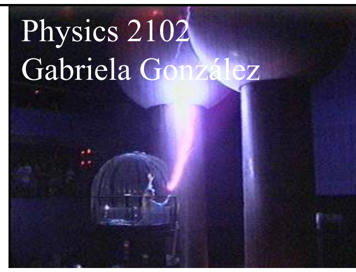
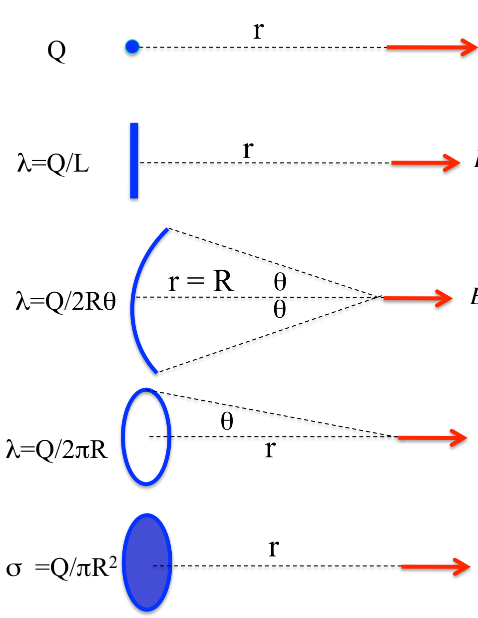


Physics 2102

Electric fields
Gauss' law

Carl Friedrich Gauss
1777-1855



$$E = \frac{kQ}{r^2}$$

$$E = \frac{kQ}{r\sqrt{r^2+(L/2)^2}} = \frac{k\lambda L}{r\sqrt{r^2+(L/2)^2}}$$

$$E = \frac{kQ}{R^2}\frac{\sin(\theta)}{\theta} = \frac{2k\lambda}{R}\sin\theta$$

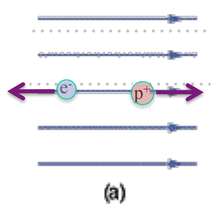
$$E = \frac{kQ}{R^2}\cos\theta = \frac{2k\lambda}{R}\frac{\cos\theta}{\pi}$$

$$E = \frac{2kQ}{R^2}\left(1-\frac{r}{\sqrt{r^2+R^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{r}{\sqrt{r^2+R^2}}\right)$$

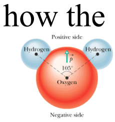
Electric field lines and forces

We want to calculate electric fields because we want to predict how charges would move in space: we want to know *forces*.

The drawings below represent electric field lines. Draw vectors representing the electric force on an electron and on a proton at the positions shown, disregarding forces between the electron and the proton.



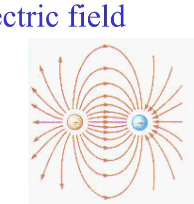
Imagine the electron-proton pair is held at a distance by a rigid bar (this is a model for a water molecule). Can you predict how the dipole will move?



Electric charges and fields

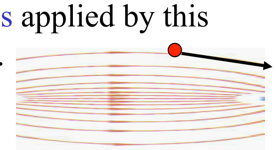
We work with two different kinds of problems, easily confused:

• **Given certain electric charges**, we calculate the electric field produced by those charges.



Example: we calculated the electric field produced by the two charges in a dipole :

• **Given an electric field**, we calculate the forces applied by this electric field on charges that come into the field.

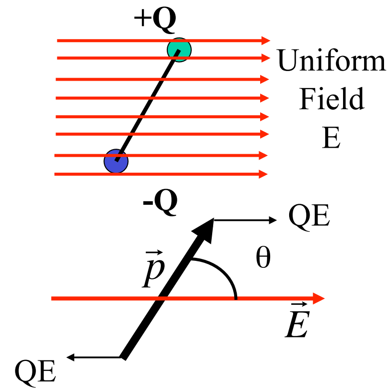


Example: forces on a single charge when immersed in the field of a dipole:
(another example: force on a dipole when immersed in a uniform field)

Electric Dipole in a Uniform Field

- **Net force on dipole = 0; center of mass stays where it is.**
- **Net TORQUE τ: INTO page. Dipole rotates to line up in direction of E.**
- **| τ | = 2(QE)(a/2)(sin θ)**
  = (Qa)(E)sinθ
  = |p| E sinθ = |p x E|
- **The dipole tends to "align" itself with the field lines.**

Distance between charges = a



$\vec{\tau} = \vec{p} \times \vec{E}$ (torque on a dipole).

Potential energy of a dipole =
Work done by the field on the dipole:

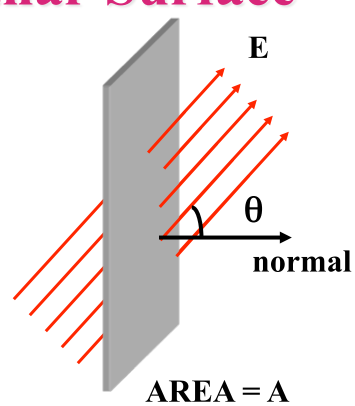
$$U = -W = -\int_{90°}^{\theta} \tau \, d\theta = \int_{90°}^{\theta} pE \sin\theta \, d\theta. \qquad U = -pE\cos\theta.$$

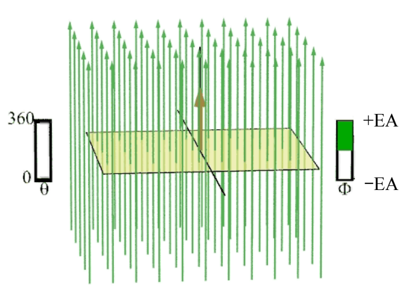
$U = -\vec{p} \cdot \vec{E}$ (potential energy of a dipole).

When is the potential energy largest?

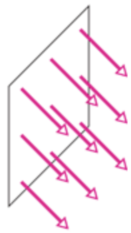
Electric Flux: Planar Surface

- Given:
  - planar surface, area A
  - uniform field E
  - E makes angle θ with NORMAL to plane
- Electric Flux: **Φ = E A cos θ**
- **Units: Nm²/C**
- Visualize: "flow of water" through surface



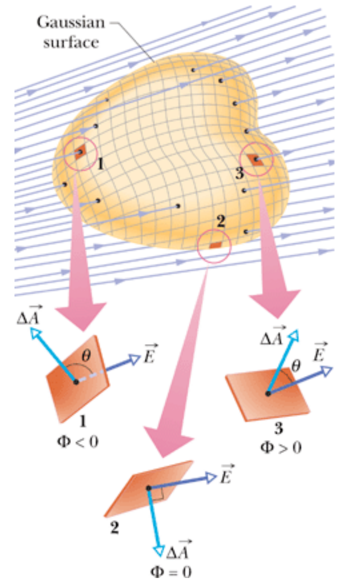


Electric Flux



$$\Phi = \int \vec{E} \bullet d\vec{A}$$

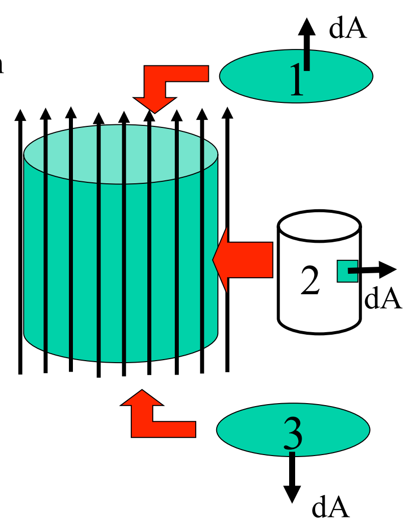
- Electric Flux
  A *surface* integral!
- **CLOSED** surfaces:
  - define the vector dA as pointing **OUTWARDS**
  - **Inward E gives negative Φ**
  - **Outward E gives positive Φ**



$\Phi = \int E \cdot dA$

Electric Flux: Example

- Closed cylinder of length L, radius R
- Uniform E parallel to cylinder axis
- What is the total electric flux through surface of cylinder?



- Note that E is NORMAL to both bottom and top cap
- E is PARALLEL to curved surface everywhere
- So: $\Phi = \Phi_1 + \Phi_2 + \Phi_3$
  $= \pi R^2 E + 0 - \pi R^2 E = 0!$
- Physical interpretation: total “inflow” = total “outflow”!

$\Phi = \int E \cdot dA$

Electric Flux: Example

- Spherical surface of radius R=1m; E is RADIALLY INWARDS and has EQUAL magnitude of 10 N/C everywhere on surface
- What is the flux through the spherical surface?

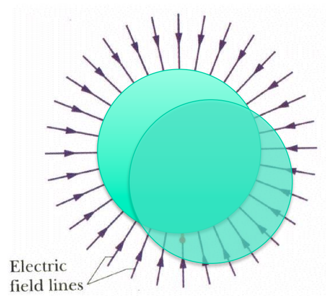
(α) $-(4/3)\pi R^2 E = -13.33\pi$ Nm^2/C

(b) $4\pi R^2 E = +40\pi$ Nm^2/C

(c) $4\pi R^2 E = -40\pi$ Nm^2/C

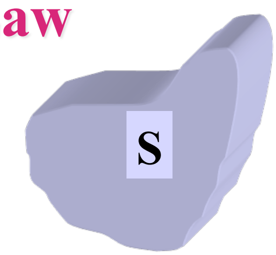
What could produce such a field?

What is the flux if the sphere is not centered on the charge?



Gauss' Law

- Consider any ARBITRARY CLOSED surface S -- NOTE: this does NOT have to be a "real" physical object!
- The TOTAL ELECTRIC FLUX through S is proportional to the TOTAL CHARGE ENCLOSED!
- The results of a complicated integral is a very simple formula: it avoids long calculations!

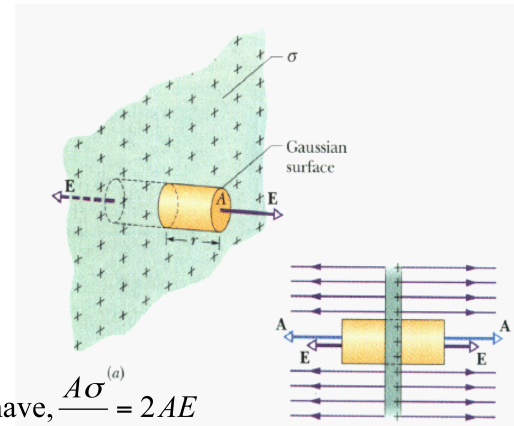


$$\Phi \equiv \oint_{\text{Surface}} \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

(One of Maxwell's 4 equations)

Gauss' Law: Example

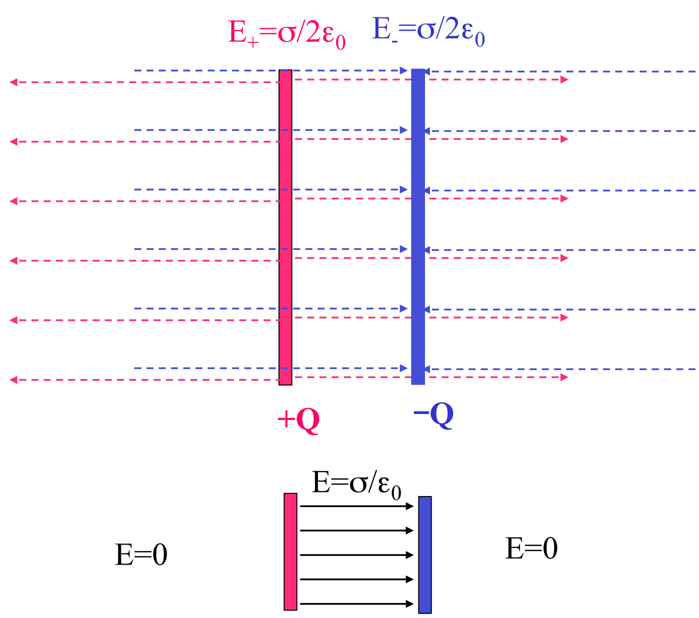
- Infinite plane with uniform charge density σ
- E is NORMAL to plane
- Construct Gaussian box as shown



Applying Gauss' law $\frac{q}{\varepsilon_0} = \Phi$, we have, $\frac{A\sigma}{\varepsilon_0} = 2AE$

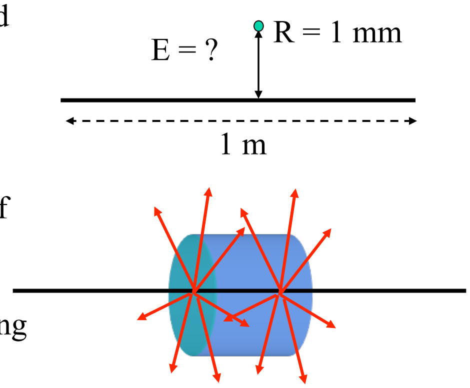
Solving for the electric field, we get $E = \frac{\sigma}{2\varepsilon_0}$

Two infinite planes



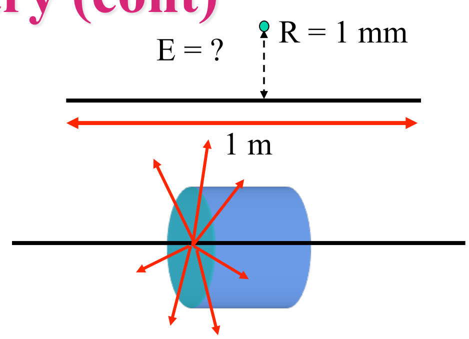
Gauss' Law: Example Cylindrical symmetry

- Charge of 10 C is uniformly spread over a line of length L = 1 m.
- Use Gauss' Law to compute magnitude of E at a perpendicular distance of 1 mm from the center of the line.
    - Approximate as infinitely long line -- E radiates outwards.
    - Choose cylindrical surface of radius R, length L co-axial with line of charge.



Gauss' Law: cylindrical symmetry (cont)

- Approximate as infinitely long line -- E radiates outwards.
- Choose cylindrical surface of radius R, length L co-axial with line of charge.



$$\Phi = |E|A = |E|2\pi RL$$

$$\Phi = \frac{q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

$$|E| = \frac{\lambda L}{2\pi\varepsilon_0 RL} = \frac{\lambda}{2\pi\varepsilon_0 R} = 2k\frac{\lambda}{R}$$

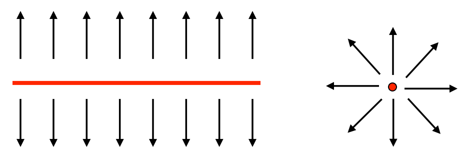
Compare with last class!

$$E_y = k\lambda\, a \int_{-L/2}^{L/2} \frac{dx}{(a^2 + x^2)^{3/2}} = k\lambda\, a \left[\frac{x}{a^2 \sqrt{x^2 + a^2}} \right]_{-L/2}^{L/2}$$

$$= \frac{2k\lambda L}{a\sqrt{4a^2 + L^2}}$$

if the line is infinitely long (L >> a)…

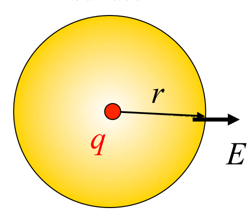
$$E_y = \frac{2k\lambda L}{a\sqrt{L^2}} = \frac{2k\lambda}{a}$$



Gauss' Law: Example
Spherical symmetry

$$\Phi \equiv \oint_{\text{Surface}} \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

- Consider a POINT charge q & pretend that you don't know Coulomb's Law
- Use Gauss' Law to compute the electric field at a distance *r* from the charge
- Use symmetry:
  - draw a spherical surface of radius R centered around the charge q
  - E has same magnitude anywhere on surface
  - E normal to surface



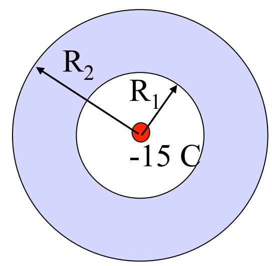
$$\Phi = |E|\, A = |E|\, 4\pi r^2$$

$$\Phi = \frac{q}{\varepsilon_0}$$

$$|E| = \frac{\Phi}{A} = \frac{q/\varepsilon_0}{4\pi r^2} = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{kq}{r^2}$$

Gauss' Law: Example

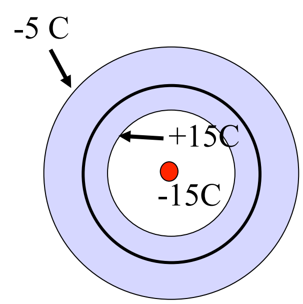
- A spherical conducting shell has an excess charge of +10 C.
- A point charge of −15 C is located at center of the sphere.
- Use Gauss' Law to calculate the charge on inner and outer surface of sphere



(a) Inner: +15 C; outer: 0
(b) Inner: 0; outer: +10 C
(c) Inner: +15 C; outer: -5 C

Gauss' Law: Example

- Inside a conductor, E = 0 under static equilibrium! Otherwise electrons would keep moving!
- Construct a Gaussian surface inside the metal as shown. (Does not have to be spherical!)
- Since E = 0 inside the metal, flux through this surface = 0
- Gauss' Law says total charge enclosed = 0
- Charge on inner surface = +15 C



Since TOTAL charge on shell is +10 C,
Charge on outer surface = +10 C − 15 C = −5 C!

Summary:

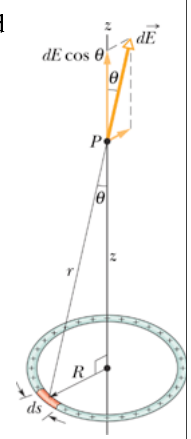
- Gauss' law: **$\Phi = \int E \cdot dA$** provides a very direct way to compute the electric flux if we know the electric field.
- In situations with symmetry, knowing the flux allows us to compute the fields reasonably easily.

Electric field of a ring

Let's calculate the field produced by a ring of radius R with total charge +Q, on a point on the axis, at a distance z from the center.

A differential ring element will have charge dq, and will produce a field d**E** with direction as shown in the figure. The magnitude of the field is $dE = kdq/r^2$.

Notice that the distance r is the same for all elements!



By symmetry, we know the field will point up, so we will only need to integrate the component $dEy = dE\cos\theta = (k\ dq/r^2)(z/r) = k(z/r^3)dq$.

Notice that the angle θ is the same for all elements, it is not an integration variable!

We integrate over the ring to get the magnitude of the total field:

$E = \int dEy = \int k(z/r^3)dq = k(z/r^3)\int dq = kQz/r^3 = kQz/(R^2+z^2)^{3/2}$

No integral table needed!

What's the field very far from the ring?

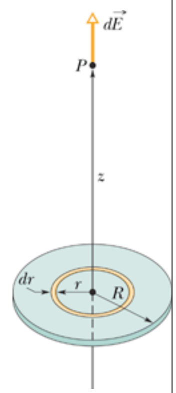
If $z >> R$, $E \sim kQz/z^3 = kQ/z^2$: of course, the field of a point charge Q.

Electric field of a disk

Let's calculate the field of a disk of radius R with charge Q, at a distance z on the axis above the disk.

First, we divide it in infinitesimal "rings", since we know the field produced by each ring.

Each ring has radius r and width dr: we will integrate on r, from 0 to R.



The charge per unit surface for the disk is $\sigma = Q/(\pi R^2)$, and the area of the ring is $dA = 2\pi r dr$, so the charge of the ring is $dq = \sigma dA = 2\pi\sigma\, rdr$.

The field of each ring points up, and has magnitude

$dE = k\, dq\, z/(r^2+z^2)^{3/2} = (1/4\pi\varepsilon_0)(2\pi\sigma\, rdr)\, z/(r^2+z^2)^{3/2}$
$= (\sigma z/4\varepsilon_0)(rdr)\, /(r^2+z^2)^{3/2}$

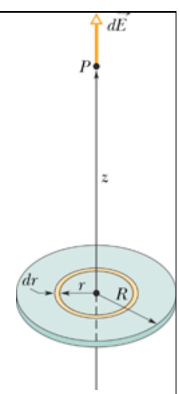
The total field is then

$E = (\sigma z/4\varepsilon_0) \int(2rdr)\, /(r^2+z^2)^{3/2} = (\sigma z/4\varepsilon_0)\, (-2/(r^2+z^2)^{1/2}\,)_0^R$

$$E = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$$

Electric field of a disk

$$E = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$$



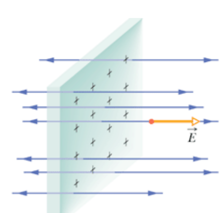
If we are very far from the disk, z>>R, E~0: of course, it gets vanishing small with distance. If we use

$$\frac{z}{\sqrt{R^2+z^2}} = \frac{1}{\sqrt{1+(R/z)^2}} \sim 1 - \frac{1}{2}(R/z)^2$$

We get $E \sim (\sigma/4\varepsilon_0)(R^2/z^2) = (Q/\pi R^2)/(4\varepsilon_0)(R^2/z^2) = kQ/z^2$. (Of course!)

If the disk is very large (or we are very close), R>>z, and $E \sim \sigma/2\varepsilon_0$

The field produced by any large charged surface is a uniform field, with magnitude $\sigma/2\varepsilon_0$.



(b)