Method of Solution:-

The Solution of exact differential equation is as by four steps method.

Step-1

We first write the preliminary result

$$F(y,t) = \int M\,dy + \psi(t)$$

Step: II.-

partially differentiating the above results w.r. to "t" and set equal to N. i.e $\frac{\partial F}{\partial t} = N$

Now we substituting the value of N in the equation.

Step: III.- Integrating the above result w.r.to dy to get the specific value of $\psi(t)$. i.e $\psi(t) = \int \psi'(t)\,dt$

Step IV.-

Combing the result of Step-I and Step III to get $F(y,t)$.

Example: No. 1 Solve the exact differential equation

$$2yt\,dy + y^2\,dt = 0$$

Solution:-

$$2yt\,dy + y^2\,dt = 0$$

here $M = 2yt$ and $N = y^2$

$$\frac{\partial M}{\partial t} = 2y \quad , \quad \frac{\partial N}{\partial y} = 2y$$

So $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$

So the equation is exact differential equation.

Step: I $F(y,t) = \int M\,dy + \psi(t)$

$$F(y,t) = \int 2yt\,dy + \psi(t) = 2t \int y\,dy + \psi(t)$$

$$= \cancel{2}t\, y^2/\cancel{2} + \psi(t) = ty^2 + \psi(t)$$

$$F(y,t) = y^2 t + \psi(t)$$

($\because$ the constant of integration automatically merged into $\psi(t)$)

Step: II:- differentiating the above result w.r.to "t"

$$\frac{\partial F}{\partial t} = y^2 + \psi'(t)$$

$$y^2 = y^2 + \psi'(t) \Rightarrow y^2 - y^2 = \psi'(t) \qquad \left(\because N = \frac{\partial F}{\partial t},\ N = y^2\right)$$

$\psi'(t) = 0$

Step:III:-

Integrating with respect to dt we have.

$\psi(t) = \int \psi'(t)\, dy = \int 0\, dt$

$\psi(t) = K$

[**Note**: derivative of a constant is equal to zero and the integration of a zero is constant.]

Step IV:-

Substituating the value of $\psi(t)$ into Step-I we get

$E(y,t) = y^2(t) + k$

the Solution of the exact differential equal should be

$E(y,t) = c$

$y^2t + k = c \Rightarrow y^2(t) = k - c$

$y^2t = c \Rightarrow y^2 = c/t \quad (\because k - c = c)$

$y = (c/t)^{1/2}$ or

$$\boxed{y(t) = (c/t)^{1/2}}$$

Ans

Example: No. 2 Solve the equation

$$(t+2y)dy + (y+3t^2)dt = 0$$

Solution:-

$$(t+2y)dy + (y+3t^2)dt = 0$$

here $M = t+2y$ and $N = y+3t^2$

$$\frac{\partial M}{\partial t} = 1 \qquad \frac{\partial N}{\partial y} = 1$$

So $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$ (which is exact diff equation)

Step: I: $F(y,t) = \int M\,dy + \psi(t)$

$$= \int (t+2y)\,dy + \psi(t)$$

$$= \int t\,dy + \int 2y\,dy + \psi(t)$$

$$F(y,t) = ty + 2y^2/2 + \psi(t)$$

$$F(y,t) = ty + y^2 + \psi(t)$$

Step: II differentiating w.r. to "t" we have

$$\frac{\partial F}{\partial t} = y + \psi'(t) \qquad \left(\because \frac{\partial F}{\partial t} = N \quad N = y+3t^2 \right)$$

$$y + 3t^2 = y + \psi'(t)$$

$$\cancel{y} - \cancel{y} + 3t^2 = \psi'(t) \implies \psi'(t) = 3t^2$$

Step: III. Integrating w.r. to "dt" we have

$$\psi(t) = \int \psi'(t)\,dt = \int 3t^2\,dt$$

$$\psi(t) = 3t^3/3 \Rightarrow \psi(t) = t^3$$

$$F(y,t) = y(t) + y^2 + t^3$$

$$(\because \psi(t) = t^3)$$

The solution of the differential equation is

$$F(y,t) = c \Rightarrow \boxed{yt + y^2 + t^3 = c}$$ Ans

Separable Variables:

If the differential equation is of the form

$$f(y,t)\,dy + g(y,t)\,dt = 0$$

Let the function f is in the variable y alone while the function g involves only the variable t. So that the equation reduce to the special form:

$$f(y)\,dy + g(t)\,dt = 0$$

In such case the variables are said to be separable.

Example: No: 1:- Solve the equation

$$3y^2dy - t\,dt = 0$$

Solution:

$$3y^2dy - t\,dt = 0$$

First of all we rewrite the equation.

$$3y^2 dy = t\,dt$$

Now integrating the both sides and equalting the results we have

$$\int 3y^2 dy = \int t\, dt$$

$$3\int y^2 dy = \int t\, dt$$

$$\cancel{3}\,\frac{y^3}{\cancel{3}} + c_1 = \frac{t^2}{2} + c_2$$

$$y^3 + c_1 = \frac{t^2}{2} + c_2$$

$$y^3 = \frac{t^2}{2} + c_2 - c_1$$

$$y^3 = \frac{t^2}{2} + c \qquad (\because c_2 - c_1 = c)$$

therefore the general solution is:

$$y(t) = \left(\tfrac{1}{2}t^2 + c\right)^{1/3} \quad \text{Ans.}$$

Example: No. 2: Solve the equation

$$2t\,dy + y\,dt = 0$$

Solution: $2t\,dy + y\,dt = 0$

dividing throughout by $2yt$. we get the separable equation.

$\frac{1}{y}dy + \frac{1}{2t}dt = 0$ or $\frac{2t\,dy + y\,dt}{2yt} = 0$

$\frac{2t\,dy}{2yt} + \frac{y\,dt}{2yt} = 0 \Rightarrow \frac{1}{y}dy + \frac{1}{2t}dt = 0$

$\frac{1}{y}dy = -\frac{1}{2t}dt$

Integrating the both sides and equating the results, we have.

$\int \frac{1}{y}dy = \int -\frac{1}{2t}dt$

$\int \frac{1}{y}dy = -\frac{1}{2}\int \frac{1}{t}dt \Rightarrow \ln y + c_1 = -\frac{1}{2}\ln t + c_2$

$\left(\because \int \frac{f'(x)}{f(x)}dx = \ln f(x) + c\right)$

$\ln y + c_1 + \frac{1}{2}\ln t - c_2 = 0$ $(\because c_2 - c_1 = c)$

$\ln y + \frac{1}{2}\ln t = c_2 - c_1$

$\ln y + \frac{1}{2}\ln t = c$

$\ln(yt^{1/2}) = c$

Taking antilog on bothsides we have.

$e^{\ln(yt^{1/2})} = e^{c}$ $(\because e^{c} = K)$

$yt^{1/2} = K \Rightarrow y = K/t^{1/2}$

$y = Kt^{-1/2}$

where k is any constant thus the Solution is

$$\boxed{\therefore y(t) = Kt^{-1/2}}$$

Ans