Part 5

Dynamic Analysis

Chapter 14

Economic Dynamics and Integral Calculus

The term *dynamics*, as applied to economic analysis, has had different meanings at different times and for different economists.[†] In standard usage today, however, the term refers to the type of analysis in which the object is either to trace and study the specific time paths of the variables or to determine whether, given sufficient time, these variables will tend to converge to certain (equilibrium) values. This type of information is important because it fills a major gap that marred our study of statics and comparative statics. In the latter, we always make the arbitrary assumption that the process of economic adjustment inevitably leads to an equilibrium. In a dynamic analysis, the question of "attainability" is to be squarely faced, rather than assumed away.

One salient feature of dynamic analysis is the *dating* of the variables, which introduces the explicit consideration of *time* into the picture. This can be done in two ways: time can be considered either as a *continuous* variable or as a *discrete* variable. In the former case, something is happening to the variable at each *point* of time (such as in continuous interest compounding); whereas in the latter, the variable undergoes a change only once within a *period* of time (e.g., interest is added only at the end of every 6 months). One of these time concepts may be more appropriate than the other in certain contexts.

We shall discuss first the continuous-time case, to which the mathematical techniques of *integral calculus* and *differential equations* are pertinent. Later, in Chaps. 17 and 18, we shall turn to the discrete-time case, which utilizes the methods of *difference equations*.

14.1 Dynamics and Integration

In a static model, generally speaking, the problem is to find the values of the endogenous variables that satisfy some specified equilibrium condition(s). Applied to the context of optimization models, the task becomes one of finding the values of the choice variables that maximize (or minimize) a specific objective function with the first-order condition serving as the equilibrium condition. In a dynamic model, by contrast, the problem

[†] Fritz Machlup, "Statics and Dynamics: Kaleidoscopic Words," *Southern Economic Journal*, October 1959, pp. 91–110; reprinted in Machlup, *Essays on Economic Semantics*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963, pp. 9–42.

involves instead the delineation of the time path of some variable, on the basis of a known pattern of change (say, a given instantaneous rate of change).

An example should make this clear. Suppose that population size H is known to change over time at the rate

$$\frac{dH}{dt} = t^{-1/2} \tag{14.1}$$

We then try to find what time path(s) of population $H = H(t)$ can yield the rate of change in (14.1).

You will recognize that, if we know the function $H = H(t)$ to begin with, the derivative dH/dt can be found by differentiation. But in the problem now confronting us, the shoe is on the other foot: we are called upon to uncover the *primitive* function from a given *derived* function, rather than the reverse. Mathematically, we now need the exact opposite of the method of differentiation, or of differential calculus.

The relevant method, known as *integration*, or *integral calculus*, will be studied in this chapter. For the time being, let us be content with the observation that the function $H(t) = 2t^{1/2}$ does indeed have a derivative of the form in (14.1), thus apparently qualifying as a solution to our problem. The trouble is that there also exist similar functions, such as $H(t) = 2t^{1/2} + 15$ or $H(t) = 2t^{1/2} + 99$ or, more generally,

$$H(t) = 2t^{1/2} + c \qquad (c = \text{an arbitrary constant}) \tag{14.2}$$

which all possess exactly the same derivative (14.1). No unique time path can be determined, therefore, unless the value of the constant c can somehow be made definite. To accomplish this, additional information must be introduced into the model, usually in the form of what is known as an *initial condition* or *boundary condition*.

If we have knowledge of the initial population $H(0)$—that is, the value of H at $t = 0$, let us say, $H(0) = 100$—then the value of the constant c can be made determinate. Setting $t = 0$ in (14.2), we get

$$H(0) = 2(0)^{1/2} + c = c$$

But if $H(0) = 100$, then $c = 100$, and (14.2) becomes

$$H(t) = 2t^{1/2} + 100 \tag{14.2'}$$

where the constant is no longer arbitrary. More generally, for any given initial population $H(0)$, the time path will be

$$H(t) = 2t^{1/2} + H(0) \tag{14.2''}$$

Thus the population size H at any point of time will, in the present example, consist of the sum of the initial population $H(0)$ and another term involving the time variable t. Such a time path indeed charts the complete itinerary of the variable H over time, and thus it truly constitutes the solution to our dynamic model. [Equation (14.1) is also a function of t. Why can't *it* be considered a solution as well?]

Simple as it is, this population example illustrates the quintessence of the problems of economic dynamics. Given the pattern of behavior of a variable over time, we seek to find a function that describes the time path of the variable. In the process, we shall encounter one or more arbitrary constants, but if we possess sufficient additional information in the form of *initial conditions*, it will be possible to definitize these arbitrary constants.

In the simpler types of problem, such as the one just cited, the solution can be found by the method of integral calculus, which deals with the process of tracing a given derivative function back to its primitive function. In more complicated cases, we can also resort to the known techniques of the closely related branch of mathematics known as *differential equations*. Since a differential equation is defined as any equation containing differential or derivative expressions, (14.1) surely qualifies as one; consequently, by finding its solution, we have in fact already solved a differential equation, albeit an exceedingly simple one.

Let us now proceed to the study of the basic concepts of integral calculus. Since we discussed differential calculus with x (rather than t) as the independent variable, for the sake of symmetry we shall use x here, too. For convenience, however, we shall in the present discussion denote the primitive and derived functions by $F(x)$ and $f(x)$, respectively, rather than distinguish them by the use of a prime.

14.2 Indefinite Integrals

The Nature of Integrals

It has been mentioned that integration is the reverse of differentiation. If differentiation of a given primitive function $F(x)$ yields the derivative $f(x)$, we can "integrate" $f(x)$ to find $F(x)$, provided appropriate information is available to definitize the arbitrary constant that will arise in the process of integration. The function $F(x)$ is referred to as an *integral* (or *antiderivative*) of the function $f(x)$. These two types of process may thus be likened to two ways of studying a family tree: *integration* involves the tracing of the parentage of the function $f(x)$, whereas *differentiation* seeks out the progeny of the function $F(x)$. But note this difference—while the (differentiable) primitive function $F(x)$ invariably produces a lone offspring, namely, a unique derivative $f(x)$, the derived function $f(x)$ is traceable to an infinite number of possible parents through integration, because if $F(x)$ is an integral of $f(x)$, then so also must be $F(x)$ plus any constant, as we saw in (14.2).

We need a special notation to denote the required integration of $f(x)$ with respect to x. The standard one is

$$\int f(x)\,dx$$

The symbol on the left—an elongated S (with the connotation of sum, to be explained later)—is called the *integral sign*, whereas the $f(x)$ part is known as the *integrand* (the function to be integrated), and the dx part—similar to the dx in the differentiation operator d/dx—reminds us that the operation is to be performed with respect to the variable x. However, you may also take $f(x)\,dx$ as a single entity and interpret it as the differential of the primitive function $F(x)$ [that is, $dF(x) = f(x)\,dx$]. Then, the integral sign in front can be viewed as an instruction to reverse the differentiation process that gave rise to the differential. With this new notation, we can write that

$$\frac{d}{dx}F(x) = f(x) \quad \Rightarrow \quad \int f(x)\,dx = F(x) + c \qquad \textbf{(14.3)}$$

where the presence of c, an arbitrary *constant of integration*, serves to indicate the multiple parentage of the integrand.

The integral $\int f(x)\,dx$ is, more specifically, known as the *indefinite integral* of $f(x)$ (as against the *definite integral* to be discussed in Sec. 14.2), because it has no definite numerical value. Because it is equal to $F(x) + c$, its value will in general vary with the value of x (even if c is definitized). Thus, like a derivative, an indefinite integral is itself a function of the variable x.

Basic Rules of Integration

Just as there are rules of derivation, we can also develop certain rules of integration. As may be expected, the latter are heavily dependent on the rules of derivation with which we are already familiar. From the following derivative formula for a power function,

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \qquad (n \neq -1)$$

for instance, we see that the expression $x^{n-1}/(n+1)$ is the primitive function for the derivative function x^n; thus, by substituting these for $F(x)$ and $f(x)$ in (14.3), we may state the result as a rule of integration.

Rule I (the power rule)

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

Example 1 Find $\int x^3 dx$. Here, we have $n = 3$, and therefore

$$\int x^3 dx = \frac{1}{4}x^4 + c$$

Example 2 Find $\int x\,dx$. Since $n = 1$, we have

$$\int x\,dx = \frac{1}{2}x^2 + c$$

Example 3 What is $\int 1\,dx$? To find this integral, we recall that $x^0 = 1$; so we can let $n = 0$ in the power rule and get

$$\int 1\,dx = x + c$$

[$\int 1\,dx$ is sometimes written simply as $\int dx$, since $1\,dx = dx$.]

Example 4 Find $\int \sqrt{x^3}\,dx$. Since $\sqrt{x^3} = x^{3/2}$, we have $n = \frac{3}{2}$; therefore,

$$\int \sqrt{x^3}\,dx = \frac{x^{5/2}}{\frac{5}{2}} + c = \frac{2}{5}\sqrt{x^5} + c$$

Example 5 Find $\int \frac{1}{x^4}dx$, $(x \neq 0)$. Since $1/x^4 = x^{-4}$, we have $n = -4$. Thus the integral is

$$\int \frac{1}{x^4}\,dx = \frac{x^{-4+1}}{-4+1} + c = -\frac{1}{3x^3} + c$$

Note that the correctness of the results of integration can always be checked by differentiation; if the integration process is correct, the derivative of the integral must be equal to the integrand.

The derivative formulas for simple exponential and logarithmic functions have been shown to be

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}\ln x = \frac{1}{x} \qquad (x > 0)$$

From these, two other basic rules of integration emerge.

Rule II (the exponential rule)

$$\int e^x\,dx = e^x + c$$

Rule III (the logarithmic rule)

$$\int \frac{1}{x}\,dx = \ln x + c \qquad (x > 0)$$

It is of interest that the integrand involved in Rule III is $1/x = x^{-1}$, which is a special form of the power function x^n with $n = -1$. This particular integrand is inadmissible under the power rule, but now is duly taken care of by the logarithmic rule.

As stated, the logarithmic rule is placed under the restriction $x > 0$, because logarithms do not exist for nonpositive values of x. A more general formulation of the rule, which can take care of negative values of x, is

$$\int \frac{1}{x}\,dx = \ln|x| + c \qquad (x \neq 0)$$

which also implies that $(d/dx)\ln|x| = 1/x$, just as $(d/dx)\ln x = 1/x$. You should convince yourself that the replacement of x (with the restriction $x > 0$) by $|x|$ (with the restriction $x \neq 0$) does not vitiate the formula in any way.

Also, as a matter of notation, it should be pointed out that the integral $\int \frac{1}{x}\,dx$ is sometimes also written as $\int \frac{dx}{x}$.

As variants of Rules II and III, we also have the following two rules.

Rule IIa

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

Rule IIIa

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c \qquad [f(x) > 0]$$

or

$$\ln|f(x)| + c \qquad [f(x) \neq 0]$$

The bases for these two rules can be found in the derivative rules in (10.20).

Rules of Operation

The three preceding rules amply illustrate the spirit underlying all rules of integration. Each rule always corresponds to a certain derivative formula. Also, an arbitrary constant is always appended at the end (even though it is to be definitized later by using a given boundary condition) to indicate that a whole family of primitive functions can give rise to the given integrand.

To be able to deal with more complicated integrands, however, we shall also find the following two rules of operation with regard to integrals helpful.

Rule IV (the integral of a sum) The integral of the sum of a finite number of functions is the sum of the integrals of those functions. For the two-function case, this means that

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

This rule is a natural consequence of the fact that

$$\underbrace{\frac{d}{dx}[F(x) + G(x)]}_{A} = \underbrace{\frac{d}{dx}F(x) + \frac{d}{dx}G(x)}_{B} = \underbrace{f(x) + g(x)}_{C}$$

Inasmuch as $A = C$, on the basis of (14.3) we can write

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + c \qquad \textbf{(14.4)}$$

But, from the fact that $B = C$, it follows that

$$\int f(x)\,dx = F(x) + c_1 \qquad \text{and} \qquad \int g(x)\,dx = G(x) + c_2$$

Thus we can obtain (by addition)

$$\int f(x)\,dx + \int g(x)\,dx = F(x) + G(x) + c_1 + c_2 \qquad \textbf{(14.5)}$$

Since the constants c, c_1, and c_2 are arbitrary in value, we can let $c = c_1 + c_2$. Then the right sides of (14.4) and (14.5) become equal, and as a consequence, their left sides must be equal also. This proves Rule IV.

Example 6 Find $\int (x^3 + x + 1)\,dx$. By Rule IV, this integral can be expressed as a sum of three integrals: $\int x^3\,dx + \int x\,dx + \int 1\,dx$. Since the values of these three integrals have previously been found in Examples 1, 2, and 3, we can simply combine those results to get

$$\int (x^3 + x + 1)\,dx = \left(\frac{x^4}{4} + c_1\right) + \left(\frac{x^2}{2} + c_2\right) + (x + c_3) = \frac{x^4}{4} + \frac{x^2}{2} + x + c$$

In the final answer, we have lumped together the three subscripted constants into a single constant c.

As a general practice, all the additive arbitrary constants of integration that emerge during the process can always be combined into a single arbitrary constant in the final answer.

Example 7 Find $\int \left(2e^{2x} + \frac{14x}{7x^2 + 5}\right) dx$. By Rule IV, we can integrate the two additive terms in the integrand separately, and then sum the results. Since the $2e^{2x}$ term is in the format of $f'(x)e^{f(x)}$ in Rule IIa, with $f(x) = 2x$, the integral is $e^{2x} + c_1$. Similarly, the other term,

$14x/(7x^2+5)$, takes the form of $f'(x)/f(x)$, with $f(x) = 7x^2 + 5 > 0$. Thus, by Rule IIIa, the integral is $\ln(7x^2+5) + c_2$. Hence we can write

$$\int \left(2e^{2x} + \frac{14x}{7x^2+5}\right) dx = e^{2x} + \ln(7x^2+5) + c$$

where we have combined c_1 and c_2 into one arbitrary constant c.

Rule V (the integral of a multiple) The integral of k times an integrand (k being a constant) is k times the integral of that integrand. In symbols,

$$\int kf(x)\,dx = k\int f(x)\,dx$$

What this rule amounts to, operationally, is that a multiplicative constant can be "factored out" of the integral sign. (*Warning:* A *variable* term *cannot* be factored out in this fashion!) To prove this rule (for the case where k is an integer), we recall that k times $f(x)$ merely means adding $f(x)$ k times; therefore, by Rule IV,

$$\int kf(x)\,dx = \int \underbrace{[f(x) + f(x) + \cdots + f(x)]}_{k\text{ terms}}\,dx$$

$$= \underbrace{\int f(x)\,dx + \int f(x)\,dx + \cdots + \int f(x)\,dx}_{k\text{ terms}} = k\int f(x)\,dx$$

Example 8 Find $\int -f(x)\,dx$. Here $k = -1$, and thus

$$\int -f(x)\,dx = -\int f(x)\,dx$$

That is, the integral of the negative of a function is the negative of the integral of that function.

Example 9 Find $\int 2x^2\,dx$. Factoring out the 2 and applying Rule I, we have

$$\int 2x^2\,dx = 2\int x^2\,dx = 2\left(\frac{x^3}{3} + c_1\right) = \frac{2}{3}x^3 + c$$

Example 10 Find $\int 3x^2\,dx$. In this case, factoring out the multiplicative constant yields

$$\int 3x^2\,dx = 3\int x^2\,dx = 3\left(\frac{x^3}{3} + c_1\right) = x^3 + c$$

Note that, in contrast to the preceding example, the term x^3 in the final answer does not have any fractional expression attached to it. This neat result is due to the fact that 3 (the multiplicative constant of the integrand) happens to be precisely equal to 2 (the power of the function) plus 1. Referring to the power rule (Rule I), we see that the multiplicative constant $(n+1)$ will in such a case cancel out the fraction $1/(n+1)$, thereby yielding $(x^{n+1} + c)$ as the answer.

In general, whenever we have an expression $(n+1)x^n$ as the integrand, there is really no need to factor out the constant $(n+1)$ and then integrate x^n; instead, we may write $x^{n+1} + c$ as the answer right away.

Example 11 Find $\int \left(5e^x - x^{-2} + \frac{3}{x}\right) dx$, $(x \neq 0)$. This example illustrates both Rules IV and V; actually, it illustrates the first three rules as well:

$$\int \left(5e^x - \frac{1}{x^2} + \frac{3}{x}\right) dx = 5\int e^x\, dx - \int x^{-2}\, dx + 3\int \frac{1}{x}\, dx \qquad \text{[by Rules IV and V]}$$

$$= (5e^x + c_1) - \left(\frac{x^{-1}}{-1} + c_2\right) + (3\ln|x| + c_3)$$

$$= 5e^x + \frac{1}{x} + 3\ln|x| + c$$

The correctness of the result can again be verified by differentiation.

Rules Involving Substitution

Now we shall introduce two more rules of integration which seek to simplify the process of integration, when the circumstances are appropriate, by a substitution of the original variable of integration. Whenever the newly introduced variable of integration makes the integration process easier than under the old, these rules will become of service.

Rule VI (the substitution rule) The integral of $f(u)(du/dx)$ with respect to the variable x is the integral of $f(u)$ with respect to the variable u:

$$\int f(u)\frac{du}{dx}\, dx = \int f(u)\, du = F(u) + c$$

where the operation $\int du$ has been substituted for the operation $\int dx$.

This rule, the integral-calculus counterpart of the chain rule, may be proved by means of the chain rule itself. Given a function $F(u)$, where $u = u(x)$, the chain rule states that

$$\frac{d}{dx}F(u) = \frac{d}{du}F(u)\frac{du}{dx} = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}$$

Since $f(u)(du/dx)$ is the derivative of $F(u)$, it follows from (14.3) that the integral (antiderivative) of the former must be

$$\int f(u)\frac{du}{dx}\, dx = F(u) + c$$

You may note that this result, in fact, follows also from the *canceling* of the two dx expressions on the left.

Example 12 Find $\int 2x(x^2+1)\, dx$. The answer to this can be obtained by first multiplying out the integrand:

$$\int 2x(x^2+1)\, dx = \int (2x^3 + 2x)dx = \frac{x^4}{2} + x^2 + c$$

but let us now do it by the substitution rule. Let $u = x^2 + 1$; then $du/dx = 2x$, or $dx = du/2x$. Substitution of $du/2x$ for dx will yield

$$\int 2x(x^2+1)\, dx = \int 2xu\frac{du}{2x} = \int u\, du = \frac{u^2}{2} + c_1$$

$$= \frac{1}{2}(x^4 + 2x^2 + 1) + c_1 = \frac{1}{2}x^4 + x^2 + c$$

where $c = \frac{1}{2} + c_1$. The same answer can also be obtained by substituting du/dx for $2x$ (instead of $du/2x$ for dx).

Example 13 Find $\int 6x^2(x^3+2)^{99}\,dx$. The integrand of this example is not easily multiplied out, and thus the substitution rule now has a better opportunity to display its effectiveness. Let $u = x^3 + 2$; then $du/dx = 3x^2$, so that

$$\int 6x^2(x^3+2)^{99}\,dx = \int \left(2\frac{du}{dx}\right) u^{99}\,dx = \int 2u^{99}\,du$$
$$= \frac{2}{100}u^{100} + c = \frac{1}{50}(x^3+2)^{100} + c$$

Example 14 Find $\int 8e^{2x+3}\,dx$. Let $u = 2x + 3$; then $du/dx = 2$, or $dx = du/2$. Hence,

$$\int 8e^{2x+3}\,dx = \int 8e^u \frac{du}{2} = 4\int e^u\,du = 4e^u + c = 4e^{2x+3} + c$$

As these examples show, this rule is of help whenever we can—by the judicious choice of a function $u = u(x)$—express the integrand (a function of x) as the product of $f(u)$ (a function of u) and du/dx (the derivative of the u function which we have chosen). However, as illustrated by the last two examples, this rule can be used also when the original integrand is transformable into a constant multiple of $f(u)(du/dx)$. This would not affect the applicability because the constant multiplier can be factored out of the integral sign, which would then leave an integrand of the form $f(u)(du/dx)$, as required in the substitution rule. When the substitution of variables results in a *variable* multiple of $f(u)(du/dx)$, say, x times the latter, however, factoring is not permissible, and this rule will be of no help. In fact, there exists no general formula giving the integral of a product of two functions in terms of the separate integrals of those functions; nor do we have a general formula giving the integral of a quotient of two functions in terms of their separate integrals. Herein lies the reason why integration, on the whole, is more difficult than differentiation and why, with complicated integrands, it is more convenient to look up the answer in prepared tables of integration formulas rather than to undertake the integration by oneself.

Rule VII (integration by parts) The integral of v with respect to u is equal to uv less the integral of u with respect to v:

$$\int v\,du = uv - \int u\,dv$$

The essence of this rule is to replace the operation $\int du$ by the operation $\int dv$.

The rationale behind this result is relatively simple. First, the product rule of differentials gives us

$$d(uv) = v\,du + u\,dv$$

If we integrate both sides of the equation (i.e., integrate each differential), we get a new equation

$$\int d(uv) = \int v\,du + \int u\,dv$$

or $\quad uv = \int v\,du + \int u\,dv \quad$ [no constant is needed on the left (why?)]

Then, by subtracting $\int u\,dv$ from both sides, the previously stated result emerges.

Example 15 Find $\int x(x+1)^{1/2}\,dx$. Unlike Examples 12 and 13, the present example is not amenable to the type of substitution used in Rule VI. (Why?) However, we may consider the given integral to be in the form of $\int v\,du$, and apply Rule VII. To this end, we shall let $v = x$, implying $dv = dx$, and also let $u = \frac{2}{3}(x+1)^{3/2}$, so that $du = (x+1)^{1/2}\,dx$. Then we can find the integral to be

$$\begin{aligned}\int x(x+1)^{1/2}\,dx &= \int v\,du = uv - \int u\,dv \\ &= \frac{2}{3}(x+1)^{3/2}x - \int \frac{2}{3}(x+1)^{3/2}\,dx \\ &= \frac{2}{3}(x+1)^{3/2}x - \frac{4}{15}(x+1)^{5/2} + c\end{aligned}$$

Example 16 Find $\int \ln x\,dx$, $(x > 0)$. We cannot apply the logarithmic rule here, because that rule deals with the integrand $1/x$, not $\ln x$. Nor can we use Rule VI. But if we let $v = \ln x$, implying $dv = (1/x)\,dx$, and also let $u = x$, so that $du = dx$, then the integration can be performed as follows:

$$\begin{aligned}\int \ln x\,dx &= \int v\,du = uv - \int u\,dv \\ &= x\ln x - \int dx = x\ln x - x + c = x(\ln x - 1) + c\end{aligned}$$

Example 17 Find $\int xe^x\,dx$. In this case, we shall simply let $v = x$, and $u = e^x$, so that $dv = dx$ and $du = e^x\,dx$. Applying Rule VII, we then have

$$\begin{aligned}\int xe^x\,dx &= \int v\,du = uv - \int u\,dv \\ &= e^x x - \int e^x\,dx = e^x x - e^x + c = e^x(x-1) + c\end{aligned}$$

The validity of this result, like those of the preceding examples, can of course be readily checked by differentiation.

EXERCISE 14.2

1. Find the following:

   (a) $\int 16x^{-3}\,dx \quad (x \neq 0)$

   (b) $\int 9x^8\,dx$

   (c) $\int (x^5 - 3x)\,dx$

   (d) $\int 2e^{-2x}\,dx$

   (e) $\int \frac{4x}{x^2+1}\,dx$

   (f) $\int (2ax + b)(ax^2 + bx)^7\,dx$

2. Find:

   (a) $\int 13e^x\,dx$

   (b) $\int \left(3e^x + \frac{4}{x}\right)dx \quad (x > 0)$

   (c) $\int \left(5e^x + \frac{3}{x^2}\right)dx \quad (x \neq 0)$

   (d) $\int 3e^{-(2x+7)}\,dx$

   (e) $\int 4xe^{x^2+3}\,dx$

   (f) $\int xe^{x^2+9}\,dx$

3. Find:

   (a) $\int \frac{3dx}{x} \quad (x \neq 0)$

   (b) $\int \frac{dx}{x-2} \quad (x \neq 2)$

   (c) $\int \frac{2x}{x^2+3}\,dx$

   (d) $\int \frac{x}{3x^2+5}\,dx$

4. Find:

   (a) $\int (x+3)(x+1)^{1/2}\,dx$

   (b) $\int x \ln x\,dx \quad (x > 0)$

5. Given n constants k_i (with $i = 1, 2, \ldots, n$) and n functions $f_i(x)$, deduce from Rules IV and V that

$$\int \sum_{i=1}^{n} k_i f_i(x)\,dx = \sum_{i=1}^{n} k_i \int f_i(x)\,dx$$

14.3 Definite Integrals

Meaning of Definite Integrals

All the integrals cited in Sec. 14.2 are of the *indefinite* variety: each is a function of a variable and, hence, possesses no definite numerical value. Now, for a given indefinite integral of a continuous function $f(x)$,

$$\int f(x)\,dx = F(x) + c$$

if we choose two values of x in the domain, say, a and b $(a < b)$, substitute them successively into the right side of the equation, and form the difference

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

we get a specific numerical value, free of the variable x as well as the arbitrary constant c. This value is called the *definite integral* of $f(x)$ from a to b. We refer to a as the *lower limit of integration* and to b as the *upper limit of integration*.

In order to indicate the limits of integration, we now modify the integral sign to the form $\int_a^b$. The evaluation of the definite integral is then symbolized in the following steps:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a) \qquad \textbf{(14.6)}$$

where the symbol $]_a^b$ (also written $|_a^b$ or $[\cdots]_a^b$) is an instruction to substitute b and a, successively, for x in the result of integration to get $F(b)$ and $F(a)$, and then take their difference, as indicated on the right of (14.6). As the first step, however, we must find the indefinite integral, although we may omit the constant c, since the latter will drop out in the process of difference-taking anyway.

Example 1 Evaluate $\int_1^5 3x^2\,dx$. Since the indefinite integral is $x^3 + c$, this definite integral has the value

$$\int_1^5 3x^2\,dx = x^3\Big]_1^5 = (5)^3 - (1)^3 = 125 - 1 = 124$$