The Uniform Distribution

38.2

Introduction

This Section introduces the simplest type of continuous probability distribution which features a continuous random variable X with probability density function $f(x)$ which assumes a constant value over a finite interval.

Prerequisites

Before starting this Section you should ...

- understand the concepts of probability
- be familiar with the concepts of expectation and variance
- be familiar with the concept of continuous probability distribution

Learning Outcomes

On completion you should be able to ...

- explain what is meant by the term uniform distribution
- calculate the mean and variance of a uniform distribution

1. The uniform distribution

The Uniform or Rectangular distribution has random variable X restricted to a finite interval $[a,b]$ and has $f(x)$ a constant over the interval. An illustration is shown in Figure 3:

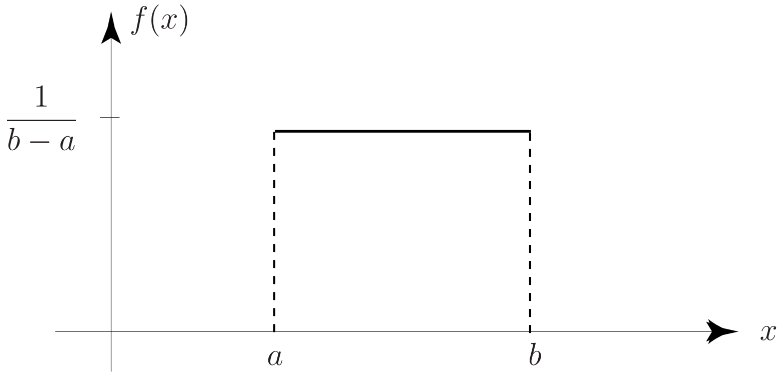

Figure 3

The function $f(x)$ is defined by:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Mean and variance of a uniform distribution

Using the definitions of expectation and variance leads to the following calculations. As you might expect, for a uniform distribution, the calculations are not difficult.
Using the basic definition of expectation we may write:

$$\begin{aligned} \mathsf{E}(X) &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_a^b x\frac{1}{b-a}\,dx = \frac{1}{2(b-a)}\left[x^2\right]_a^b \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

Using the formula for the variance, we may write:

$$\begin{aligned} \mathsf{V}(X) &= \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 \\ &= \int_a^b x^2.\frac{1}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2 = \frac{1}{3(b-a)}\left[x^3\right]_a^b - \left(\frac{b+a}{2}\right)^2 \\ &= \frac{b^3-a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\ &= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

Key Point 3

The **Uniform** random variable X whose density function $f(x)$ is defined by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ \\ 0 & \text{otherwise} \end{cases}$$

has expectation and variance given by the formulae

$$\text{E}(X) = \frac{b+a}{2} \qquad \text{and} \qquad \text{V}(X) = \frac{(b-a)^2}{12}$$

Example 2

The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval $[0, 25]$. Write down the formula for the probability density function $f(x)$ of the random variable X representing the current. Calculate the mean and variance of the distribution and find the cumulative distribution function $F(x)$.

Solution

Over the interval $[0, 25]$ the probability density function $f(x)$ is given by the formula

$$f(x) = \begin{cases} \dfrac{1}{25-0} = 0.04, & 0 \le x \le 25 \\ \\ 0 & \text{otherwise} \end{cases}$$

Using the formulae developed for the mean and variance gives

$$\text{E}(X) = \frac{25+0}{2} = 12.5 \text{ mA} \qquad \text{and} \qquad \text{V}(X) = \frac{(25-0)^2}{12} = 52.08 \text{ mA}^2$$

The cumulative distribution function is obtained by integrating the probability density function as shown below.

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

Hence, choosing the three distinct regions $x < 0$, $0 \le x \le 25$ and $x > 25$ in turn gives:

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{25} & 0 \le x \le 25 \\ 1 & x > 25 \end{cases}$$

Task

The thickness x of a protective coating applied to a conductor designed to work in corrosive conditions follows a uniform distribution over the interval $[20, 40]$ microns. Find the mean, standard deviation and cumulative distribution function of the thickness of the protective coating. Find also the probability that the coating is less than 35 microns thick.

Your solution

Answer

Over the interval $[20, 40]$ the probability density function $f(x)$ is given by the formula

$$f(x) = \begin{cases} 0.05, & 20 \leq x \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

Using the formulae developed for the mean and variance gives

$$\mathsf{E}(X) = 10 \ \mu\text{m} \qquad \text{and} \qquad \sigma = \sqrt{\mathsf{V}(X)} = \frac{20}{\sqrt{12}} = 5.77 \ \mu\text{m}$$

The cumulative distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

Hence, choosing appropriate ranges for x, the cumulative distribution function is obtained as:

$$F(x) = \begin{cases} 0, & x < 20 \\ \dfrac{x-20}{20} & 20 \leq x \leq 40 \\ 1 & x \geq 40 \end{cases}$$

Hence the probability that the coating is less than 35 microns thick is

$$F(x < 35) = \frac{35 - 20}{20} = 0.75$$

Exercises

1. In the manufacture of petroleum the distilling temperature (T°C) is crucial in determining the quality of the final product. T can be considered as a random variable uniformly distributed over 150°C to 300°C. It costs £C_1 to produce 1 gallon of petroleum. If the oil distills at temperatures less than 200°C the product sells for £C_2 per gallon. If it distills at a temperature greater than 200°C it sells for £C_3 per gallon. Find the expected net profit per gallon.

2. Packages have a nominal net weight of 1 kg. However their actual net weights have a uniform distribution over the interval 980 g to 1030 g.

   (a) Find the probability that the net weight of a package is less than 1 kg.

   (b) Find the probability that the net weight of a package is less than w g, where $980 < w < 1030$.

   (c) If the net weights of packages are independent, find the probability that, in a sample of five packages, all five net weights are less than wg and hence find the probability density function of the weight of the heaviest of the packages. (Hint: all five packages weigh less than w g if and only if the heaviest weighs less that w g).

Answers

1.

$$\mathsf{P}(X < 200) = 50 \times \frac{1}{150} = \frac{1}{3} \qquad \mathsf{P}(X > 200) = \frac{2}{3}$$

Let F be a random variable defining profit.
F can take two values £$(C_2 - C_1)$ or £$(C_3 - C_1)$

x	$C_2 - C_1$	$C_3 - C_1$
$\mathsf{P}(F = x)$	$^1/_3$	$^2/_3$

$$\mathsf{E}(F) = \left[\frac{C_2 - C_1}{3}\right] + \frac{2}{3}[C_3 - C_1] = \frac{C_2 - 3C_1 + 2C_3}{3}$$

2.

(a) The required probability is $\mathsf{P}(W < 1000) = \dfrac{1000 - 98}{1030 - 980} = \dfrac{20}{50} = 0.4$

(b) The required probability is $\mathsf{P}(W < w) = \dfrac{w - 980}{1030 - 980} = \dfrac{w - 980}{50}$

(c) The probability that all five weigh less than w g is $\left(\dfrac{w - 980}{50}\right)^5$ so the pdf of the heaviest is

$$\frac{d}{dw}\left(\frac{w - 980}{50}\right)^5 = \frac{5}{50}\left(\frac{w - 980}{50}\right)^4 = 0.1\left(\frac{w - 980}{50}\right)^4 \quad \text{for } 980 < w < 1030.$$