its value when

(*a*) $f(z) = 1/z$; (*b*) $f(z) = x^2 + iy^2$; (*c*) $f(z) = z \operatorname{Im} z$.

Ans. (*a*) $f'(z) = -1/z^2 \ (z \neq 0)$; (*b*) $f'(x + ix) = 2x$; (*c*) $f'(0) = 0$.

4. Use the theorem in Sec. 24 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

(*a*) $f(z) = 1/z^4 \quad (z \neq 0)$;
(*b*) $f(z) = e^{-\theta}\cos(\ln r) + i\,e^{-\theta}\sin(\ln r) \quad (r > 0, 0 < \theta < 2\pi)$.

Ans. (*b*) $f'(z) = i\,\dfrac{f(z)}{z}$.

5. Solve equations (2), Sec. 24 for u_x and u_y to show that

$$u_x = u_r\cos\theta - u_\theta\,\frac{\sin\theta}{r}, \quad u_y = u_r\sin\theta + u_\theta\,\frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that in Sec. 24 equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 24, are the Cauchy–Riemann equations in polar form.

6. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0\exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 5, together with the polar form (6), Sec. 24, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 23 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

7. (*a*) With the aid of the polar form (6), Sec. 24, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 6.

(*b*) Use the expression for $f'(z_0)$ in part *(a)* to show that the derivative of the function $f(z) = 1/z \ (z \neq 0)$ in Exercise 3(*a*) is $f'(z) = -1/z^2$.

8. (*a*) Recall (Sec. 6) that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y}\right).$$

(*b*) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part (*a*), to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the ***complex form*** $\partial f/\partial \overline{z} = 0$ of the Cauchy–Riemann equations.

25. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is ***analytic in an open set*** S if it has a derivative everywhere in that set. It is ***analytic at a point*** z_0 if it is analytic in some neighborhood of z_0.*

Note how it follows that if f is analytic at a point z_0, it must be analytic at *each* point in some neighborhood of z_0. If we should speak of a function that is analytic in a set S that is not open, it is to be understood that f is analytic in an open set containing S.

An ***entire*** function is a function that is analytic at each point in the entire plane.

EXAMPLES. The function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane since its derivative $f'(z) = -1/z^2$ exists at such a point. But the function $f(z) = |z|^2$ is not analytic anywhere since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 3, Sec. 19.) Finally, since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

A necessary, but by no means sufficient, condition for a function to be analytic in a domain D is clearly the continuity of f throughout D. (See the statement in italics near the end of Sec. 19.) Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 23 and 24.

Other useful sufficient conditions are obtained from the rules for differentiation in Sec. 20. The derivatives of the sum and product of two functions exist wherever the functions themselves have derivatives. Thus, *if two functions are analytic in a domain D, their sum and their product are both analytic in D.* Similarly, *their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.* In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.

From the chain rule for the derivative of a composite function, we find that *a composition of two analytic functions is analytic.* More precisely, suppose that a function $f(z)$ is analytic in a domain D and that the image (Sec. 13) of D under the transformation $w = f(z)$ is contained in the domain of definition of a function $g(w)$. Then the composition $g[f(z)]$ is analytic in D, with derivative

$$\frac{d}{dz} g[f(z)] = g'[f(z)]f'(z).$$

The following property of analytic functions is especially useful, in addition to being expected.

Theorem. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

We start the proof by writing $f(z) = u(x, y) + iv(x, y)$. Assuming that $f'(z) = 0$ in D, we note that $u_x + iv_x = 0$; and, in view of the Cauchy–Riemann equations, $v_y - iu_y = 0$. Consequently,

$$u_x = u_y = 0 \quad \text{and} \quad v_x = v_y = 0$$

at each point in D.

Next, we show that $u(x, y)$ is constant along any line segment L extending from a point P to a point P' and lying entirely in D. We let s denote the distance along L from the point P and let $\mathbf{U}$ denote the unit vector along L in the direction of increasing s (see Fig. 30). We know from calculus that the directional derivative du/ds can be written as the dot product

$$\text{(1)} \qquad \frac{du}{ds} = (\text{grad } u) \cdot \mathbf{U},$$

where grad u is the gradient vector

$$\text{(2)} \qquad \text{grad } u = u_x\mathbf{i} + u_y\mathbf{j}.$$

Because u_x and u_y are zero everywhere in D, grad u is evidently the zero vector at all points on L. Hence it follows from equation (1) that the derivative du/ds is zero along L; and this means that u is constant on L.

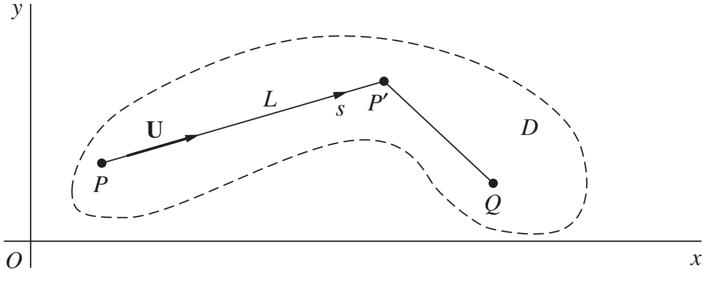

FIGURE 30

Finally, since there is always a finite number of such line segments, joined end to end, connecting any two points P and Q in D (Sec. 12), the values of u at P and Q must be the same. We may conclude, then, that there is a real constant a such that $u(x, y) = a$ throughout D. Similarly, $v(x, y) = b$; and we find that $f(z) = a + bi$ at each point in D. That is, $f(z) = c$ where c is the constant $c = a + bi$.

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a ***singular point,*** or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic. Singular points will play an important role in our development of complex analysis in chapters to follow.

26. FURTHER EXAMPLES

As pointed out in Sec. 25, it is often possible to determine where a given function $f(z)$ is analytic by simply recalling various differentiation rules in Sec. 20.

EXAMPLE 1. The quotient

$$f(z) = \frac{z^2 + 3}{(z + 1)(z^2 + 5)}$$

is evidently analytic throughout the z plane except for the singular points $z = -1$ and $z = \pm\sqrt{5}\,i$. The analyticity is due to the existence of familiar differentiation rules, which need to be applied only if an expression for $f'(z)$ is actually wanted.

When a function is given in terms of its component functions u and v, its analyticity can be determined by direct application of the Cauchy–Riemann equations.

EXAMPLE 2. If $f(z) = \sin x \cosh y + i \cos x \sinh y$, the component functions are

$$u(x, y) = \sin x \cosh y \quad \text{and} \quad v(x, y) = \cos x \sinh y.$$

Because

$$u_x = \cos x \cosh y = v_y \quad \text{and} \quad u_y = \sin x \sinh y = -v_x$$

everywhere, it is clear from the theorem in Sec. 23 that f is entire. In fact, according to that theorem,

$$f'(z) = u_x + i v_x = \cos x \cosh y - i \sin x \sinh y. \tag{1}$$

It is straightforward to show that $f'(z)$ is also entire by writing expression (1) as

$$f'(z) = U(x, y) + i V(x, y)$$

where

$$U(x, y) = \cos x \cosh y \quad \text{and} \quad V(x, y) = -\sin x \sinh y.$$

For then

$$U_x = -\sin x \cosh y = V_y \quad \text{and} \quad U_y = \cos x \sinh y = -V_x.$$

Furthermore,

$$f''(z) = U_x + i V_x = -(\sin x \cosh y + i \cos x \sinh y) = -f(z).$$

The next two examples serve to illustrate how the Cauchy–Riemann equations can be used to obtain various properties of analytic functions.

EXAMPLE 3. Suppose that a function $f(z) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are *both* analytic in a domain D. Let us show that $f(z)$ must, then, be constant throughout D.

To do this, we write $\overline{f(z)} = U(x, y) + V(x, y)$ where

$$U(x, y) = u(x, y) \quad \text{and} \quad V(x, y) = -v(x, y). \tag{2}$$

Because of the analyticity of $f(z)$, the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{3}$$

hold in D; and the analyticity of $\overline{f(z)}$ in D tells us that

$$U_x = V_y, \quad U_y = -V_x. \tag{4}$$

In view of relations (2), equations (4) can also be written

$$u_x = -v_y, \quad u_y = v_x. \tag{5}$$

By adding corresponding sides of the first of equations (3) and (5), we find that $u_x = 0$ in D. Similarly, subtraction involving corresponding sides of the second of equations (3) and (5) reveals that $v_x = 0$. According to expression (8) in Sec. 25, then,

$$f'(z) = u_x + iv_x = 0 + i0 = 0;$$

and it follows from the theorem in Sec. 25 that $f(z)$ is constant throughout D.

EXAMPLE 4. As in Example 3, we consider a function f that is analytic throughout a given domain D. Assuming further that the modulus $|f(z)|$ is constant throughout D, one can prove that $f(z)$ must be constant there too. This result is needed to obtain an important result later on in Chap. 4 (Sec. 59).

The proof is accomplished by writing

$$|f(z)| = c \quad \text{for all } z \text{ in } D, \tag{6}$$

where c is a real constant. If $c = 0$, it follows that $f(z) = 0$ everywhere in D. If $c \neq 0$, the property $z\bar{z} = |z|^2$ of complex numbers tells us that

$$f(z)\overline{f(z)} = c^2 \neq 0$$

and hence that $f(z)$ is never zero in D. So

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{for all } z \text{ in } D,$$

and it follows from this that $\overline{f(z)}$ is analytic everywhere in D. The main result in Example 3 just above thus ensures that $f(z)$ is constant throughout D.

EXERCISES

1. Apply the theorem in Sec. 23 to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$; ($b$) $f(z) = \cosh x \cos y + i \sinh x \sin y$;

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; ($d$) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 21, show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$; (b) $f(z) = 2xy + i(x^2 - y^2)$;

(c) $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere else:

(a) $f(z) = \dfrac{2z + 1}{z(z^2 + 1)}$; ($b$) $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$;

(c) $f(z) = \dfrac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)}$.

Ans. (a) $z = 0, \pm i$; (b) $z = 1, 2$; (c) $z = -2, -1 \pm i$.

5. According to Example 2, Sec. 24, the function

$$g(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is analytic in its domain of definition, with derivative

$$g'(z) = \frac{1}{2\,g(z)}.$$

Show that the composite function $G(z) = g(2z - 2 + i)$ is analytic in the half plane $x > 1$, with derivative

$$G'(z) = \frac{1}{g(2z - 2 + i)}.$$

Suggestion: Observe that $\text{Re}(2z - 2 + i) > 0$ when $x > 1$.

6. Use results in Sec. 24 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0, y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0, y > 0$.

7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughout D.

27. HARMONIC FUNCTIONS

A real-valued function H of two real variables x and y is said to be ***harmonic*** in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as ***Laplace's equation***.

Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

EXAMPLE 1. It is easy to verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi, y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig. 31. More precisely, it satisfies all of the conditions

$$T_{xx}(x, y) + T_{yy}(x, y) = 0,$$

$$T(0, y) = 0, \qquad T(\pi, y) = 0,$$

$$T(x, 0) = \sin x, \qquad \lim_{y \to \infty} T(x, y) = 0,$$

which describe steady temperatures $T(x, y)$ in a thin homogeneous plate in the xy plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

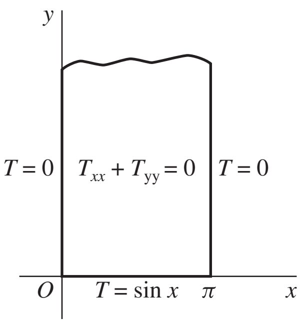

FIGURE 31

The use of the theory of functions of a complex variable in discovering solutions, such as the one in Example 1, of temperature and other problems is described in considerable detail later on in Chap. 10 and in parts of chapters following it.* That theory is based on the theorem below, which provides a source of harmonic functions.

Theorem. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

To show this, we need a result that is to be proved in Chap. 4 (Sec. 57). Namely, if a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all orders at that point.

Assuming that f is analytic in D, we start with the observation that the first-order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations throughout D:

$$u_x = v_y, \quad u_y = -v_x. \tag{2}$$

Differentiating both sides of these equations with respect to x, we have

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}. \tag{3}$$

Likewise, differentiation with respect to y yields

$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}. \tag{4}$$

Now, by a theorem in advanced calculus,† the continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. It then follows from equations (3) and (4) that

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

That is, u and v are harmonic in D.

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*Another important method is developed in the authors' "*Fourier Series and Boundary Value Problems*," 8th ed., 2012.

†See, for instance, A. E. Taylor and W. R. Mann, "*Advanced Calculus*," 3d ed., pp. 199–201, 1983.

EXAMPLE 2. The function $f(z) = e^{-y} \sin x - ie^{-y} \cos x$ is entire, as is shown in Exercise 1(c), Sec. 26. Hence its real component, which is the temperature function $T(x, y) = e^{-y} \sin x$ in Example 1, must be harmonic in every domain of the xy plane.

EXAMPLE 3. Since the function $f(z) = 1/z^2$ is analytic at every nonzero point z and since

$$\frac{1}{z^2} = \frac{1}{z^2} \cdot \frac{\bar{z}^2}{\bar{z}^2} = \frac{\bar{z}^2}{(z\,\bar{z})^2} = \frac{\bar{z}^2}{|z^2|^2} = \frac{(x^2 - y^2) - i2xy}{(x^2 + y^2)^2},$$

the two functions

$$u(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad \text{and} \qquad v(x, y) = -\frac{2xy}{(x^2 + y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

Further discussion of harmonic functions related to the theory of functions of a complex variable appears in Chaps. 9 and 10, where they are needed in solving physical problems, such as in Example 1 here.

EXERCISES

1. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (Sec. 24) and assuming continuity of partial derivatives, show that throughout D the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the ***polar form of Laplace's equation***. Show that the same is true of the function $v(r, \theta)$.

2. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of ***level curves*** $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the pair of equations $u(x, y) = c_1$ and $v(x, y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

3. Show that when $f(z) = z^2$, the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the component functions are the hyperbolas indicated in Fig. 32. Note the orthogonality of the two families, described in Exercise 2. Observe that the curves $u(x, y) = 0$ and $v(x, y) = 0$ intersect at the origin but are not, however, orthogonal to each other. Why is this fact in agreement with the result in Exercise 2?