

## Gauss' Law

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### Preamble

In this lecture we learn a simple and powerful technique for calculating the electric field for situations involving a high degree of symmetry.

### Flux

Flux,  $\Phi$ , is a measure of the amount of a vector field,  $\mathbf{F}$ , passing through a surface with area  $\mathbf{A}$  (direction defined perpendicular to the plane of the surface):

$$\Phi = \mathbf{F} \cdot \mathbf{A} = FA \cos \theta$$

where  $\theta$  is the angle between the field direction and the normal to the area surface.

As one example, consider the flow of air or water through a surface. The volume of air passing through the surface per unit time is given by:

$$\Phi = \mathbf{v} \cdot \mathbf{A} = vA \cos \theta$$

Obviously this flow rate depends on the orientation of the surface.

As another example, consider the light from the Sun impinging onto the surface of the Earth. Obviously land near the equator receives a large light flux than that at the poles.

In this course, we will consider the flux of electric and magnetic fields (essentially light!) passing through a surface.

## Flux Through a Closed Surface

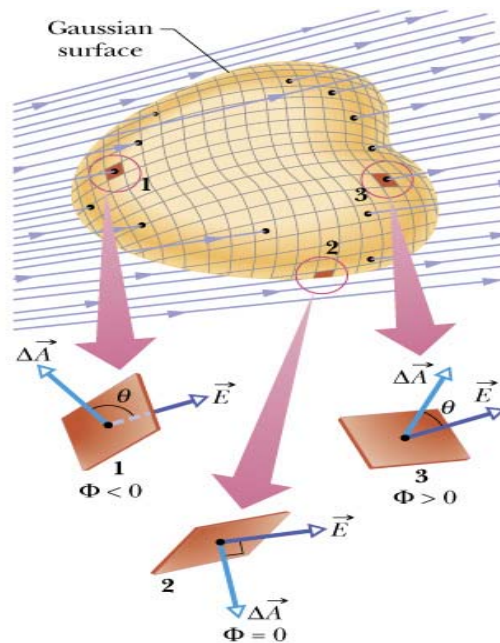


Fig. from HRW 7/e

Consider the flux of electric field through a closed surface. Let's break the surface into a large number ( $N$ ) of small elements of area  $\Delta\mathbf{A}$  that are nearly flat. By convention, let's take the direction of the area vector to be perpendicular to the surface and pointing out.

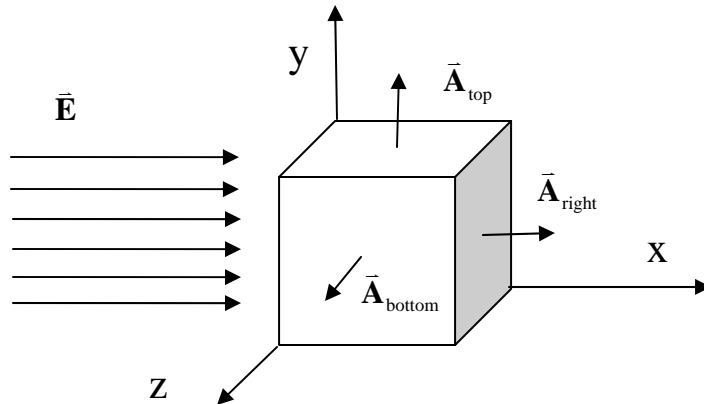
The flux of electric field passing through this closed surface is then:

$$\Phi = \sum_{i=1}^N \mathbf{E} \cdot \Delta\mathbf{A}$$

If we let  $N \rightarrow \infty$  and  $\Delta\mathbf{A} \rightarrow 0$ , then the small elements become infinitesimal and we can write:

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A}$$

This is a closed surface integral. It is a 2-dimensional integral over a 3-dimensional surface. The "circle" in the integral sign denotes an integral over an entire closed surface (versus just a single face).

**Example**

Consider a square cube immersed in a constant electric field:  $\vec{E} = E_0 \hat{x}$ . Calculate the flux through each face.

$\vec{A}_i$  is the area vector pointing out from cube face  $i$ , which has magnitude  $|\vec{A}_i| = s^2$ , where  $s$  is the side length of the cube.

Note that  $\vec{E} \cdot \vec{A}_{\text{top}} = 0$  since  $\vec{E} \perp \vec{A}_{\text{top}}$

$$\vec{E} \cdot \vec{A}_{\text{bottom}} = 0$$

Also:  $\vec{E} \cdot \vec{A}_{\text{front}} = 0$

$$\vec{E} \cdot \vec{A}_{\text{back}} = 0$$

What is left is:

$$\vec{E} \cdot \vec{A}_{\text{right}} = |\vec{E}| |\vec{A}_{\text{right}}| = E_0 s^2 \text{ since } \vec{E} \parallel \vec{A}_{\text{right}}$$

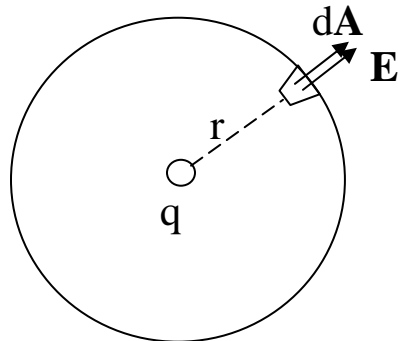
$$\vec{E} \cdot \vec{A}_{\text{left}} = -|\vec{E}| |\vec{A}_{\text{left}}| = -E_0 s^2 \text{ since } \vec{E} \text{ opposite } \vec{A}_{\text{left}}$$

$$\Rightarrow \Phi = \sum_{i=1}^6 \vec{E} \cdot \vec{A}_i = E_0 s^2 - E_0 s^2 = 0 \text{ sum over all 6 faces}$$

Thus, the total flux into the cube is balanced by the flux going out. This will hold for any arbitrary shape of the surface, provided the electric field is constant

## Flux of a Point Charge

Consider a spherical shell surface of radius  $r$  enclosing a point charge at the center.



Note that  $d\vec{A}$  points out from the closed surface, and that since  $\vec{E}$  points radially as well,  $\vec{E} \parallel d\vec{A}$  so that  $\vec{E} \cdot d\vec{A} = |\vec{E}| |d\vec{A}|$

By Coulomb's Law,  $|\vec{E}| = K \frac{q}{r^2}$

So the flux through the spherical surface is:

$$\begin{aligned}\Phi &= \oint_S \vec{E} \cdot d\vec{A} = \oint_S |\vec{E}| |d\vec{A}| \\ &= |\vec{E}| \oint_S |d\vec{A}| \text{ since } |\vec{E}| \text{ is constant at fixed radius}\end{aligned}$$

$$\oint_S |d\vec{A}| = 4\pi r^2 \text{ surface area of sphere}$$

To see this explicitly, let's do the surface integral:  $\oint_S |d\vec{A}| = \int_0^\pi r^2 \sin \theta d\theta \int_0^{2\pi} d\phi = 4\pi r^2$

So, the flux is:

$$\Phi = |\vec{E}| 4\pi r^2 = K \frac{q}{r^2} 4\pi r^2 = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \frac{4\pi r^2}{1}$$

$$\Rightarrow \Phi = \frac{q}{\epsilon_0}$$

Note that this is a constant, independent of the size of the sphere, and only dependent on the amount of charge enclosed. In fact, it turns out that it does not even depend on the shape of the surface!

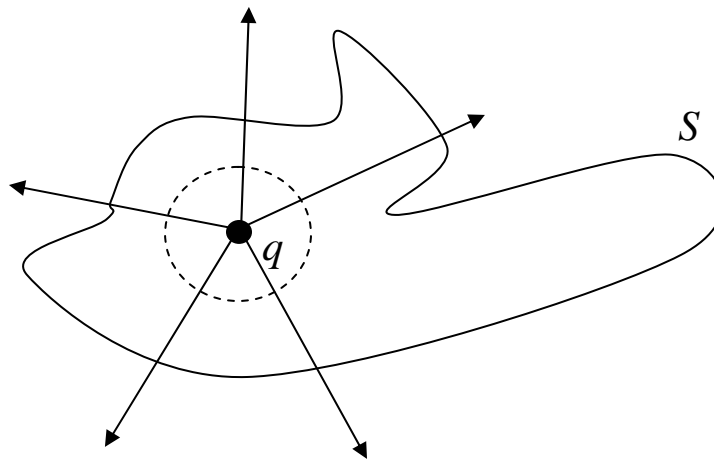
## Gauss' Law

Gauss' Law generalizes the previous example, and states that:

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0}$$

So the amount of flux passing through any closed surface  $S$  depends only on the net amount of electric charge enclosed.

Why is this general case true? Well, an arbitrary surface can always be inscribed by a sphere within it. And the flux through that sphere, calculated in the previous example, must be equal to the flux intercepted by the outer surface  $S$ .



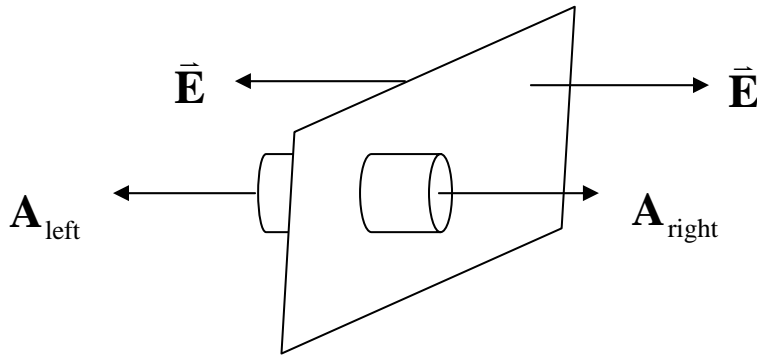
Gauss' Law relates the electric flux through a closed surface to the net charge enclosed. It was derived as a consequence of Coulomb's Law, but in fact is completely equivalent to Coulomb's Law. It can form an alternative starting point for calculating electric fields.

## Applications of Gauss' Law

Gauss' Law is a powerful technique to calculate the electric field for situations exhibiting a high degree of symmetry.

### Infinite Sheet of Charge

Let's calculate the electric field from an infinite sheet of charge, with a charge density of  $\sigma$  (measured in  $C/m^2$ ).



By symmetry, we expect  $\vec{E}$  to point perpendicular to the surface for a very large sheet (and far from the edges). We would not expect any other angle, because why would any particular direction be preferred over any other for such a symmetric situation?

Now consider a closed surface (which we will call a Gaussian surface) that extends through the sheet of charge. The sides, here assumed cylindrical, are chosen to be perpendicular to the sheet. So  $\vec{E} \cdot \vec{A}_{side} = 0$ . The caps are parallel to the sheet, so  $\vec{E} \cdot \vec{A}_{left} = \vec{E} \cdot \vec{A}_{right}$  since both vectors always point in the same direction. The total flux through the surface is thus:

$$\Phi = \vec{E} \cdot \vec{A}_{left} + \vec{E} \cdot \vec{A}_{right} = 2EA$$

By Gauss' Law,  $\Phi = \frac{q_{enc}}{\epsilon_0}$ , where  $q_{enc} = \sigma A$ . So:

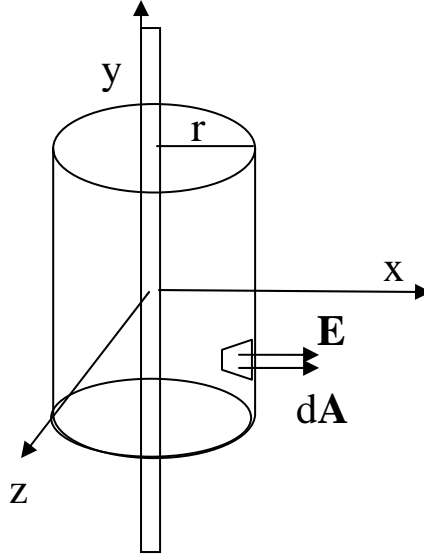
$$\Phi = 2EA = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

This is exactly the same solution for the magnitude of the electric field from an infinite sheet that we obtained by painstaking integration of rings of charge! And as noted before, the electric field is a constant and does not depend on the distance from the sheet.

## Infinite Line of Charge

Let's calculate the electric field a distance  $r$  from a line of electric charge infinite in extent with charge density  $\lambda$  (measured in C/m).



By symmetry, we expect that  $E$  points in the radial direction (no preferred direction):

$$\mathbf{E} = E(r)\hat{\mathbf{r}}$$

Choose a Gaussian surface with cylindrical geometry, with top and bottom caps aligned such that their area vectors point in the same direction as the line of charge (and perpendicular to the electric field):  $\mathbf{E} \cdot \mathbf{A}_{\text{top}} = \mathbf{E} \cdot \mathbf{A}_{\text{bottom}} = 0$ .

So the total flux leaves through the sides:

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = E(r) \oint_S |d\mathbf{A}| = E(r) 2\pi r h$$

since  $|\mathbf{E}|$  is constant at a fixed radius. Thus,

$$\Phi = E(r) 2\pi r h = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\lambda h}{\epsilon_0}$$

$$\Rightarrow \boxed{E(r) = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2K\lambda}{r}}$$

Same as derived by integration in the last lecture.

## Spheres of Charge

Let's calculate the electric field outside a spherically charged shell.

The radius of the sphere is  $R$ . A total charge  $Q$  is spread uniformly on the surface, so the charge density per unit area is  $\sigma = \frac{Q}{4\pi R^2}$

By symmetry we expect that the electric field points radially:  $\mathbf{E} = E(r)\hat{\mathbf{r}}$

For our Gaussian surface choose another sphere with radius  $r > R$ , centered on the charged sphere.

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = E(r) \oint_S |d\mathbf{A}|$$

since  $|\mathbf{E}|$  is constant at a fixed radius. The surface integral is just the integral of the surface area of a sphere, so:

$$\begin{aligned} \Phi &= \frac{Q}{\epsilon_0} = E(r)4\pi r^2 \\ \Rightarrow E(r) &= \frac{Q}{4\pi\epsilon_0 r^2} = K \frac{Q}{r^2} \end{aligned}$$

Exactly the same field strength as if all the charge on the sphere were placed at the center of the sphere as a point charge! This result applies to a solid sphere of total charge  $Q$  as well.

The situation changes for the electric field *inside* a spherical shell.

For  $r < R$ , then there is no net enclosed charge, and  $E(r) = 0$ . No force acts on any point charge placed anywhere inside a charged shell (forces from all infinitesimal charges in the shell balances).

For the electric field *inside* a solid sphere of total charge  $Q$ , then there is a net enclosed charge. First assume that the total charge of a solid sphere is spread uniformly throughout its volume. The volume charge density is therefore:

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \quad (\text{measured in C/m}^3).$$

The charge enclosed for  $r < R$  is:



$$q_{\text{enc}} = \rho \frac{4}{3} \pi r^3 = \frac{Q}{\frac{4}{3} \pi R^3} \frac{4}{3} \pi r^3 = Q \frac{r^3}{R^3}$$

So Gauss' Law tells us:

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = E(r) 4\pi r^2 = \frac{Q \frac{r^3}{R^3}}{\epsilon_0}$$

$$\Rightarrow E(r) = \frac{Q}{4\pi\epsilon_0 R^3} r$$

That is, the electric field rises linearly from 0 at the center of the sphere to the magnitude obtained at the surface of the sphere (which is the same as that from a point charge at a distance  $R$ ).

## Electric Fields inside Conductors

The previous examples of continuous charge distributions have implicitly assumed that the infinitesimal charges were not free to move—locked in place within the medium. A conductor, on the other hand, has some electrical charges free to move (the valence electrons in the metal). So a conductor can carry an electric current (units: amps, C/s).

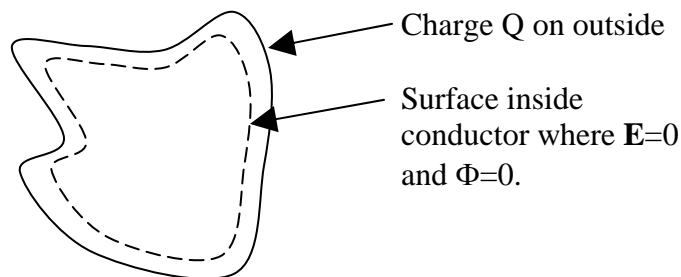
In electrostatic situations, which means that the charges have reached static equilibrium inside the conductor, we must have:

$$\vec{E} = 0.$$

If the field were not zero, that would imply a force acting on the charge carriers ( $\mathbf{F} = q\mathbf{E} = m\mathbf{a}$ ), which would cause them to accelerate. Thus the situation would still be dynamically in motion and not have reached equilibrium. But once equilibrium has been achieved (very quickly unless an electromotive force maintains an electric potential difference across the conductor—see next chapter), the field must be zero.

Now let's explore the consequences of this observation. If  $\vec{E} = 0$  inside the conductor, then the flux of electric field passing through any closed surface inside the conductor is also zero:

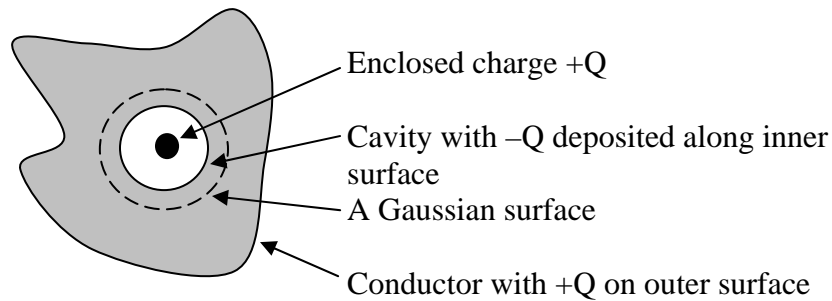
$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = 0$$



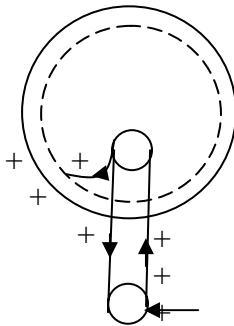
So by Gauss' Law, there can be no net enclosed charge inside the surface. Therefore, for conductors, all deposited charge must be on the outside of the conductor.

The same applies even if cavities exist inside the conducting volume.

Now electric charge could exist inside a non-conducting cavity within the conductor, but in that case "screening" charges (of opposite sign to the original charge) must move to the inner surface of the conductor in order to "screen" the electric field arising from this inner charge so that  $\mathbf{E}=0$  in the conductor. (Thus for a Gaussian surface containing this inner surface and the charge, the net charge would be zero and by Gauss' Law, the flux and field would be zero as well). Now because the conductor was presumably balanced in charge, the distribution of charge on the inner surface will result in an excess of charge on the outer surface of the conductor equal in magnitude and sign to the enclosed charge.



A Van de Graaff accelerator exploits this phenomenon that the charge inside a conductor always moves to the outside surface, no matter how much charge is already there. By “scraping” static charge from a material by friction, we can transfer this tiny amount of charge via a belt to the inside of a conducting shell (which by the property above must not have any charge on its surface). A fraction of this charge will then be transferred to the shell, and then immediately it will move to the outside surface of the shell where it can accumulate continuously to very large values.

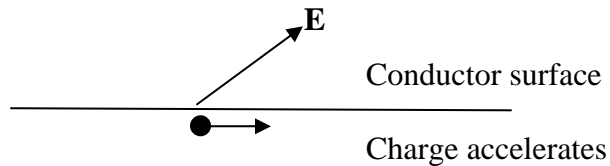


This phenomenon that  $E=0$  inside a conductor also has implications for electromagnetic waves. If a radio transmitter (such as a cell phone) is placed inside a metal box, the electromagnetic fields it generates cannot penetrate through the metal. It is shielded. In fact, the metal need not be completely solid: a screen or even chicken wire works fairly well at shielding electric fields.

Such a box is known as a Faraday cage. You can make one yourself by wrapping your cell phone in aluminum foil and then trying to call its phone number. The phone will not ring even when the phone is on. You might also notice this effect if you happen to be talking on your cell phone while you enter an elevator that is completely surrounded by metal. Once the doors close, call is over!

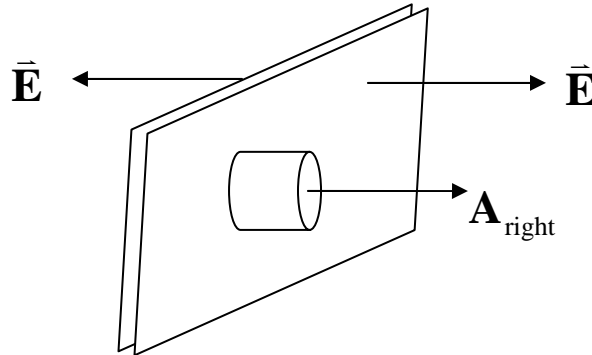
## Electric Fields on the Surface of Conductors

There are also constraints on the direction of the electric field at the surface of a conductor in electrostatic equilibrium. Outside the conductor, we must have that the electric field direction is perpendicular to the surface of the conductor (parallel to the area vector). If it were not, there would be a component of the electric field parallel to the surface, and that component would cause the free charges in the conductor to accelerate (thus not being in equilibrium). After a short amount of time, all charges will redistribute themselves and the electric field in this configuration will always be perpendicular to the surface.



### Conducting sheet of Charge

Let's now calculate the magnitude of the electric field arising from electric charge applied to the surface of a conductor (we know the direction will be perpendicular to the surface).



We solve this problem in a manner similar to the static sheet of charge solved earlier, except that now we do not extend the Gaussian surface through the sheet, but stop it inside the conductor (it will always have some thickness). The flux through the sides of the Gaussian surface is zero because the electric field and area vectors are perpendicular. The flux through the inner left cap is also zero because  $\mathbf{E}=0$  inside a conductor. Thus the total flux only arises through the rightmost cap:

$$\Phi = \sum_i \mathbf{E} \cdot \mathbf{A}_i = \mathbf{E} \cdot \mathbf{A}_{\text{right}} = EA = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

So we have for the magnitude of the electric field:

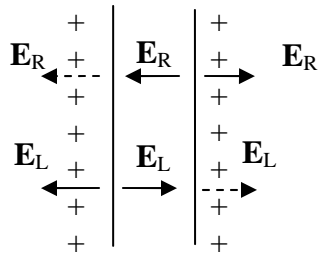
$$\Rightarrow E = \frac{\sigma}{\epsilon_0}$$

Note that this is exactly two times larger than the same expression for a non-conducting sheet of charge with charge density  $\sigma$ .

To understand why this is the case, note that if a charge  $Q$  is deposited on the surface of a conductor, it will spread and uniformly cover *both* sides of the conducting sheet. The charge density on each side will be:

$$\sigma = \frac{(Q/2)}{A} = \frac{Q}{2A}$$

Now let's examine the contribution to the total electric field from each side of the sheet separately, and then add the two together to get the total field.



Here you can consider each surface as a sheet of charge on a non-conducting surface (it is infinitesimally thin). The magnitude of the field from these surfaces is:

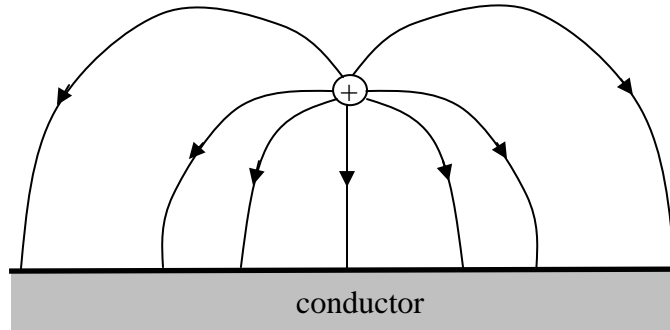
$$E_R = E_L = \frac{\sigma}{2\epsilon_0} = \frac{Q}{4\epsilon_0 A}$$

If we assume that the sheets are large compared to the thickness of the conductor, the fields from each surface cancel inside the conductor (oppositely directed fields). Outside the conductor, the fields add in the same direction, so the magnitude is:

$$E_{TOT} = E_L + E_R = 2 \frac{Q}{4\epsilon_0 A} = \frac{Q}{\epsilon_0 2A} = \frac{\sigma}{\epsilon_0}$$

## Method of Image Charges

We argued that the electric field must always be perpendicular to the surface of a conductor in equilibrium. That means any charge placed near the surface of a conductor must have field lines that stop at the surface of the conductor perpendicular to the surface.



There is a powerful trick to solve for the shape of the field lines for such electrostatic situations known as the method of image charges. The trick is to place an imaginary opposite sign charge of equal magnitude to the first charge behind the surface of the conductor (obviously where we cannot actually place a physically charge).

For the example above, we have actually solved this before! It's an electric dipole.

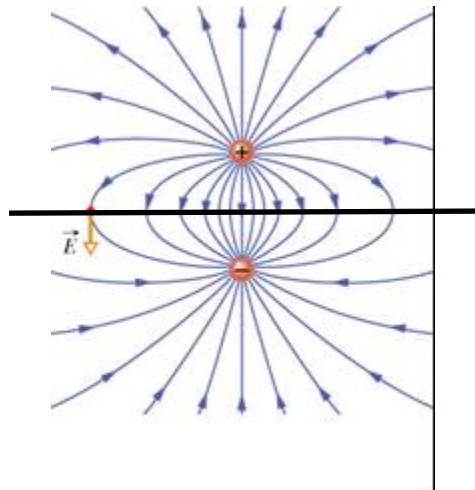


Fig. from HRW 7/e

This technique can be applied to other charge and conductor distributions. In fact, images of images may even be necessary if a charge is placed between two conductors.