CHAPTER

1

COMPLEX NUMBERS

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

1. SUMS AND PRODUCTS

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the ***complex plane***, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x, 0)$ on the ***real axis***, we write $x = (x, 0)$; and it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called ***pure imaginary numbers*** when $y \neq 0$. The y axis is then referred to as the ***imaginary axis***.

It is customary to denote a complex number (x, y) by z, so that (see Fig. 1)

$$z = (x, y). \tag{1}$$

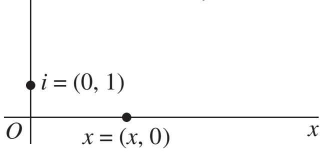

FIGURE 1

The real numbers x and y are, moreover, known as the ***real and imaginary parts*** of z, respectively, and we write

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z. \tag{2}$$

Two complex numbers z_1 and z_2 are ***equal*** whenever they have the same real parts and the same imaginary parts. Thus the statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex, or z, plane.

The ***sum*** $z_1 + z_2$ and ***product*** $z_1 z_2$ of two complex numbers

$$z_1 = (x_1, y_1) \quad \text{and} \quad z_2 = (x_1, y_1)$$

are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2). \tag{4}$$

Note that the operations defined by means of equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$
$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and if we think of a real number as either x or $(x, 0)$ and let i denote the pure imaginary number (0,1), as shown in Fig. 1, it is clear that*

$$z = x + iy. \tag{5}$$

Also, with the convention that $z^2 = zz$, $z^3 = z^2 z$, etc., we have

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

$$i^2 = -1. \tag{6}$$

Because $(x, y) = x + iy$, definitions (3) and (4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \tag{7}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2). \tag{8}$$

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*In electrical engineering, the letter j is used instead of i.

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs. Also, observe how equation (8) tells us that *any complex number times zero is zero*. More precisely,

$$z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0$$

for any $z = x + iy$.

2. BASIC ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

The commutative laws

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \tag{1}$$

and the associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3) \tag{2}$$

follow easily from the definitions in Sec. 1 of addition and multiplication of complex numbers and the fact that real numbers have corresponding properties. The same is true of the distributive law

$$z(z_1 + z_2) = z z_1 + z z_2. \tag{3}$$

EXAMPLE. If

$$z_1 = (x_1, y_1) \quad \text{and} \quad z_2 = (x_2, y_2),$$

then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

According to the commutative law for multiplication, $iy = yi$. Hence one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1 z_2 z_3$ is well defined without parentheses, as is the case with real numbers.

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z \tag{4}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 8).

There is associated with each complex number $z = (x, y)$ an additive inverse

$$-z = (-x, -y), \tag{5}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation

$$(x, y) + (u, v) = (0, 0)$$

implies that

$$u = -x \quad \text{and} \quad v = -y.$$

For any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (4), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \quad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \qquad (z \neq 0). \tag{6}$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression (6).

EXERCISES

1. Verify that

(*a*) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$;
(*b*) $(2, -3)(-2, 1) = (-1, 8)$;
(*c*) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

2. Show that

(a) $\operatorname{Re}(iz) = -\operatorname{Im} z$;

(b) $\operatorname{Im}(iz) = \operatorname{Re} z$.

3. Show that $(1+z)^2 = 1 + 2z + z^2$.

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

5. Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

6. Verify

(a) the associative law for addition of complex numbers, stated at the beginning of Sec. 2;

(b) the distributive law (3), Sec. 2.

7. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

8. (a) Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

9. Use $-1 = (-1, 0)$ and $z = (x, y)$ to show that $(-1)z = -z$.

10. Use $i = (0, 1)$ and $y = (y, 0)$ to verify that $-(iy) = (-i)y$. Thus show that the additive inverse of a complex number $z = x + iy$ can be written $-z = -x - iy$ without ambiguity.

11. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}\right)$.

3. FURTHER ALGEBRAIC PROPERTIES

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described in Sec. 2. Inasmuch as such properties continue to be anticipated because they also apply to real numbers, the reader can easily pass to Sec. 4 without serious disruption.

We begin with the observation that the existence of multiplicative inverses enables us to show that *if a product $z_1 z_2$ is zero, then so is at least one of the factors z_1 and z_2.* For suppose that $z_1 z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists; and any complex number times zero is zero (Sec. 1). Hence

$$z_2 = z_2 \cdot 1 = z_2\left(z_1 z_1^{-1}\right) = \left(z_1^{-1} z_1\right) z_2 = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0.$$

That is, if $z_1z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both of the numbers z_1 and z_2 are zero. Another way to state this result is that *if two complex numbers z_1 and z_2 are nonzero, then so is their product z_1z_2.*

Subtraction and division are defined in terms of additive and multiplicative inverses:

$$z_1 - z_2 = z_1 + (-z_2), \tag{1}$$

$$\frac{z_1}{z_2} = z_1z_2^{-1} \qquad (z_2 \neq 0). \tag{2}$$

Thus, in view of expressions (5) and (6) in Sec. 2,

$$z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2) \tag{3}$$

and

$$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right) \qquad (z_2 \neq 0) \tag{4}$$

when $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Using $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one can write expressions (3) and (4) here as

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{5}$$

and

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0). \tag{6}$$

Although expression (6) is not easy to remember, it can be obtained by writing (see Exercise 7)

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}, \tag{7}$$

multiplying out the products in the numerator and denominator on the right, and then using the property

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1z_3^{-1} + z_2z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \qquad (z_3 \neq 0). \tag{8}$$

The motivation for starting with equation (7) appears in Sec. 5.

EXAMPLE. The method is illustrated below:

$$\frac{4 + i}{2 - 3i} = \frac{(4 + i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{5 + 14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

There are some expected properties involving quotients that follow from the relation

$$\frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0), \tag{9}$$

which is equation (2) when $z_1 = 1$. Relation (9) enables us, for instance, to write equation (2) in the form

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0). \tag{10}$$

Also, by observing that (see Exercise 3)

$$(z_1z_2)(z_1^{-1}z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1 \qquad (z_1 \neq 0, z_2 \neq 0),$$

and hence that $z_1^{-1}z_2^{-1} = (z_1z_2)^{-1}$, one can use relation (9) to show that

$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = z_1^{-1}z_2^{-1} = (z_1z_2)^{-1} = \frac{1}{z_1z_2} \qquad (z_1 \neq 0, z_2 \neq 0). \tag{11}$$

Another useful property, to be derived in the exercises, is

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1z_2}{z_3z_4} \qquad (z_3 \neq 0, z_4 \neq 0). \tag{12}$$

Finally, we note that the ***binomial formula*** involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \qquad (n = 1, 2, \ldots) \tag{13}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof is left as an exercise. Because addition of complex numbers is commutative, the binomial formula can, of course, be written

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \qquad (n = 1, 2, \ldots). \tag{14}$$

EXERCISES

1. Reduce each of these quantities to a real number:

(*a*) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$; (*b*) $\dfrac{5i}{(1-i)(2-i)(3-i)}$; (*c*) $(1-i)^4$.

Ans. (*a*) $-\dfrac{2}{5}$; (*b*) $-\dfrac{1}{2}$; (*c*) -4.

2. Show that

$$\frac{1}{1/z} = z \qquad (z \neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4).$$

4. Prove that if $z_1 z_2 z_3 = 0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1 z_2) z_3 = 0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \qquad (z_3 \neq 0,\ z_4 \neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2} \qquad (z_2 \neq 0,\ z \neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

Suggestion: When $n = m + 1$, write

$$\begin{aligned}
(z_1 + z_2)^{m+1} &= (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1)\sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m-k} \\
&= \sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m}\binom{m}{k} z_1^{k+1} z_2^{m-k}
\end{aligned}$$

and replace k by $k - 1$ in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

4. VECTORS AND MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.