not exist. [Note that it is not sufficient to simply consider nonzero points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

6. Prove statement (8) in Theorem 2 of Sec. 16 using

(a) Theorem 1 in Sec. 16 and properties of limits of real-valued functions of two real variables;

(b) definition (2), Sec. 15, of limit.

7. Use definition (2), Sec. 15, of limit to prove that

$$\text{if} \quad \lim_{z\to z_0} f(z) = w_0, \quad \text{then} \quad \lim_{z\to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how inequality (2), Sec. 5, enables one to write

$$||f(z)| - |w_0|| \le |f(z) - w_0|.$$

8. Write $\Delta z = z - z_0$ and show that

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{\Delta z\to 0} f(z_0 + \Delta z) = w_0.$$

9. Show that

$$\lim_{z\to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z\to z_0} f(z) = 0$$

and if there exists a positive number M such that $|g(z)| \le M$ for all z in some neighborhood of z_0.

10. Use the theorem in Sec. 17 to show that

(a) $\displaystyle\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$; ($b$) $\displaystyle\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$; ($c$) $\displaystyle\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$.

11. With the aid of the theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \ne 0),$$

(a) $\displaystyle\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;

(b) $\displaystyle\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\displaystyle\lim_{z\to -d/c} T(z) = \infty$ if $c \ne 0$.

12. State why limits involving the point at infinity are unique.

13. Show that a set S is unbounded (Sec. 12) if and only if every neighborhood of the point at infinity contains at least one point in S.

19. DERIVATIVES

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The ***derivative*** of f at z_0 is the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{1}$$

and the function f is said to be ***differentiable*** at z_0 when $f'(z_0)$ exists.

By expressing the variable z in definition (1) in terms of the new complex variable

$$\Delta z = z - z_0 \quad (z \neq z_0),$$

one can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2}$$

Because f is defined throughout a neighborhood of z_0, the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small (Fig. 27).

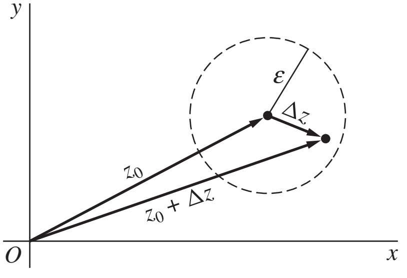

FIGURE 27

When taking form (2) of the definition of derivative, we often drop the subscript on z_0 and introduce the number

$$\Delta w = f(z + \Delta z) - f(z),$$

which denotes the change in the value $w = f(z)$ of f corresponding to a change Δz in the point at which f is evaluated. Then, if we write dw/dz for $f'(z)$, equation (2) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{3}$$

EXAMPLE 1. Suppose that $f(z) = 1/z$. At each nonzero point z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{1}{z + \Delta z} - \frac{1}{z} \right) \frac{1}{\Delta z} = \lim_{\Delta z \to 0} \frac{-1}{(z + \Delta z)\, z},$$

provided these limits exist; and properties of limits in Sec. 16 tell us that

$$\frac{dw}{dz} = -\frac{1}{z^2}, \quad \text{or} \quad f'(z) = -\frac{1}{z^2},$$

when $z \neq 0$.

EXAMPLE 2. If $f(z) = \overline{z}$, then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}. \tag{4}$$

If the limit of $\Delta w/\Delta z$ exists, it can be found by letting the point $\Delta z = (\Delta x, \Delta y)$ approach the origin $(0, 0)$ in the Δz plane in any manner. In particular, as Δz approaches

$(0, 0)$ horizontally through the points $(\Delta x, 0)$ on the real axis (Fig. 28),

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z.$$

In that case, expression (4) tells us that

$$\frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = 1.$$

Hence if the limit of $\Delta w/\Delta z$ exists, its value must be unity. However, when Δz approaches $(0, 0)$ vertically through the points $(0, \Delta y)$ on the imaginary axis, so that

$$\overline{\Delta z} = \overline{0 + i\,\Delta y} = 0 - i\,\Delta y = -(0 + i\,\Delta y) = -\Delta z,$$

we find from expression (4) that

$$\frac{\Delta w}{\Delta z} = \frac{-\Delta z}{\Delta z} = -1.$$

Hence the limit must be -1 if it exists. Since limits are unique (Sec. 15), it follows that dw/dz does not exist anywhere.

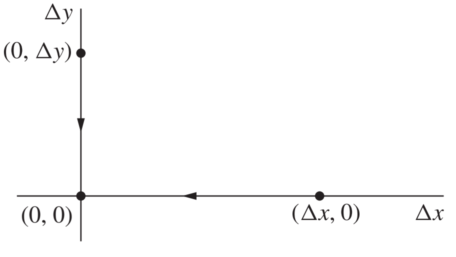

FIGURE 28

EXAMPLE 3. Consider the real-valued function $f(z) = |z|^2$. Here

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\overline{z + \Delta z}) - z\overline{z}}{\Delta z};$$

and since $\overline{z + \Delta z} = \overline{z} + \overline{\Delta z}$, this becomes

$$\frac{\Delta w}{\Delta z} = \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}. \tag{5}$$

Proceeding as in Example 2, where horizontal and vertical approaches of Δz toward the origin gave us

$$\overline{\Delta z} = \Delta z \quad \text{and} \quad \overline{\Delta z} = -\Delta z,$$

respectively, we have the expressions

$$\frac{\Delta w}{\Delta z} = \overline{z} + \Delta z + z \quad \text{when} \quad \Delta z = (\Delta x, 0)$$

and

$$\frac{\Delta w}{\Delta z} = \overline{z} - \overline{\Delta z} - z \quad \text{when} \quad \Delta z = (0, \Delta y).$$

Hence if the limit of $\Delta w/\Delta z$ exists as Δz tends to zero, the uniqueness of limits, used in Example 2, tells us that

$$\overline{z} + z = \overline{z} - z,$$

or that $z = 0$. Evidently, then, dw/dz cannot exist if $z \neq 0$.

To show that dw/dz does, in fact, exist at $z = 0$, we need only observe that expression (5) reduces to

$$\frac{\Delta w}{\Delta z} = \overline{\Delta z}$$

when $z = 0$. We conclude, therefore, that dw/dz exists *only* at $z = 0$, its value there being 0.

Example 3 illustrates the following three facts, the first two of which may be surprising.

(a) A function $f(z) = u(x, y) + iv(x, y)$ can be differentiable at a point $z = (x, y)$ but nowhere else in any neighborhood of that point.

(b) Since $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$ when $f(z) = |z|^2$, one can see that the real and imaginary components of a function of a complex variable can have continuous partial derivatives of all orders at a point $z = (x, y)$ and yet the function of z may not be differentiable there.

(c) Because the component functions $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$ of the function $f(z) = |z|^2$ are continuous everywhere in the plane, it is also evident that the continuity of a function of a complex variable at a point does not imply the existence of its derivative there. More precisely, the components

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0$$

of $f(z) = |z|^2$ are continuous at each nonzero point $z = (x, y)$ but $f'(z)$ does not exist there. It is, however, true that *the existence of the derivative of a function at a point implies the continuity of the function at that point.* To see this, we assume that $f'(z_0)$ exists and write

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0,$$

from which it follows that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This is the statement of continuity of f at z_0 (Sec. 18).

Geometric interpretations of derivatives of functions of a complex variable are not as immediate as they are for derivatives of functions of a real variable. We defer the development of such interpretations until Chap. 9.

20. RULES FOR DIFFERENTIATION

The definition of derivative in Sec. 19 is formally the same as the definition in calculus when z is substituted for x. Hence the basic differentiation rules given below can be derived from the definition in Sec. 19 by the same steps as the ones used in calculus. In stating such rules, we shall use either

$$\frac{d}{dz}\, f(z) \quad \text{or} \quad f'(z),$$

depending on which notation is more convenient.

Let c be a complex constant, and let f be a function whose derivative exists at a point z. It is easy to show that

$$\frac{d}{dz}c = 0, \quad \frac{d}{dz}z = 1, \quad \frac{d}{dz}[cf(z)] = cf'(z). \tag{1}$$

Also, if n is a positive integer,

$$\frac{d}{dz}z^n = nz^{n-1}. \tag{2}$$

This rule remains valid when n is a negative integer, provided that $z \neq 0$.

If the derivatives of two functions f and g exist at a point z, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z), \tag{3}$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z); \tag{4}$$

and, when $g(z) \neq 0$,

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}. \tag{5}$$

Let us derive rule (4). To do this, we write the following expression for the change in the product $w = f(z)g(z)$:

$$\begin{aligned}\Delta w &= f(z + \Delta z)g(z + \Delta z) - f(z)g(z)\\ &= f(z)[g(z + \Delta z) - g(z)] + [f(z + \Delta z) - f(z)]g(z + \Delta z).\end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\,\frac{g(z + \Delta z) - g(z)}{\Delta z} + \frac{f(z + \Delta z) - f(z)}{\Delta z}\, g(z + \Delta z);$$

and, letting Δz tend to zero, we arrive at the desired rule for the derivative of $f(z)g(z)$. Here we have used the fact that g is continuous at the point z, since $g'(z)$ exists; thus $g(z + \Delta z)$ tends to $g(z)$ as Δz tends to zero (see Exercise 8, Sec. 18).

There is also a chain rule for differentiating composite functions. Suppose that f has a derivative at z_0 and that g has a derivative at the point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]f'(z_0). \tag{6}$$

If we write $w = f(z)$ and $W = g(w)$, so that $W = F(z)$, the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}.$$

EXAMPLE. To find the derivative of $(1-4z^2)^3$, one can write $w = 1-4z^2$ and $W = w^3$. Then

$$\frac{d}{dz}(1-4z^2)^3 = 3w^2(-8z) = -24z(1-4z^2)^2.$$

To start the derivation of rule (6), choose a specific point z_0 at which $f'(z_0)$ exists. Write $w_0 = f(z_0)$ and also assume that $g'(w_0)$ exists. There is, then, some ε neighborhood $|w - w_0| < \varepsilon$ of w_0 such that for all points w in that neighborhood, we can define a function Φ having the values $\Phi(w_0) = 0$ and

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad \text{when} \quad w \neq w_0. \tag{7}$$

Note that in view of the definition of derivative,

$$\lim_{w \to w_0} \Phi(w) = 0. \tag{8}$$

Hence Φ is continuous at w_0.

Now expression (7) can be put in the form

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \qquad (|w - w_0| < \varepsilon), \tag{9}$$

which is valid even when $w = w_0$; and since $f'(z_0)$ exists and f is therefore continuous at z_0, we can choose a positive number δ such that the point $f(z)$ lies in the ε neighborhood $|w - w_0| < \varepsilon$ of w_0 if z lies in the δ neighborhood $|z - z_0| < \delta$ of z_0. Thus it is legitimate to replace the variable w in equation (9) by $f(z)$ when z is any point in the neighborhood $|z - z_0| < \delta$. With that substitution, and with $w_0 = f(z_0)$, equation (9) becomes

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\}\frac{f(z) - f(z_0)}{z - z_0} \qquad (0 < |z - z_0| < \delta), \tag{10}$$

where we must stipulate that $z \neq z_0$ so that we are not dividing by zero. As already noted, f is continuous at z_0 and Φ is continuous at the point $w_0 = f(z_0)$. Hence the composition $\Phi[f(z)]$ is continuous at z_0; and since $\Phi(w_0) = 0$,

$$\lim_{z \to z_0} \Phi[f(z)] = 0.$$

So equation (10) becomes equation (6) in the limit as z approaches z_0.

EXERCISES

1. Use definition (3), Sec. 19, to give a direct proof that

$$\frac{dw}{dz} = 2z \quad \text{when} \quad w = z^2.$$

2. Use results in Sec. 20 to find $f'(z)$ when

(*a*) $f(z) = 3z^2 - 2z + 4$; (*b*) $f(z) = (2z^2 + i)^5$;

(*c*) $f(z) = \dfrac{z-1}{2z+1} \quad \left(z \neq -\dfrac{1}{2}\right)$; (*d*) $f(z) = \dfrac{(1+z^2)^4}{z^2} \quad (z \neq 0)$.

3. Using results in Sec. 20, show that

(*a*) a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n ($n \geq 1$) is differentiable everywhere, with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1};$$

(*b*) the coefficients in the polynomial $P(z)$ in part (*a*) can be written

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}.$$

4. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use definition (1), Sec. 19, of derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

5. Derive expression (3), Sec. 20, for the derivative of the sum of two functions.

6. Derive expression (2), Sec. 20, for the derivative of z^n when n is a positive integer by using

(*a*) mathematical induction and expression (4), Sec. 20, for the derivative of the product of two functions;

(*b*) definition (3), Sec. 19, of derivative and the binomial formula (Sec. 3).

7. Prove that expression (2), Sec. 20, for the derivative of z^n remains valid when n is a negative integer ($n = -1, -2, \ldots$), provided that $z \neq 0$.

Suggestion: Write $m = -n$ and use the rule for the derivative of a quotient of two functions.

8. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when

(*a*) $f(z) = \operatorname{Re} z$; (*b*) $f(z) = \operatorname{Im} z$.

9. Let f denote the function whose values are

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $\Delta x\,\Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane (Fig. 29). Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Exercise 5, Sec. 18, as well as Example 2, Sec. 19.)

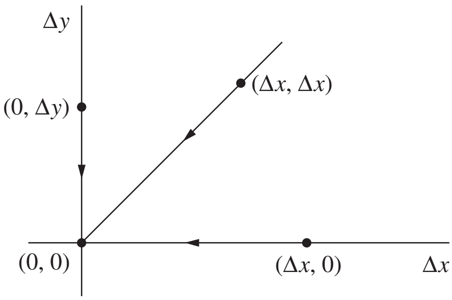

FIGURE 29

10. With the aid of the binomial formula (13) in Sec. 3, point out why each of the functions

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2 - 1)^n \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial (Sec. 13) of degree n*. (We use the convention that the derivative of order zero of a function is the function itself.)

21. CAUCHY–RIEMANN EQUATIONS

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

Starting with the assumption that $f'(z_0)$ exists, we write

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y,$$

*These are called ***Legendre polynomials*** and are important in applied mathematics. See, for instance, Chap. 10 of the authors' book (2012), listed in the Bibliography.

and

$$\Delta w = f(z_0 + \Delta z) - f(z_0),$$

which is the same as

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)] - [u(x_0, y_0) + iv(x_0, y_0)].$$

This last equation enables us to write

$$(2) \quad \frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0 y_0)}{\Delta x + i\Delta y}.$$

Now it is important to keep in mind that expression (2) remains valid as $(\Delta x, \Delta y)$ tends to $(0, 0)$ in any manner that we may choose.

Horizontal approach

In particular, write $\Delta y = 0$ and let $(\Delta x,\ 0)$ tend to $(0,\ 0)$ horizontally. Then, in view of Theorem 1 in Sec. 16, equation (2) tells us that

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0 y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0 y_0)}{\Delta x}.$$

That is,

$$(3) \qquad f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Vertical approach

We might have set $\Delta x = 0$ in equation (2) and taken a vertical approach. In that case, we find from Theorem 1 in Sec. 16 and equation (2) that

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0 y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0 y_0)}{i\Delta y},$$

or, because $1/i = -i$,

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0 y_0)}{\Delta y} - i \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0 y_0)}{\Delta y}.$$

It now follows that

$$(4) \qquad f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0),$$

where the partial derivatives of u and v are, this time, with respect to y. Note that equation (4) can also be written in the form

$$(5) \qquad f'(z_0) = -i[u_y(x_0, y_0) + iv_y(x_0, y_0)].$$

Expressions (3) and (4) not only give $f'(z_0)$ in terms of partial derivatives of the component functions u and v but, in view of the uniqueness of limits (Sec. 15), they also provide necessary conditions for the existence of $f'(z_0)$. To obtain those conditions, we need only equate the real parts and then the imaginary parts in expressions

(3) and (4) to see that the existence of $f'(z_0)$ requires that

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -\,v_x(x_0, y_0). \tag{6}$$

Equations (6) are the ***Cauchy–Riemann equations***, so named in honor of the French mathematician A. L. Cauchy (1789–1857), who discovered and used them, and in honor of the German mathematician G. F. B. Riemann (1826–1866), who made them fundamental in his development of the theory of functions of a complex variable.

We summarize the above results as follows.

Theorem. *Suppose that*

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{7}$$

there. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x, \tag{8}$$

where these partial derivatives are to be evaluated at (x_0, y_0).

22. EXAMPLES

Before we continue our discussion of the Cauchy–Riemann equations, we pause here to illustrate their use and to motivate further discussion of them.

EXAMPLE 1. In Exercise 1, Sec. 20, we showed that the function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

is differentiable everywhere and that $f'(z) = 2z$. To verify that the Cauchy–Riemann equations are satisfied everywhere, write

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Thus

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x.$$

Moreover, according to equation (8) in Sec. 21,

$$f'(z) = 2x + i2y = 2(x + iy) = 2z.$$

Since the Cauchy–Riemann equations are necessary conditions for the existence of the derivative of a function f at a point z_0, they can often be used to locate points at which f does *not* have a derivative.

EXAMPLE 2. When $f(z) = |z|^2$, we have

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0.$$

If the Cauchy–Riemann equations are to hold at a point (x, y), it follows that $2x = 0$ and $2y = 0$, or that $x = y = 0$. Consequently, $f'(z)$ does not exist at any nonzero point, as we already know from Example 3 in Sec. 19. Note that the theorem just proved does not ensure the existence of $f'(0)$. The theorem in the next section will, however, do this.

In Example 2, we considered a function $f(z)$ whose component functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations at the origin and whose derivative $f'(0)$ exists there. It is possible, however, to have a function $f(z)$ whose component functions satisfy the Cauchy–Riemann equations at the origin but whose derivative $f'(0)$ does *not* exist. This is illustrated in our next example.

EXAMPLE 3. If the function $f(z) = u(x, y) + iv(x, y)$ is defined by means of the equations

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0, \\ 0 & \text{when } z = 0, \end{cases}$$

its real and imaginary components are [see Exercise 2(*b*), Sec. 14]

$$u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}$$

when $(x, y) \neq (0, 0)$. Also, $u(0, 0) = 0$ and $v(0, 0) = 0$.

Because

$$u_x(0, 0) = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

and

$$v_y(0, 0) = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1,$$

we find that the first Cauchy–Riemann equation $u_x = v_y$ is satisfied at $z = 0$. Likewise, it is easy to show that $u_y = 0 = -v_x$ when $z = 0$. But, as was shown in Exercise 9, Sec. 20, $f'(0)$ fails to exist.

23. SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

As pointed out in Example 3, Sec. 22, satisfaction of the Cauchy–Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. But, with certain continuity conditions, we have the following useful theorem.

Theorem. *Let the function*

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some ε *neighborhood of a point* $z_0 = x_0 + iy_0$, *and suppose that*

(*a*) *the first-order partial derivatives of the functions* u *and* v *with respect to* x *and* y *exist everywhere in the neighborhood;*

(*b*) *those partial derivatives are continuous at* (x_0, y_0) *and satisfy the Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0),

Then $f'(z_0)$ *exists, its value being*

$$f'(z_0) = u_x + iv_x$$

where the right-hand side is to be evaluated at (x_0, y_0).

To prove the theorem, we assume that conditions (*a*) and (*b*) in its hypothesis are satisfied and write $\Delta z = \Delta x + i\Delta y$, where $0 < |\Delta z| < \varepsilon$, as well as

$$\Delta w = f(z_0 + \Delta z) - f(z_0).$$

Thus

$$\Delta w = \Delta u + i\Delta v, \tag{1}$$

where

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

and

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

The assumption that the first-order partial derivatives of u and v are continuous at the point (x_0, y_0) enables us to write*

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \tag{2}$$

and

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y, \tag{3}$$

*See, for instance, W. Kaplan, "*Advanced Calculus*," 5th ed., pp. 86ff, 2003.

where ε_1, ε_2, ε_3, and ε_4 tend to zero as $(\Delta x, \Delta y)$ approaches (0, 0) in the Δz plane. Substitution of expressions (2) and (3) into equation (1) now tells us that

$$\begin{aligned}(4)\qquad \Delta w = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ + i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y].\end{aligned}$$

Because the Cauchy–Riemann equations are assumed to be satisfied at (x_0, y_0), one can replace $u_y(x_0, y_0)$ by $-v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ by $u_x(x_0, y_0)$ in equation (4) and then divide through by the quantity $\Delta z = \Delta x + i\Delta y$ to get

$$(5)\qquad \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}.$$

But $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$, according to inequalities (3) in Sec. 4, and so

$$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \le 1.$$

Consequently,

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| \le |\varepsilon_1 + i\varepsilon_3| \le |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right| \le |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_2| + |\varepsilon_4|;$$

and this means that the last two terms on the right in equation (5) tend to zero as the variable $\Delta z = \Delta x + i\Delta y$ approaches zero. The expression for $f'(z_0)$ in the statement of the theorem is now established.

EXAMPLE 1. Consider the function

$$f(z) = e^x e^{iy} = e^x \cos y + i e^x \sin y,$$

where $z = x + iy$ and y is to be taken in radians when $\cos y$ and $\sin y$ are evaluated. Here

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere and since these derivatives are everywhere continuous, the conditions in the above theorem are satisfied at all points in the complex plane. Thus $f'(z)$ exists everywhere, and

$$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y.$$

Note that $f'(z) = f(z)$ for all z.

EXAMPLE 2. It also follows from our theorem that the function $f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0,$$

has a derivative at $z = 0$. In fact, $f'(0) = 0 + i0 = 0$. We saw in Example 2, Sec. 22, that this function *cannot* have a derivative at any nonzero point since the Cauchy–Riemann equations are not satisfied at such points. (See also Example 3, Sec. 19.)

EXAMPLE 3. When using the theorem in this section to find a derivative at a point z_0, one must be careful not to use the expression for $f'(z)$ in the statement of the theorem before the *existence* of $f'(z)$ at z_0 is established.

Consider, for instance, the function

$$f(z) = x^3 + i(1-y)^3.$$

Here

$$u(x, y) = x^3 \quad \text{and} \quad v(x, y) = (1-y)^3,$$

and it would be a mistake to say that $f'(z)$ exists everywhere and that

$$f'(z) = u_x + iv_x = 3x^2. \tag{6}$$

To see this, we observe that the first Cauchy–Riemann equation $u_x = v_y$ can hold only if

$$x^2 + (1-y)^2 = 0 \tag{7}$$

and that the second equation $u_y = -v_x$ is always satisfied. Condition (7) thus tells us that $f'(z)$ can exist only when $x = 0$ and $y = 1$. In view of equation (6), then, our theorem tells us that $f'(z)$ exists *only* when $z = i$, in which case $f'(i) = 0$.

24. POLAR COORDINATES

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

to restate the theorem in Sec. 23 in polar coordinates.

Depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta} \qquad (z \neq 0)$$

when $w = f(z)$, the real and imaginary components of $w = u + iv$ are expressed in terms of either the variables x and y or r and θ. Suppose that the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighborhood of a given nonzero point z_0 and are continuous at z_0. The first-order partial derivatives of u and v with respect to r and θ also have those properties, and the chain rule for differentiating real-valued functions of two real variables can be used to write them in terms of the ones with respect to x and y. More precisely, since

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$