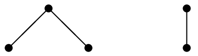
EXERCISES

1.5. Draw the Hasse diagrams for all possible orders on sets of $3, 4, 5$ elements.

1.6. Draw the Hasse diagram for the set of positive divisors of 210 when ordered by divisibility.

1.7. If p, q are distinct primes and m, n are positive integers, draw the Hasse diagram for the set of positive divisors of $p^m q^n$, ordered by divisibility.

1.8. Let P_1 and P_2 be the ordered sets with Hasse diagrams



Draw the Hasse diagrams of $P_1 \times P_2$ and $P_2 \times P_1$ under the cartesian order (Example 1.6). Comment on the result.

1.9. With P_1 and P_2 as in Exercise 1.8, draw the Hasse diagrams of $P_1 \times P_2$ and $P_2 \times P_1$ under the lexicographic order (Exercise 1.1), and those of the vertical sums $P_1 \oplus P_2$ and $P_2 \oplus P_1$ (Exercise 1.3).

1.2 Order-preserving mappings

Definition If $(A; \leqslant_1)$ and $(B; \leqslant_2)$ are ordered sets then we say that a mapping $f : A \to B$ is **isotone** (or **order-preserving**) if

$$(\forall x, y \in A) \quad x \leqslant_1 y \Rightarrow f(x) \leqslant_2 f(y);$$

and is **antitone** (or **order-inverting**) if

$$(\forall x, y \in A) \quad x \leqslant_1 y \Rightarrow f(x) \geqslant_2 f(y).$$

Example 1.12 If E is a non-empty set and $A \subseteq E$ then $f_A : \mathbb{P}(E) \to \mathbb{P}(E)$ given by $f_A(X) = A \cap X$ is isotone. If X' is the complement of X in E then the assignment $X \mapsto X'$ defines an antitone mapping on $\mathbb{P}(E)$.

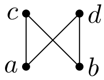
Example 1.13 Given $f : E \to F$ consider the induced **direct image map** $f^{\to} : \mathbb{P}(E) \to \mathbb{P}(F)$ defined for every $X \subseteq E$ by $f^{\to}(X) = \{f(x) \mid x \in X\}$ and the induced **inverse image map** $f^{\leftarrow} : \mathbb{P}(F) \to \mathbb{P}(E)$ defined for every $Y \subseteq F$ by $f^{\leftarrow}(Y) = \{x \in E \mid f(x) \in Y\}$. Each of these mappings is isotone.

We shall now give a natural interpretation of isotone mappings. For this purpose we require the following notions.

Definition By a **down-set** (or **hereditary subset**) of an ordered set $(E; \leqslant)$ we shall mean a subset D of E with the property that if $x \in D$ and $y \in E$ is such that $y \leqslant x$ then $y \in D$. We include the empty subset of E as a down-set. By a **principal down-set** we shall mean a down-set of the form $x^{\downarrow} = \{y \in E \mid y \leqslant x\}$. Dually, we define an **up-set** to be a subset U such that if $x \in U$ and $y \in E$ is such that $y \geqslant x$ then $y \in U$; and a **principal up-set** to be an up-set of the form $x^{\uparrow} = \{y \in E \mid y \geqslant x\}$.

Example 1.14 In the chain $\mathbb{Q}^+$ of positive rationals the set $\{q \in \mathbb{Q}^+ \mid q^2 \leqslant 2\}$ is a down-set that is not principal.

Example 1.15 If A and B are down-sets of an ordered set E then clearly so also are $A \cap B$ and $A \cup B$. This is not true in general for principal down-sets. For example, in



we have $c^{\downarrow} \cap d^{\downarrow} = \{a, b\} = a^{\downarrow} \cup b^{\downarrow}$.

Isotone mappings are characterised by the following properties.

Theorem 1.2 *If E, F are ordered sets and if $f : E \to F$ is any mapping then the following statements are equivalent*:

(1) *f is isotone*;
(2) *the inverse image of every principal down-set of F is a down-set of E*;
(3) *the inverse image of every principal up-set of F is an up-set of E.*

Proof (1) $\Rightarrow$ (2): Suppose that f is isotone. Let $y \in F$ and let $A = f^{\leftarrow}(y^{\downarrow})$. If $A \neq \emptyset$ let $x \in A$. Then for every $z \in E$ with $z \leqslant x$ we have $f(z) \leqslant f(x) \leqslant y$ whence $z \in A$. Thus A is a down-set of E.

(2) $\Rightarrow$ (1): For every $x \in E$ we have $x \in f^{\leftarrow}[f(x)^{\downarrow}]$. By (2) this is a down-set of E, so if $y \in E$ is such that $y \leqslant x$ we have $y \in f^{\leftarrow}[f(x)^{\downarrow}]$. It follows that $f(y) \leqslant f(x)$ and therefore f is isotone.

(1) $\Leftrightarrow$ (3): This follows from the above by the Principle of Duality. $\square$

1.3 Residuated mappings

In view of the above natural result, we now investigate under what conditions the inverse image of a principal down-set is also a principal down-set. The outcome will be a type of mapping that will play an important role in the sequel.

Theorem 1.3 *If E, F are ordered sets then the following conditions concerning $f : E \to F$ are equivalent*:

(1) *the inverse image under f of every principal down-set of F is a principal down-set of E*;

(2) *f is isotone and there is an isotone mapping $g : F \to E$ such that $g \circ f \geqslant \mathrm{id}_E$ and $f \circ g \leqslant \mathrm{id}_F$.*

Proof (1) $\Rightarrow$ (2): If (1) holds then it follows from Theorem 1.2 that f is isotone. In symbolic form, (1) becomes

$$(\forall y \in F)(\exists x \in E)\ \ f^{\leftarrow}(y^{\downarrow}) = x^{\downarrow}.$$

Now for every given $y \in F$ this element x is clearly unique, so we can define a mapping $g : F \to E$ by setting $g(y) = x$. Since $f^{\leftarrow}$ is isotone it follows that so is g. For this mapping g we have $g(y) \in g(y)^{\downarrow} = x^{\downarrow} = f^{\leftarrow}(y^{\downarrow})$, so $f[g(y)] \leqslant y$ for all $y \in F$ and therefore $f \circ g \leqslant \mathrm{id}_F$; and $x \in f^{\leftarrow}[f(x)^{\downarrow}] = g[f(x)]^{\downarrow}$ so that $x \leqslant g[f(x)]$ for all $x \in E$ and therefore $g \circ f \geqslant \mathrm{id}_E$.

(2) $\Rightarrow$ (1): If (2) holds then on the one hand we have

$$f(x) \leqslant y \;\Rightarrow\; x \leqslant g[f(x)] \leqslant g(y),$$

and on the other we have

$$x \leqslant g(y) \;\Rightarrow\; f(x) \leqslant f[g(y)] \leqslant y.$$

It follows from these observations that $f(x) \leqslant y$ if and only if $x \leqslant g(y)$ and therefore $f^{\leftarrow}(y^{\downarrow}) = g(y)^{\downarrow}$ from which (1) follows. $\square$

Definition A mapping $f : E \to F$ that satisfies either of the equivalent conditions of Theorem 1.3 is said to be **residuated**.

We note in particular that if $f : E \to F$ is a residuated mapping then an isotone mapping $g : F \to E$ which is such that $g \circ f \geqslant \mathrm{id}_E$ and $f \circ g \leqslant \mathrm{id}_F$ is in fact *unique*. To see this, suppose that g and $g^\star$ are each isotone and satisfy these properties. Then $g = \mathrm{id}_E \circ g \leqslant (g^\star \circ f) \circ g = g^\star \circ (f \circ g) \leqslant g^\star \circ \mathrm{id}_F = g^\star$. Similarly, $g^\star \leqslant g$ and therefore $g = g^\star$.

We shall denote this unique g by f^+ and call it the **residual** of f.

It is clear from the above that $f : E \to F$ is residuated if and only if, for every $y \in F$, there exists

$$f^+(y) = \max f^{\leftarrow}(y^{\downarrow}) = \max\{x \in E \mid f(x) \leqslant y\}.$$

Moreover, $f^+ \circ f \geqslant \mathrm{id}_E$ and $f \circ f^+ \leqslant \mathrm{id}_F$.

Example 1.16 Simple calculations reveal that if $f : E \to F$ then the direct image map $f^{\rightarrow} : \mathbb{P}(E) \to \mathbb{P}(F)$ is residuated with residual $f^{\leftarrow} : \mathbb{P}(F) \to \mathbb{P}(E)$.

Example 1.17 If E is any set and $A \subseteq E$ then $\lambda_A : \mathbb{P}(E) \to \mathbb{P}(E)$ defined by $\lambda_A(X) = A \cap X$ is residuated with residual λ_A^+ given by $\lambda_A^+(Y) = Y \cup A'$.

Example 1.18 For $m \in \mathbb{N}\backslash 0$ define $f_m : \mathbb{N} \to \mathbb{N}$ by $f_m(n) = mn$. Then f_m is residuated with $f_m^+(p) = \lfloor \frac{p}{m} \rfloor$ where $\lfloor q \rfloor$ denotes the integer part of $q \in \mathbb{Q}$.

Example 1.19 Every bounded operator f on a Hilbert space H induces a residuated mapping on the set of closed subspaces of H, namely that given by $M \mapsto [f^{\rightarrow}(M)]^{\perp\perp}$.

Example 1.20 If S is a semigroup, define a multiplication on $\mathbb{P}(S)$ by

$$XY = \begin{cases} \{xy \mid x \in X,\, y \in Y\} & \text{if } X, Y \neq \emptyset; \\ \emptyset & \text{otherwise.} \end{cases}$$

Then multiplication by a fixed subset of S is a residuated mapping on $\mathbb{P}(S)$.

Example 1.21 If R is a commutative ring with a 1 then multiplication by a fixed ideal of R is a residuated mapping on the ordered set $(I(R); \subseteq)$ of ideals.

The notion of a residuated mapping has its roots in investigations by Certaine [37], Ward and Dilworth [112], and Dilworth [43] into multiplicative ideal theory which relates directly to Example 1.21.

EXERCISES

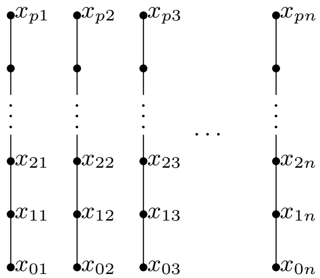
1.10. For each integer $n \geqslant 1$ let $\mathbf{n}$ denote the chain $1 < 2 < 3 < \cdots < n$. Prove that a mapping $f : \mathbf{n} \to \mathbf{m}$ is residuated if and only if it is isotone and $f(1) = 1$.

1.11. Let E be a bounded ordered set with bottom element 0 and top element 1. Given $e \in E$, let $\alpha_e, \beta_e : E \to E$ be defined by

$$\alpha_e(x) = \begin{cases} 0 & \text{if } x = 0; \\ e & \text{otherwise,} \end{cases} \qquad \beta_e(x) = \begin{cases} 0 & \text{if } x \leqslant e; \\ 1 & \text{otherwise.} \end{cases}$$

Prove that α_e and β_e are residuated and determine their residuals.

1.12. Let E be the ordered set given by the Hasse diagram



Show that the mapping $f : E \to E$ given by

$$f(x_{ij}) = \begin{cases} x_{0,j+1} & \text{if } i = 0,\ j \neq n; \\ x_{01} & \text{if } i = 0,\ j = n; \\ x_{i-1,j+1} & \text{if } i \neq 0,\ j \neq n; \\ x_{i-1,1} & \text{if } i \neq 0,\ j = n. \end{cases}$$

is residuated and determine f^+.

For every non-empty set E the residuated mappings on $\mathbb{P}(E)$ are completely described in the following result.

Theorem 1.4 *Let E be a non-empty set and let R be a binary relation on E. Then the mapping $\xi_R : \mathbb{P}(E) \to \mathbb{P}(E)$ given by the prescription*

$$\xi_R(A) = \{y \in E \mid (\exists x \in A)\ \ (x, y) \in R\}$$

is residuated. Moreover, every residuated mapping $f : \mathbb{P}(E) \to \mathbb{P}(E)$ is of this form for some binary relation R on E.

Proof Let $i : \mathbb{P}(E) \to \mathbb{P}(E)$ be the antitone mapping that sends each subset of E to its complement. Consider the isotone mapping $\xi_R^+ = i \circ \xi_{R^d} \circ i$. It is readily verified (draw pictures!) that $\xi_R \circ \xi_R^+ \leqslant \mathrm{id}$ and $\xi_R^+ \circ \xi_R \geqslant \mathrm{id}$, whence ξ_R^+ is the residual of ξ_R.

To see that every residuated mapping $f : \mathbb{P}(E) \to \mathbb{P}(E)$ is of this form for some binary relation R on E, consider the relation R_f defined on E by

$$(x, y) \in R_f \iff y \in f(\{x\}).$$

Observe that $\xi_{R_f}(\{x\}) = \{y \in E \mid (x, y) \in R_f\} = f(\{x\})$, so that f and ξ_{R_f} agree on singletons.

Now if $k : \mathbb{P}(E) \to \mathbb{P}(E)$ is any residuated mapping then, since it is isotone, for every non-empty subset A of E we have $k(A) = k\big(\bigcup_{x\in A}\{x\}\big) = \bigcup_{x\in A} k(\{x\})$. In fact, if $B = \bigcup_{x\in A} k(\{x\})$ then clearly $k(A) \supseteq B$. On the other hand, $k(\{x\}) \subseteq B$ for every $x \in A$ and so $\{x\} \subseteq k^+(B)$ whence $A = \bigcup_{x\in A}\{x\} \subseteq k^+(B)$ and therefore $k(A) \subseteq B$. The resulting equality, applied to both f and ξ_{R_f}, together with the fact that f and ξ_{R_f} agree on singletons now gives $f(A) = \bigcup_{x\in A} f(\{x\}) = \bigcup_{x\in A} \xi_{R_f}(\{x\}) = \xi_{R_f}(A)$ whence we obtain $f = \xi_{R_f}$. □

Particular properties of residuated mappings are the following.

Theorem 1.5 *If $f : E \to F$ is residuated then*

$$f \circ f^+ \circ f = f \quad \textit{and} \quad f^+ \circ f \circ f^+ = f^+.$$

Proof Since f is isotone, it follows from Theorem 1.3 that $f\circ f^+\circ f \geqslant f\circ \mathrm{id}_E = f$, and that $f \circ f^+ \circ f \leqslant \mathrm{id}_F \circ f = f$, from which the first equality follows. The second is established similarly. □

Theorem 1.6 *If $f : E \to F$ and $g : F \to G$ are residuated mappings then so is $g \circ f : E \to G$, and $(g \circ f)^+ = f^+ \circ g^+$.*

Proof Clearly, $g \circ f$ and $f \circ g$ are isotone. Moreover,

$$(f^+ \circ g^+) \circ (g \circ f) \geqslant f^+ \circ \mathrm{id}_F \circ f = f^+ \circ f \geqslant \mathrm{id}_E;$$
$$(g \circ f) \circ (f^+ \circ g^+) \leqslant g \circ \mathrm{id}_F \circ g^+ = g \circ g^+ \leqslant \mathrm{id}_G\,.$$

Thus, by the uniqueness of residuals, $(g \circ f)^+$ exists and is $f^+ \circ g^+$. □

Corollary *For every ordered set E the set* $\mathrm{Res}\,E$ *of residuated mappings $f : E \to E$ forms a semigroup, as does the set* $\mathrm{Res}^+ E$ *of residual mappings $f^+ : E \to E$.* □

EXERCISES

1.13. If $f, g : E \to E$ are residuated prove that $f \leqslant g \iff g^+ \leqslant f^+$. Deduce that the semigroups $\mathrm{Res}\,E$ and $\mathrm{Res}^+ E$ are anti-isomorphic.

1.14. If $f : E \to E$ is residuated prove that $f = f^+ \iff f^2 = \mathrm{id}_E$.

1.15. If $f : E \to F$ is residuated prove that the following are equivalent:

(1) $f^+ \circ f = \mathrm{id}_E$; (2) f is injective; (3) f^+ is surjective.

1.16. If E has a top element 1 prove that the mapping $\Theta : \mathrm{Res}\,E \to E$ given by $\Theta(f) = f(1)$ is residuated, with residual the mapping $\Psi : E \to \mathrm{Res}\,E$ given by $\Psi(e) = \alpha_e$ where α_e is defined in Exercise 1.11.

1.17. Let S be a semigroup with a zero element 0. Let $\mathbb{P}_0(S)$ be the set of all subsets of S that contain 0. For each $A \in \mathbb{P}_0(S)$ let $\lambda_A : \mathbb{P}_0(S) \to \mathbb{P}_0(S)$ be given by

$$\lambda_A(X) = AX = \{ax \mid a \in A, x \in X\}.$$

Prove that λ_A is residuated and determine λ_A^+. Do likewise for the mapping $\rho_A : \mathbb{P}_0(S) \to \mathbb{P}_0(S)$ given by $\rho_A(X) = XA$.