Chapter 7

Rules of Differentiation and Their Use in Comparative Statics

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem of finding the derivative of some function $y = f(x)$, provided only an infinitesimal change in x is being considered. Even though the derivative dy/dx is defined as the limit of the difference quotient $q = g(v)$ as $v \to 0$, it is by no means necessary to undertake the process of limit-taking each time the derivative of a function is sought, for there exist various rules of differentiation (derivation) that will enable us to obtain the desired derivatives directly. Instead of going into comparative-static models immediately, therefore, let us begin by learning some rules of differentiation.

7.1 Rules of Differentiation for a Function of One Variable

First, let us discuss three rules that apply, respectively, to the following types of function of a single independent variable: $y = k$ (constant function) and $y = x^n$ and $y = cx^n$ (power functions). All these have smooth, continuous graphs and are therefore differentiable everywhere.

Constant-Function Rule

The derivative of a constant function $y = k$, or $f(x) = k$, is identically zero, i.e., is zero for all values of x. Symbolically, this rule may be stated as: Given $y = f(x) = k$, the derivative is

$$\frac{dy}{dx} = \frac{dk}{dx} = 0 \qquad \text{or} \qquad f'(x) = 0$$

Alternatively, we may state the rule as: Given $y = f(x) = k$, the derivative is

$$\frac{d}{dx}y = \frac{d}{dx}f(x) = \frac{d}{dx}k = 0$$

where the derivative symbol has been separated into two parts, d/dx on the one hand, and y [or $f(x)$ or k] on the other. The first part, d/dx, is an *operator symbol*, which instructs us to perform a particular mathematical operation. Just as the operator symbol $\sqrt{\ }$ instructs us to take a square root, the symbol d/dx represents an instruction to take the derivative of, or to differentiate, (some function) with respect to the variable x. The function to be operated on (to be differentiated) is indicated in the second part; here it is $y = f(x) = k$.

The proof of the rule is as follows. Given $f(x) = k$, we have $f(N) = k$ for any value of N. Thus the value of $f'(N)$—the value of the derivative at $x = N$—as defined in (6.13) is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{k - k}{x - N} = \lim_{x \to N} 0 = 0$$

Moreover, since N represents any value of x at all, the result $f'(N) = 0$ can be immediately generalized to $f'(x) = 0$. This proves the rule.

It is important to distinguish clearly between the statement $f'(x) = 0$ and the similar-looking but different statement $f'(x_0) = 0$. By $f'(x) = 0$, we mean that the derivative function f' has a zero value for *all* values of x; in writing $f'(x_0) = 0$, on the other hand, we are merely associating the zero value of the derivative with a particular value of x, namely, $x = x_0$.

As discussed before, the derivative of a function has its geometric counterpart in the slope of the curve. The graph of a constant function, say, a fixed-cost function $C_F = f(Q) = \$1{,}200$, is a horizontal straight line with a zero slope throughout. Correspondingly, the derivative must also be zero for all values of Q:

$$\frac{d}{dQ}C_F = \frac{d}{dQ}1200 = 0$$

Power-Function Rule

The derivative of a power function $y = f(x) = x^n$ is nx^{n-1}. Symbolically, this is expressed as

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \text{or} \qquad f'(x) = nx^{n-1} \tag{7.1}$$

Example 1 The derivative of $y = x^3$ is $\frac{dy}{dx} = \frac{d}{dx}x^3 = 3x^2$.

Example 2 The derivative of $y = x^9$ is $\frac{d}{dx}x^9 = 9x^8$.

This rule is valid for any real-valued power of x; that is, the exponent can be any real number. But we shall prove it only for the case where n is some positive integer. In the simplest case, that of $n = 1$, the function is $f(x) = x$, and according to the rule, the derivative is

$$f'(x) = \frac{d}{dx}x = 1(x^0) = 1$$

The proof of this result follows easily from the definition of $f'(N)$ in (6.14'). Given $f(x) = x$, the derivative value at any value of x, say, $x = N$, is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x - N}{x - N} = \lim_{x \to N} 1 = 1$$

Since N represents any value of x, it is permissible to write $f'(x) = 1$. This proves the rule for the case of $n = 1$. As the graphical counterpart of this result, we see that the function $y = f(x) = x$ plots as a 45° line, and it has a slope of $+1$ throughout.

For the cases of larger integers, $n = 2, 3, \ldots$, let us first note the following identities:

$$\frac{x^2 - N^2}{x - N} = x + N \qquad \text{[2 terms on the right]}$$

$$\frac{x^3 - N^3}{x - N} = x^2 + Nx + N^2 \qquad \text{[3 terms on the right]}$$

$$\vdots$$

$$\frac{x^n - N^n}{x - N} = x^{n-1} + Nx^{n-2} + N^2x^{n-3} + \cdots + N^{n-1}$$
$$\text{[}n\text{ terms on the right]} \qquad \textbf{(7.2)}$$

On the basis of (7.2), we can express the derivative of a power function $f(x) = x^n$ at $x = N$ as follows:

$$\begin{aligned} f'(N) &= \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x^n - N^n}{x - N} \\ &= \lim_{x \to N} (x^{n-1} + Nx^{n-2} + \cdots + N^{n-1}) && \text{[by (7.2)]} \\ &= \lim_{x \to N} x^{n-1} + \lim_{x \to N} Nx^{n-2} + \cdots + \lim_{x \to N} N^{n-1} && \text{[sum limit theorem]} \\ &= N^{n-1} + N^{n-1} + \cdots + N^{n-1} && \text{[a total of } n \text{ terms]} \\ &= nN^{n-1} && \textbf{(7.3)} \end{aligned}$$

Again, N is any value of x; thus this last result can be generalized to

$$f'(x) = nx^{n-1}$$

which proves the rule for n, any positive integer.

As mentioned previously, this rule applies even when the exponent n in the power expression x^n is not a positive integer. The following examples serve to illustrate its application to the latter cases.

Example 3 Find the derivative of $y = x^0$. Applying (7.1), we find

$$\frac{d}{dx} x^0 = 0(x^{-1}) = 0$$

Example 4 Find the derivative of $y = 1/x^3$. This involves the reciprocal of a power, but by rewriting the function as $y = x^{-3}$, we can again apply (7.1) to get the derivative:

$$\frac{d}{dx} x^{-3} = -3x^{-4} \qquad \left[= \frac{-3}{x^4}\right]$$

Example 5 Find the derivative of $y = \sqrt{x}$. A square root is involved in this case, but since $\sqrt{x} = x^{1/2}$, the derivative can be found as follows:

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} \qquad \left[= \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x}\right]$$

Derivatives are themselves functions of the independent variable x. In Example 1, for instance, the derivative is $dy/dx = 3x^2$, or $f'(x) = 3x^2$, so that a different value of x will result in a different value of the derivative, such as

$$f'(1) = 3(1)^2 = 3 \qquad f'(2) = 3(2)^2 = 12$$

These specific values of the derivative can be expressed alternatively as

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 \qquad \left.\frac{dy}{dx}\right|_{x=2} = 12$$

but the notations $f'(1)$ and $f'(2)$ are obviously preferable because of their simplicity.

It is of the utmost importance to realize that, to find the derivative values $f'(1)$, $f'(2)$, etc., we must *first* differentiate the function $f(x)$, to get the derivative function $f'(x)$, and *then* let x assume specific values in $f'(x)$. To substitute specific values of x into the primitive function $f(x)$ prior to differentiation is definitely not permissible. As an illustration, if we let $x = 1$ in the function of Example 1 before differentiation, the function will degenerate into $y = x = 1$—a constant function—which will yield a zero derivative rather than the correct answer of $f'(x) = 3x^2$.

Power-Function Rule Generalized

When a multiplicative constant c appears in the power function, so that $f(x) = cx^n$, its derivative is

$$\frac{d}{dx}cx^n = cnx^{n-1} \qquad \text{or} \qquad f'(x) = cnx^{n-1}$$

This result shows that, in differentiating cx^n, we can simply retain the multiplicative constant c intact and then differentiate the term x^n according to (7.1).

Example 6 Given $y = 2x$, we have $dy/dx = 2x^0 = 2$.

Example 7 Given $f(x) = 4x^3$, the derivative is $f'(x) = 12x^2$.

Example 8 The derivative of $f(x) = 3x^{-2}$ is $f'(x) = -6x^{-3}$.

For a proof of this new rule, consider the fact that for any value of x, say, $x = N$, the value of the derivative of $f(x) = cx^n$ is

$$f'(N) = \lim_{x\to N}\frac{f(x) - f(N)}{x - N} = \lim_{x\to N}\frac{cx^n - cN^n}{x - N} = \lim_{x\to N} c\left(\frac{x^n - N^n}{x - N}\right)$$

$$= \lim_{x\to N} c \lim_{x\to N}\frac{x^n - N^n}{x - N} \qquad \text{[product limit theorem]}$$

$$= c\lim_{x\to N}\frac{x^n - N^n}{x - N} \qquad \text{[limit of a constant]}$$

$$= cnN^{n-1} \qquad \text{[from (7.3)]}$$

In the view that N is any value of x, this last result can be generalized immediately to $f'(x) = cnx^{n-1}$, which proves the rule.

EXERCISE 7.1

1. Find the derivative of each of the following functions:
   (a) $y = x^{12}$ (c) $y = 7x^5$ (e) $w = -4u^{1/2}$
   (b) $y = 63$ (d) $w = 3u^{-1}$ (f) $w = 4u^{1/4}$
2. Find the following:
   (a) $\frac{d}{dx}(-x^{-4})$ (c) $\frac{d}{dw}5w^4$ (e) $\frac{d}{du}au^b$
   (b) $\frac{d}{dx}9x^{1/3}$ (d) $\frac{d}{dx}cx^2$ (f) $\frac{d}{du} - au^{-b}$
3. Find $f'(1)$ and $f'(2)$ from the following functions:
   (a) $y = f(x) = 18x$ (c) $f(x) = -5x^{-2}$ (e) $f(w) = 6w^{1/3}$
   (b) $y = f(x) = cx^3$ (d) $f(x) = \frac{3}{4}x^{4/3}$ (f) $f(w) = -3w^{-1/6}$
4. Graph a function $f(x)$ that gives rise to the derivative function $f'(x) = 0$. Then graph a function $g(x)$ characterized by $g'(x_0) = 0$.

7.2 Rules of Differentiation Involving Two or More Functions of the Same Variable

The three rules presented in Sec. 7.1 are each concerned with a single given function $f(x)$. Now suppose that we have two *differentiable* functions of the same variable x, say, $f(x)$ and $g(x)$, and we want to differentiate the sum, difference, product, or quotient formed with these two functions. In such circumstances, are there appropriate rules that apply? More concretely, given two functions—say, $f(x) = 3x^2$ and $g(x) = 9x^{12}$—how do we get the derivative of, say, $3x^2 + 9x^{12}$, or the derivative of $(3x^2)(9x^{12})$?

Sum-Difference Rule

The derivative of a sum (difference) of two functions is the sum (difference) of the derivatives of the two functions:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

The proof of this again involves the application of the definition of a derivative and of the various limit theorems. We shall omit the proof and, instead, merely verify its validity and illustrate its application.

Example 1

From the function $y = 14x^3$, we can obtain the derivative $dy/dx = 42x^2$. But $14x^3 = 5x^3 + 9x^3$, so that y may be regarded as the sum of two functions $f(x) = 5x^3$ and $g(x) = 9x^3$. According to the sum rule, we then have

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 + 9x^3) = \frac{d}{dx}5x^3 + \frac{d}{dx}9x^3 = 15x^2 + 27x^2 = 42x^2$$

which is identical with our earlier result.

This rule, which we stated in terms of two functions, can easily be extended to more functions. Thus, it is also valid to write

$$\frac{d}{dx}[f(x) \pm g(x) \pm h(x)] = f'(x) \pm g'(x) \pm h'(x)$$

Example 2

The function cited in Example 1, $y = 14x^3$, can be written as $y = 2x^3 + 13x^3 - x^3$. The derivative of the latter, according to the sum-difference rule, is

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 13x^3 - x^3) = 6x^2 + 39x^2 - 3x^2 = 42x^2$$

which again checks with the previous answer.

This rule is of great practical importance. With it at our disposal, it is now possible to find the derivative of any polynomial function, since the latter is nothing but a sum of power functions.

Example 3

$$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$$

Example 4

$$\frac{d}{dx}(7x^4 + 2x^3 - 3x + 37) = 28x^3 + 6x^2 - 3 + 0 = 28x^3 + 6x^2 - 3$$

Note that in Examples 3 and 4 the constants c and 37 do not really produce any effect on the derivative, because the derivative of a constant term is zero. In contrast to the *multiplicative* constant, which is retained during differentiation, the *additive* constant drops out. This fact provides the mathematical explanation of the well-known economic principle that the fixed cost of a firm does not affect its marginal cost. Given a short-run total-cost function

$$C = Q^3 - 4Q^2 + 10Q + 75$$

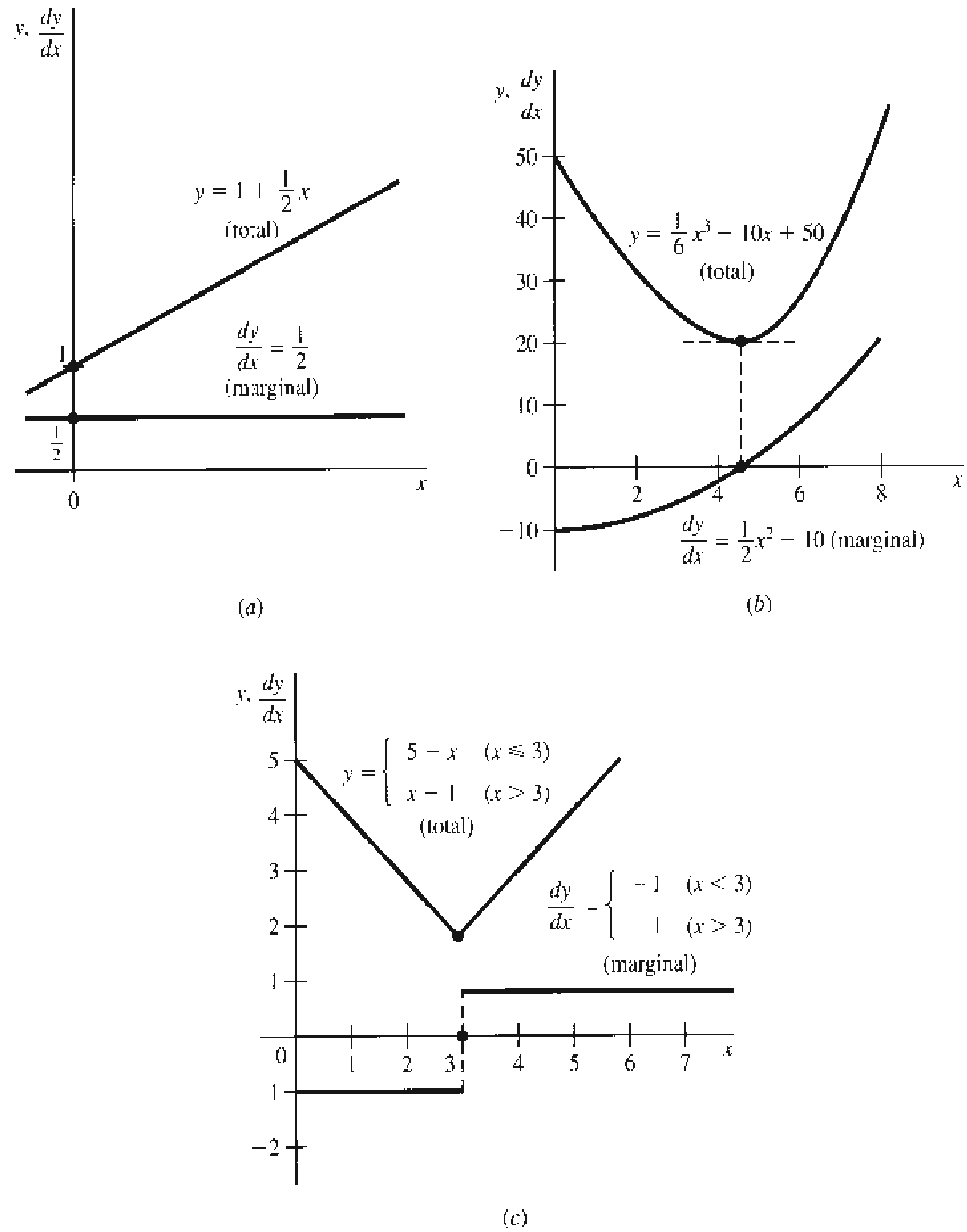
the marginal-cost function (for infinitesimal output change) is the limit of the quotient $\Delta C/\Delta Q$, or the derivative of the C function:

$$\frac{dC}{dQ} = 3Q^2 - 8Q + 10$$

whereas the fixed cost is represented by the additive constant 75. Since the latter drops out during the process of deriving dC/dQ, the magnitude of the fixed cost obviously cannot affect the marginal cost.

In general, if a primitive function $y = f(x)$ represents a *total* function, then the derivative function dy/dx is its *marginal* function. Both functions can, of course, be plotted against the variable x graphically; and because of the correspondence between the derivative of a function and the slope of its curve, for each value of x the marginal function should show the slope of the total function at that value of x. In Fig. 7.1a, a linear (constant-slope) total function is seen to have a constant marginal function. On the other hand, the nonlinear (varying-slope) total function in Fig. 7.1b gives rise to a curved marginal function, which lies below (above) the horizontal axis when the total function is negatively (positively) sloped. And, finally, the reader may note from Fig. 7.1c (cf. Fig. 6.5) that

FIGURE 7.1



"nonsmoothness" of a total function will result in a gap (discontinuity) in the marginal or derivative function. This is in sharp contrast to the everywhere-smooth total function in Fig. 7.1*b* which gives rise to a continuous marginal function. For this reason, the *smoothness* of a *primitive* function can be linked to the *continuity* of its *derivative* function. In particular, instead of saying that a certain function is smooth (and differentiable) everywhere, we may alternatively characterize it as a function with a continuous derivative function, and refer to it as a *continuously differentiable* function.

The following notations are often used to denote the continuity and the continuous differentiability of a function f:

$$f \in C^{(0)} \quad \text{or} \quad f \in C: \qquad f \text{ is continuous}$$

$$f \in C^{(1)} \quad \text{or} \quad f \in C': \qquad f \text{ is continuously differentiable}$$

where $C^{(0)}$, or simply C, is the symbol for the set of all continuous functions, and $C^{(1)}$, or C', is the symbol for the set of all continuously differentiable functions.

Product Rule

The derivative of the product of two (differentiable) functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$
$$= f(x)g'(x) + g(x)f'(x) \qquad \textbf{(7.4)}$$

It is also possible, of course, to rearrange the terms and express the rule as

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \qquad \textbf{(7.4')}$$

Example 5

Find the derivative of $y = (2x + 3)(3x^2)$. Let $f(x) = 2x + 3$ and $g(x) = 3x^2$. Then it follows that $f'(x) = 2$ and $g'(x) = 6x$, and according to (7.4) the desired derivative is

$$\frac{d}{dx}[(2x + 3)(3x^2)] = (2x + 3)(6x) + (3x^2)(2) = 18x^2 + 18x$$

This result can be checked by first multiplying out $f(x)g(x)$ and then taking the derivative of the product polynomial. The product polynomial is in this case $f(x)g(x) = (2x + 3)(3x^2) = 6x^3 + 9x^2$, and direct differentiation does yield the same derivative, $18x^2 + 18x$.

The important point to remember is that the derivative of a product of two functions is *not* the simple product of the two separate derivatives. Instead, it is a weighted sum of $f'(x)$ and $g'(x)$, the weights being $g(x)$ and $f(x)$, respectively. Since this differs from what intuitive generalization leads one to expect, let us produce a proof for (7.4). According to (6.13), the value of the derivative of $f(x)g(x)$ when $x = N$ should be

$$\frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = \lim_{x\to N} \frac{f(x)g(x) - f(N)g(N)}{x - N} \qquad \textbf{(7.5)}$$

But, by adding *and* subtracting $f(x)g(N)$ in the numerator (thereby leaving the original magnitude unchanged), we can transform the quotient on the right of (7.5) as follows:

$$\frac{f(x)g(x) - f(x)g(N) + f(x)g(N) - f(N)g(N)}{x - N}$$
$$= f(x)\frac{g(x) - g(N)}{x - N} + g(N)\frac{f(x) - f(N)}{x - N}$$

Substituting this for the quotient on the right of (7.5) and taking its limit, we then get

$$\frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = \lim_{x\to N} f(x) \lim_{x\to N} \frac{g(x) - g(N)}{x - N}$$
$$+ \lim_{x\to N} g(N) \lim_{x\to N} \frac{f(x) - f(N)}{x - N} \qquad \textbf{(7.5')}$$

The four limit expressions in (7.5′) are easily evaluated. The first one is $f(N)$, and the third is $g(N)$ (limit of a constant). The remaining two are, according to (6.13), respectively, $g'(N)$ and $f'(N)$. Thus (7.5′) reduces to

$$\frac{d}{dx}[f(x)g(x)]\Big|_{x=N} = f(N)g'(N) + g(N)f'(N) \qquad \textbf{(7.5}''\textbf{)}$$

And, since N represents any value of x, (7.5″) remains valid if we replace every N symbol by x. This proves the rule.

As an extension of the rule to the case of *three* functions, we have

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \qquad [\text{cf. } (7.4')] \qquad \textbf{(7.6)}$$

In words, the derivative of the product of three functions is equal to the product of the second and third functions times the derivative of the first, plus the product of the first and third functions times the derivative of the second, plus the product of the first and second functions times the derivative of the third. This result can be derived by the repeated application of (7.4). First treat the product $g(x)h(x)$ as a single function, say, $\phi(x)$, so that the original product of three functions will become a product of *two* functions, $f(x)\phi(x)$. To this, (7.4) is applicable. After the derivative of $f(x)\phi(x)$ is obtained, we may reapply (7.4) to the product $g(x)h(x) \equiv \phi(x)$ to get $\phi'(x)$. Then (7.6) will follow. The details are left to you as an exercise.

The validity of a rule is one thing; its serviceability is something else. Why do we need the product rule when we can resort to the alternative procedure of multiplying out the two functions $f(x)$ and $g(x)$ and then taking the derivative of the product directly? One answer to this question is that the alternative procedure is applicable only to *specific* (numerical or parametric) functions, whereas the product rule is applicable even when the functions are given in the *general* form. Let us illustrate with an economic example.

Finding Marginal-Revenue Function from Average-Revenue Function

If we are given an average-revenue (AR) function in specific form,

$$\text{AR} = 15 - Q$$

the marginal-revenue (MR) function can be found by first multiplying AR by Q to get the total-revenue (R) function:

$$R \equiv \text{AR} \cdot Q = (15 - Q)Q = 15Q - Q^2$$

and then differentiating R:

$$\text{MR} \equiv \frac{dR}{dQ} = 15 - 2Q$$

But if the AR function is given in the general form $\text{AR} = f(Q)$, then the total-revenue function will also be in a general form:

$$R \equiv \text{AR} \cdot Q = f(Q) \cdot Q$$

and therefore the "multiply out" approach will be to no avail. However, because R is a product of two functions of Q, namely, $f(Q)$ and Q itself, the product rule can be put to work. Thus we can differentiate R to get the MR function as follows:

$$\text{MR} \equiv \frac{dR}{dQ} = f(Q) \cdot 1 + Q \cdot f'(Q) = f(Q) + Qf'(Q) \qquad \textbf{(7.7)}$$

However, can such a general result tell us anything significant about the MR? Indeed it can. Recalling that $f(Q)$ denotes the AR function, let us rearrange (7.7) and write

$$\text{MR} - \text{AR} = \text{MR} - f(Q) = Qf'(Q) \qquad \textbf{(7.7')}$$

This gives us an important relationship between MR and AR: namely, they will always differ by the amount $Qf'(Q)$.

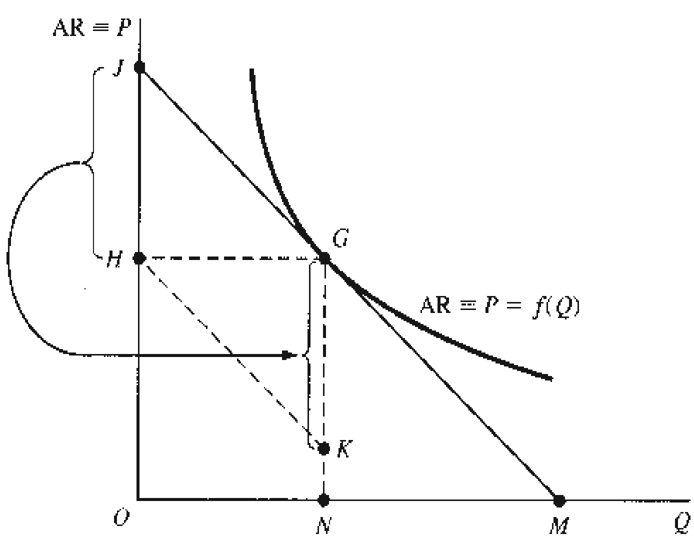
It remains to examine the expression $Qf'(Q)$. Its first component Q denotes output and is always nonnegative. The other component, $f'(Q)$, represents the slope of the AR curve plotted against Q. Since "average revenue" and "price" are but different names for the same thing:

$$\text{AR} \equiv \frac{R}{Q} \equiv \frac{PQ}{Q} \equiv P$$

the AR curve can also be regarded as a curve relating price P to output Q: $P = f(Q)$. Viewed in this light, the AR curve is simply the *inverse* of the demand curve for the product of the firm, i.e., the demand curve plotted after the P and Q axes are reversed. Under pure competition, the AR curve is a horizontal straight line, so that $f'(Q) = 0$ and, from (7.7′), $\text{MR} - \text{AR} = 0$ for all possible values of Q. Thus the MR curve and the AR curve must coincide. Under imperfect competition, on the other hand, the AR curve is normally downward-sloping, as in Fig. 7.2, so that $f'(Q) < 0$ and, from (7.7′), $\text{MR} - \text{AR} < 0$ for all positive levels of output. In this case, the MR curve must lie below the AR curve.

The conclusion just stated is *qualitative* in nature; it concerns only the relative positions of the two curves. But (7.7′) also furnishes the *quantitative* information that the MR curve will fall short of the AR curve at any output level Q by precisely the amount $Qf'(Q)$. Let us look at Fig. 7.2 again and consider the particular output level N. For that output, the

FIGURE 7.2



expression $Qf'(Q)$ specifically becomes $Nf'(N)$; if we can find the magnitude of $Nf'(N)$ in the diagram, we shall know how far below the average-revenue point G the corresponding marginal-revenue point must lie.

The magnitude of N is already specified. And $f'(N)$ is simply the slope of the AR curve at point G (where $Q = N$), that is, the slope of the tangent line JM measured by the ratio of two distances OJ/OM. However, we see that $OJ/OM = HJ/HG$; besides, distance HG is precisely the amount of output under consideration, N. Thus the distance $Nf'(N)$, by which the MR curve must lie below the AR curve at output N, is

$$Nf'(N) = HG\frac{HJ}{HG} = HJ$$

Accordingly, if we mark a vertical distance $KG = HJ$ directly below point G, then point K must be a point on the MR curve. (A simple way of accurately plotting KG is to draw a straight line passing through point H and parallel to JG; point K is where that line intersects the vertical line NG.)

The same procedure can be used to locate other points on the MR curve. All we must do, for any chosen point G' on the curve, is first to draw a tangent to the AR curve at G' that will meet the vertical axis at some point J'. Then draw a horizontal line from G' to the vertical axis, and label the intersection with the axis as H'. If we mark a vertical distance $K'G' = H'J'$ directly below point G', then the point K' will be a point on the MR curve. This is the graphical way of deriving an MR curve from a given AR curve. Strictly speaking, the accurate drawing of a tangent line requires a knowledge of the value of the derivative at the relevant output, that is, $f'(N)$; hence the graphical method just outlined cannot quite exist by itself. An important exception is the case of a linear AR curve, where the tangent to any point on the curve is simply the given line itself, so that there is in effect no need to draw any tangent at all. Then the graphical method will apply in a straightforward way.

Quotient Rule

The derivative of the quotient of two functions, $f(x)/g(x)$, is

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

In the numerator of the right-hand expression, we find two product terms, each involving the derivative of only one of the two original functions. Note that $f'(x)$ appears in the positive term, and $g'(x)$ in the negative term. The denominator consists of the square of the function $g(x)$; that is, $g^2(x) \equiv [g(x)]^2$.

Example 6

$$\frac{d}{dx}\left(\frac{2x-3}{x+1}\right) = \frac{2(x+1) - (2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$$

Example 7

$$\frac{d}{dx}\left(\frac{5x}{x^2+1}\right) = \frac{5(x^2+1) - 5x(2x)}{(x^2+1)^2} = \frac{5(1-x^2)}{(x^2+1)^2}$$

Example 8

$$\frac{d}{dx}\left(\frac{ax^2+b}{cx}\right) = \frac{2ax(cx) - (ax^2+b)(c)}{(cx)^2}$$

$$= \frac{c(ax^2-b)}{(cx)^2} = \frac{ax^2-b}{cx^2}$$

This rule can be proved as follows. For any value of $x = N$, we have

$$\frac{d}{dx}\frac{f(x)}{g(x)}\bigg|_{x=N} = \lim_{x\to N}\frac{f(x)/g(x) - f(N)/g(N)}{x-N} \tag{7.8}$$

The quotient expression following the limit sign can be rewritten in the form

$$\frac{f(x)g(N) - f(N)g(x)}{g(x)g(N)}\,\frac{1}{x-N}$$

By adding *and* subtracting $f(N)g(N)$ in the numerator and rearranging, we can further transform the expression to

$$\frac{1}{g(x)g(N)}\left[\frac{f(x)g(N) - f(N)g(N) + f(N)g(N) - f(N)g(x)}{x-N}\right]$$

$$= \frac{1}{g(x)g(N)}\left[g(N)\,\frac{f(x)-f(N)}{x-N} - f(N)\frac{g(x)-g(N)}{x-N}\right]$$

Substituting this result into (7.8) and taking the limit, we then have

$$\frac{d}{dx}\frac{f(x)}{g(x)}\bigg|_{x=N} = \lim_{x\to N}\frac{1}{g(x)g(N)}\left[\lim_{x\to N} g(N)\lim_{x\to N}\frac{f(x)-f(N)}{x-N}\right.$$
$$\left. - \lim_{x\to N} f(N)\lim_{x\to N}\frac{g(x)-g(N)}{x-N}\right]$$
$$= \frac{1}{g^2(N)}[g(N)f'(N) - f(N)g'(N)] \qquad \text{[by (6.13)]}$$

which can be generalized by replacing the symbol N with x, because N represents any value of x. This proves the quotient rule.

Relationship Between Marginal-Cost and Average-Cost Functions

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total-cost function $C = C(Q)$, the average-cost (AC) function is a quotient of two functions of Q, since $\text{AC} \equiv C(Q)/Q$, defined as long as $Q > 0$. Therefore, the rate of change of AC with respect to Q can be found by differentiating AC:

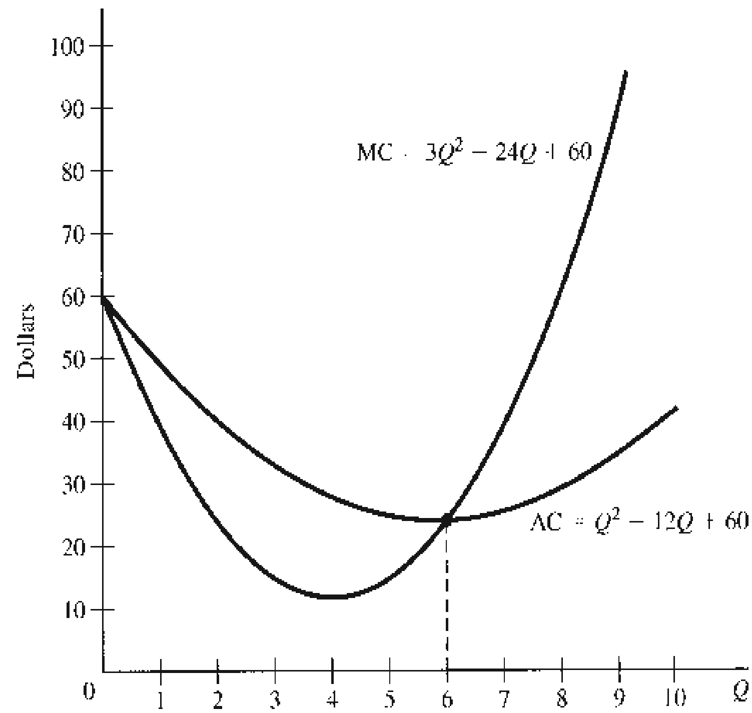
$$\frac{d}{dQ}\frac{C(Q)}{Q} = \frac{[C'(Q)\cdot Q - C(Q)\cdot 1]}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] \tag{7.9}$$

From this it follows that, for $Q > 0$,

$$\frac{d}{dQ}\frac{C(Q)}{Q} \gtreqless 0 \qquad \text{if} \qquad C'(Q) \gtreqless \frac{C(Q)}{Q} \tag{7.10}$$

Since the derivative $C'(Q)$ represents the marginal-cost (MC) function, and $C(Q)/Q$ represents the AC function, the economic meaning of (7.10) is: The slope of the AC

FIGURE 7.3



curve will be positive, zero, or negative if and only if the marginal-cost curve lies above, intersects, or lies below the AC curve. This is illustrated in Fig. 7.3, where the MC and AC functions plotted are based on the specific total-cost function

$$C = Q^3 - 12Q^2 + 60Q$$

To the left of $Q = 6$, AC is declining, and thus MC lies below it; to the right, the opposite is true. At $Q = 6$, AC has a slope of zero, and MC and AC have the same value.†

The qualitative conclusion in (7.10) is stated explicitly in terms of cost functions. However, its validity remains unaffected if we interpret $C(Q)$ as *any other* differentiable total function, with $C(Q)/Q$ and $C'(Q)$ as its corresponding average and marginal functions. Thus this result gives us a *general* marginal-average relationship. In particular, we may point out, the fact that MR lies below AR when AR is downward-sloping, as discussed in connection with Fig. 7.2, is nothing but a special case of the general result in (7.10).

† Note that (7.10) does *not* state that, when AC is negatively sloped, MC must also be negatively sloped; it merely says that AC must exceed MC in that circumstance. At $Q = 5$ in Fig. 7.3, for instance, AC is declining but MC is rising, so that their slopes will have opposite signs.

EXERCISE 7.2

1. Given the total-cost function $C = Q^3 - 5Q^2 + 12Q + 75$, write out a variable-cost (VC) function. Find the derivative of the VC function, and interpret the economic meaning of that derivative.
2. Given the average-cost function $AC = Q^2 - 4Q + 174$, find the MC function. Is the given function more appropriate as a long-run or a short-run function? Why?

3. Differentiate the following by using the product rule:
   (*a*) $(9x^2 - 2)(3x - 1)$ (*c*) $x^2(4x + 6)$ (*e*) $(2 - 3x)(1 + x)(x + 2)$
   (*b*) $(3x + 10)(6x^2 - 7x)$ (*d*) $(ax - b)(cx^2)$ (*f*) $(x^2 - 3)x^{-1}$
4. (*a*) Given $AR = 60 - 3Q$, plot the average-revenue curve, and then find the MR curve by the method used in Fig. 7.2.
   (*b*) Find the total-revenue function and the marginal-revenue function mathematically from the given AR function.
   (*c*) Does the graphically derived MR curve in (*a*) check with the mathematically derived MR function in (*b*)?
   (*d*) Comparing the AR and MR functions, what can you conclude about their relative slopes?
5. Provide a mathematical proof for the general result that, given a *linear* average curve, the corresponding marginal curve must have the same vertical intercept but will be twice as steep as the average curve.
6. Prove the result in (7.6) by first treating $g(x)h(x)$ as a single function, $g(x)h(x) \equiv \phi(x)$, and then applying the product rule (7.4).
7. Find the derivatives of:
   (*a*) $(x^2 + 3)/x$ (*c*) $6x/(x + 5)$
   (*b*) $(x + 9)/x$ (*d*) $(ax^2 + b)/(cx + d)$
8. Given the function $f(x) = ax + b$, find the derivatives of:
   (*a*) $f(x)$ (*b*) $xf(x)$ (*c*) $1/f(x)$ (*d*) $f(x)/x$
9. (*a*) Is it true that $f \in C' \Rightarrow f \in C$?
   (*b*) Is it true that $f \in C \Rightarrow f \in C'$?
10. Find the marginal and average functions for the following total functions and graph the results.
    Total-cost function:
    (*a*) $C = 3Q^2 + 7Q + 12$
    Total-revenue function:
    (*b*) $R = 10Q - Q^2$
    Total-product function:
    (*c*) $Q = aL + bL^2 - cL^3 \quad (a, b, c > 0)$

7.3 Rules of Differentiation Involving Functions of Different Variables

In Sec. 7.2, we discussed the rules of differentiation of a sum, difference, product, or quotient of two (or more) differentiable functions of the same variable. Now we shall consider cases where there are two or more differentiable functions, each of which has a *distinct* independent variable.

Chain Rule

If we have a differentiable function $z = f(y)$, where y is in turn a differentiable function of another variable x, say, $y = g(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y, times the derivative of y with respect to x. Expressed symbolically,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x) \tag{7.11}$$

This rule, known as the *chain rule*, appeals easily to intuition. Given a Δx, there must result a corresponding Δy via the function $y = g(x)$, but this Δy will in turn bring about a Δz via the function $z = f(y)$. Thus there is a "chain reaction" as follows:

$$\Delta x \xrightarrow{\text{via } g} \Delta y \xrightarrow{\text{via } f} \Delta z$$

The two links in this chain entail two difference quotients, $\Delta y/\Delta x$ and $\Delta z/\Delta y$, but when they are multiplied, the Δy will cancel itself out, and we end up with

$$\frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} = \frac{\Delta z}{\Delta x}$$

a difference quotient that relates Δz to Δx. If we take the limit of these difference quotients as $\Delta x \to 0$ (which implies $\Delta y \to 0$), each difference quotient will turn into a derivative; i.e., we shall have $(dz/dy)(dy/dx) = dz/dx$. This is precisely the result in (7.11).

In view of the function $y = g(x)$, we can express the function $z = f(y)$ as $z = f[g(x)]$, where the contiguous appearance of the two function symbols f and g indicates that this is a *composite function* (function of a function). It is for this reason that the chain rule is also referred to as the *composite-function rule* or *function-of-a-function rule*.

The extension of the chain rule to three or more functions is straightforward. If we have $z = f(y)$, $y = g(x)$, and $x = h(w)$, then

$$\frac{dz}{dw} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw} = f'(y)g'(x)h'(w)$$

and similarly for cases in which more functions are involved.

Example 1

If $z = 3y^2$, where $y = 2x + 5$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 6y(2) = 12y = 12(2x+5)$$

Example 2

If $z = y - 3$, where $y = x^3$, then

$$\frac{dz}{dx} = 1(3x^2) = 3x^2$$

Example 3

The usefulness of this rule can best be appreciated when we must differentiate a function such as $z = (x^2 + 3x - 2)^{17}$. Without the chain rule at our disposal, dz/dx can be found only via the laborious route of first multiplying out the 17th-power expression. With the chain rule, however, we can take a shortcut by defining a new, *intermediate* variable $y = x^2 + 3x - 2$, so that we get in effect two functions linked in a chain:

$$z = y^{17} \qquad \text{and} \qquad y = x^2 + 3x - 2$$

The derivative dz/dx can then be found as follows:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 17y^{16}(2x+3) = 17(x^2+3x-2)^{16}(2x+3)$$

Example 4 Given a total-revenue function of a firm $R = f(Q)$, where output Q is a function of labor input L, or $Q = g(L)$, find dR/dL. By the chain rule, we have

$$\frac{dR}{dL} = \frac{dR}{dQ}\frac{dQ}{dL} = f'(Q)g'(L)$$

Translated into economic terms, dR/dQ is the MR function and dQ/dL is the marginal-physical-product-of-labor (MPP_L) function. Similarly, dR/dL has the connotation of the marginal-revenue-product-of-labor (MRP_L) function. Thus the result shown constitutes the mathematical statement of the well-known result in economics that $MRP_L = MR \cdot MPP_L$.

Inverse-Function Rule

If the function $y = f(x)$ represents a one-to-one mapping, i.e., if the function is such that each value of y is associated with a unique value of x, the function f will have an *inverse function* $x = f^{-1}(y)$ (read: "x is an inverse function of y"). Here, the symbol f^{-1} is a function symbol which, like the derivative-function symbol f', signifies a function related to the function f; it does *not* mean the reciprocal of the function $f(x)$.

What the existence of an inverse function essentially means is that, in this case, not only will a given value of x yield a unique value of y [that is, $y = f(x)$], but also a given value of y will yield a unique value of x. To take a nonnumerical instance, we may exemplify the one-to-one mapping by the mapping from the set of all husbands to the set of all wives in a monogamous society. Each husband has a unique wife, and each wife has a unique husband. In contrast, the mapping from the set of all fathers to the set of all sons is not one-to-one, because a father may have more than one son, albeit each son has a unique father.

When x and y refer specifically to numbers, the property of one-to-one mapping is seen to be unique to the class of functions known as *strictly monotonic* (or *monotone*) *functions*. Given a function $f(x)$, if successively larger values of the independent variable x *always* lead to successively larger values of $f(x)$, that is, if

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

then the function f is said to be a *strictly increasing* function. If successive increases in x *always* lead to successive decreases in $f(x)$, that is, if

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

on the other hand, the function is said to be a *strictly decreasing* function. In either of these cases, an inverse function f^{-1} exists.†

A practical way of ascertaining the strict monotonicity of a given function $y = f(x)$ is to check whether the derivative $f'(x)$ always adheres to the same algebraic sign (not zero) for all values of x. Geometrically, this means that its slope is either always upward or always

† By omitting the adverb *strictly*, we can define *monotonic* (or *monotone*) functions as follows: An *increasing function* is a function with the property that

$$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \qquad \text{[with the weak inequality } \geq]$$

and a *decreasing function* is one with the property that

$$x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \qquad \text{[with the weak inequality } \leq]$$

Note that, under this definition, an ascending (descending) step function qualifies as an increasing (decreasing) function, despite the fact that its graph contains horizontal segments. Since such functions do not have a one-to-one mapping, they do not have inverse functions.

downward. Thus a firm's demand curve $Q = f(P)$ that has a negative slope throughout is strictly decreasing. As such, it has an inverse function $P = f^{-1}(Q)$, which, as mentioned previously, gives the average-revenue curve of the firm, since $P \equiv \text{AR}$.

Example 5 The function

$$y = 5x + 25$$

has the derivative $dy/dx = 5$, which is positive regardless of the value of x; thus the function is strictly increasing. It follows that an inverse function exists. In the present case, the inverse function is easily found by solving the given equation $y = 5x + 25$ for x. The result is the function

$$x = \frac{1}{5}y - 5$$

It is interesting to note that this inverse function is also strictly increasing, because $dx/dy = \frac{1}{5} > 0$ for all values of y.

Generally speaking, if an inverse function exists, the original and the inverse functions must both be strictly monotonic. Moreover, if f^{-1} is the inverse function of f, then f must be the inverse function of f^{-1}; that is, f and f^{-1} must be inverse functions of each other.

It is easy to verify that the graph of $y = f(x)$ and that of $x = f^{-1}(y)$ are one and the same, only with the axes reversed. If one lays the x axis of the f^{-1} graph over the x axis of the f graph (and similarly for the y axis), the two curves will coincide. On the other hand, if the x axis of the f^{-1} graph is laid over the y axis of the f graph (and vice versa), the two curves will become *mirror images* of each other with reference to the 45° line drawn through the origin. This mirror-image relationship provides us with an easy way of graphing the inverse function f^{-1}, once the graph of the original function f is given. (You should try this with the two functions in Example 5.)

For inverse functions, the rule of differentiation is

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

This means that the derivative of the inverse function is the reciprocal of the derivative of the original function; as such, dx/dy must take the same sign as dy/dx, so that if f is strictly increasing (decreasing), then so must be f^{-1}.

As a verification of this rule, we can refer back to Example 5, where dy/dx was found to be 5, and dx/dy equal to $\frac{1}{5}$. These two derivatives are indeed reciprocal to each other and have the same sign.

In that simple example, the inverse function is relatively easy to obtain, so that its derivative dx/dy can be found directly from the inverse function. As Example 6 shows, however, the inverse function is sometimes difficult to express explicitly, and thus direct differentiation may not be practicable. The usefulness of the inverse-function rule then becomes more fully apparent.

Example 6 Given $y = x^5 + x$, find dx/dy. First of all, since

$$\frac{dy}{dx} = 5x^4 + 1 > 0$$

for any value of x, the given function is strictly increasing, and an inverse function exists. To solve the given equation for x may not be such an easy task, but the derivative of the inverse function can nevertheless be found quickly by use of the inverse-function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4 + 1}$$

The inverse-function rule is, strictly speaking, applicable only when the function involved is a one-to-one mapping. In fact, however, we do have some leeway. For instance, when dealing with a U-shaped curve (not strictly monotonic), we may consider the downward- and the upward-sloping segments of the curve as representing two *separate* functions, each with a restricted domain, and each being strictly monotonic in the restricted domain. To each of these, the inverse-function rule can then again be applied.

EXERCISE 7.3

1. Given $y = u^3 + 2u$, where $u = 5 - x^2$, find dy/dx by the chain rule.
2. Given $w = ay^2$ and $y = bx^2 + cx$, find dw/dx by the chain rule.
3. Use the chain rule to find dy/dx for the following:
   (*a*) $y = (3x^2 - 13)^3$ (*b*) $y = (7x^3 - 5)^9$ (*c*) $y = (ax + b)^5$
4. Given $y = (16x + 3)^{-2}$, use the chain rule to find dy/dx. Then rewrite the function as $y = 1/(16x + 3)^2$ and find dy/dx by the quotient rule. Are the answers identical?
5. Given $y = 7x + 21$, find its inverse function. Then find dy/dx and dx/dy, and verify the inverse-function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.
6. Are the following functions strictly monotonic?
   (*a*) $y = -x^6 + 5 \quad (x > 0)$
   (*b*) $y = 4x^5 + x^3 + 3x$
   For each strictly monotonic function, find dx/dy by the inverse-function rule.

7.4 Partial Differentiation

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter. Therefore, as a final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n) \tag{7.12}$$

where the variables x_i $(i = 1, 2, \ldots, n)$ are all *independent* of one another, so that each can vary by itself without affecting the others. If the variable x_1 undergoes a change Δx_1 while