4. Two Dimensional Transformations

In many applications, changes in orientations, size, and shape are accomplished with geometric transformations that alter the coordinate descriptions of objects.

Basic geometric transformations are:
- Translation
- Rotation
- Scaling

Other transformations:
- Reflection
- Shear

4.1 Basic Transformations

Translation

We translate a 2D point by adding translation distances, tx and ty, to the original coordinate position (x,y):

$$x' = x + t_x,\ y' = y + t_y$$

Alternatively, translation can also be specified by the following transformation matrix:

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Then we can rewrite the formula as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For example, to translate a triangle with vertices at original coordinates (10,20), (10,10), (20,10) by t_x=5, t_y=10, we compute as followings:

Translation of vertex (10,20):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 1*10+0*20+5*1 \\ 0*10+1*20+10*1 \\ 0*10+0*20+1*1 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \\ 1 \end{bmatrix}$$

Translation of vertex (10,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 1*10+0*10+5*1 \\ 0*10+1*10+10*1 \\ 0*10+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 1 \end{bmatrix}$$

Translation of vertex (20,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 1*20+0*10+5*1 \\ 0*20+1*10+10*1 \\ 0*20+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \\ 1 \end{bmatrix}$$

The resultant coordinates of the triangle vertices are (15,30), (15,20), and (25,20) respectively.

Exercise: translate a triangle with vertices at original coordinates (10,25), (5,10), (20,10) by t_x=15, t_y=5. Roughly plot the original and resultant triangles.

Rotation About the Origin

To rotate an object about the origin (0,0), we specify the rotation angle ?. Positive and negative values for the rotation angle define counterclockwise and clockwise rotations respectively. The followings is the computation of this rotation for a point:

x' = x cos ? - y sin ?
y' = x sin ? + y cos ?

Alternatively, this rotation can also be specified by the following transformation matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we can rewrite the formula as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For example, to rotate a triange about the origin with vertices at original coordinates (10,20), (10,10), (20,10) by 30 degrees, we compute as followings:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation of vertex (10,20):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866*10+(-0.5)*20+0*1 \\ 0.5*10+0.866*20+0*1 \\ 0*10+0*20+1*1 \end{bmatrix} = \begin{bmatrix} -1.34 \\ 22.32 \\ 1 \end{bmatrix}$$

Rotation of vertex (10,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866*10+(-0.5)*10+0*1 \\ 0.5*10+0.866*10+0*1 \\ 0*10+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 3.66 \\ 13.66 \\ 1 \end{bmatrix}$$

Rotation of vertex (20,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866*20+(-0.5)*10+0*1 \\ 0.5*20+0.866*10+0*1 \\ 0*20+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 12.32 \\ 18.66 \\ 1 \end{bmatrix}$$

The resultant coordinates of the triangle vertices are (-1.34,22.32), (3.6,13.66), and (12.32,18.66) respectively.

Exercise: Rotate a triange with vertices at original coordinates (10,20), (5,10), (20,10) by 45 degrees. Roughly plot the original and resultant triangles.

Scaling With Respect to the Origin

We scale a 2D object with respect to the origin by setting the scaling factors s_x and s_y, which are multiplied to the original vertex coordinate positions (x,y):

$$x' = x * s_x,\ y' = y * s_y$$

Alternatively, this scaling can also be specified by the following transformation matrix:

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we can rewrite the formula as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For example, to scale a triange with respect to the origin, with vertices at original coordinates (10,20), (10,10), (20,10) by s_x=2, s_y=1.5, we compute as followings:

Scaling of vertex (10,20):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 2*10+0*20+0*1 \\ 0*10+1.5*20+0*1 \\ 0*10+0*20+1*1 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 1 \end{bmatrix}$$

Scaling of vertex (10,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 2*10+0*10+0*1 \\ 0*10+1.5*10+0*1 \\ 0*10+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 1 \end{bmatrix}$$

Scaling of vertex (20,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 2*20+0*10+0*1 \\ 0*20+1.5*10+0*1 \\ 0*20+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 40 \\ 15 \\ 1 \end{bmatrix}$$

The resultant coordinates of the triangle vertices are (20,30), (20,15), and (40,15) respectively.

Exercise: Scale a triange with vertices at original coordinates (10,25), (5,10), (20,10) by s_x=1.5, s_y=2, with respect to the origin. Roughly plot the original and resultant triangles.

4.2 Concatenation Properties of Composite Matrix

I. Matrix multiplication is associative:

$$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$$

Therefore, we can evaluate matrix products using these associative grouping.
For example, we have a triangle, we want to rotate it with the matrix B, then we translate it with matrix A.
Then, for a vertex of that triangle represented as C, we compute its transformation as:

$$C' = A \cdot (B \cdot C)$$

But we can also change the computation method as:

$$C' = (A \cdot B) \cdot C$$

The advantage of computing it using $C' = (A \cdot B) \cdot C$ instead of $C' = A \cdot (B \cdot C)$ is that, for computing the 3 vertices of the triangle, C_1, C_2, C_3, the computation time is shortened:

Using $C' = A \cdot (B \cdot C)$:
1. compute $B \cdot C_1$ and put the result into I_1
2. compute $A \cdot I_1$ and put the result into C_1'
3. compute $B \cdot C_2$ and put the result into I_2
4. compute $A \cdot I_2$ and put the result into C_2'
5. compute $B \cdot C_3$ and put the result into I_3
6. compute $A \cdot I_3$ and put the result into C_3'

Using $C' = (A \cdot B) \cdot C$:
1. compute $A \cdot B$ and put the result into M
2. compute $M \cdot C_1$ and put the result into C_1'
3. compute $M \cdot C_2$ and put the result into C_2'
4. compute $M \cdot C_3$ and put the result into C_3'

Example: Rotate a triangle with vertices (10,20), (10,10), (20,10) about the origin by 30 degrees and then translate it by t_x=5, t_y=10,

We compute the rotation matrix:

$$B = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And we compute the translation matrix:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, we compute M=A·B

$$M = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1*0.866+0*0.5+5*0 & 1*-0.5+0*0.866+5*0 & 1*0+0*0+5*1 \\ 0*0.866+1*0.5+10*0 & 0*-0.5+1*0.866+10*0 & 0*0+1*0+10*1 \\ 0*0.866+0*0.5+1*0 & 0*-0.5+0*0.866+1*0 & 0*0+0*0+1*1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.866 & -0.5 & 5 \\ 0.5 & 0.866 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, we compute the transformations of the 3 vertices:
Transformation of vertex (10,20):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 5 \\ 0.5 & 0.866 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866*10+(-0.5)*20+5*1 \\ 0.5*10+0.866*20+10*1 \\ 0*10+0*20+1*1 \end{bmatrix} = \begin{bmatrix} 3.66 \\ 32.32 \\ 1 \end{bmatrix}$$

Transformation of vertex (10,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 5 \\ 0.5 & 0.866 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866*10+(-0.5)*10+5*1 \\ 0.5*10+0.866*10+10*1 \\ 0*10+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 8.66 \\ 23.66 \\ 1 \end{bmatrix}$$

Transformation of vertex (20,10):

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 5 \\ 0.5 & 0.866 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866*20+(-0.5)*10+5*1 \\ 0.5*20+0.866*10+10*1 \\ 0*20+0*10+1*1 \end{bmatrix} = \begin{bmatrix} 17.32 \\ 28.66 \\ 1 \end{bmatrix}$$

The resultant coordinates of the triangle vertices are (3.66,32.32), (8.66,23.66), and (17.32,28.66) respectively.

II. Matrix multiplication may not be commutative:

A·B may not equal to B·A

This means that if we want to translate and rotate an object, we must be careful about the order in which the composite matrix is evaluated. Using the previous example, if you compute C' = (A·B)·C, you are rotating the triangle with B first, then translate it with A, but if you compute C' = (B·A)·C, you are translating it with A first, then rotate it with B. The result is different.

Exercise: Translate a triangle with vertices (10,20), (10,10), (20,10) by t_x=5, t_y=10 and then rotate it about the origin by 30 degrees. Compare the result with the one obtained previously: (3.66,32.32), (8.66,23.66), and (17.32,28.66) by plotting the original triangle together with these 2 results.

4.3 Composite Transformation Matrix

Translations

By common sense, if we translate a shape with 2 successive translation vectors: (t_{x1}, t_{y1}) and (t_{x2}, t_{y2}), it is equal to a single translation of (t_{x1}+ t_{x2}, t_{y1}+ t_{y2}).
This additive property can be demonstrated by composite transformation matrix:

$$\begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1*1+0*0+t_{x1}*0 & 1*0+0*1+t_{x1}*0 & 1*t_{x2}+0*t_{y2}+t_{x1}*1 \\ 0*1+1*0+t_{y1}*0 & 0*0+1*1+t_{y1}*0 & 0*t_{x2}+1*t_{y2}+t_{y1}*1 \\ 0*1+0*0+1*0 & 0*0+0*1+1*0 & 0*t_{x2}+0*t_{u2}+1*1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & t_{x1}+t_{x2} \\ 0 & 1 & t_{y1}+t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

This demonstrates that 2 successive translations are additive.

Rotations

By common sense, if we rotate a shape with 2 successive rotation angles: ? and a, about the origin, it is equal to rotating the shape once by an angle ? + a about the origin.
Similarly, this additive property can be demonstrated by composite transformation matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\alpha + (-\sin\theta)*\sin\alpha + 0*0 & \cos\theta*(-\sin\alpha) + (-\sin\theta)*\cos\alpha + 0*0 & \cos\theta*0 + (-\sin\theta)*0 + 0*1 \\ \sin\theta\cos\alpha + \cos\theta*\sin\alpha + 0*0 & \sin\theta*(-\sin\alpha) + \cos\theta*\cos\alpha + 0*0 & \sin\theta*0 + \cos\theta*0 + 0*1 \\ 0*\cos\alpha + 0*\sin\alpha + 1*0 & 0*(-\sin\alpha) + 0*\cos\alpha + 1*0 & 0*0 + 0*0 + 1*1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & -(\cos\theta\sin\alpha + \sin\theta\cos\alpha) & 0 \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha & -\sin\theta\sin\alpha + \cos\theta\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) & 0 \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This demonstrates that 2 successive rotations are additive.

Scalings With Respect to the Origin

By common sense, if we scale a shape with 2 successive scaling factor: (s_{x1}, s_{y1}) and (s_{x2}, s_{y2}), with respect to the origin, it is equal to a single scaling of (s_{x1}* s_{x2}, s_{y1}* s_{y2}) with respect to the origin.
This multiplicative property can be demonstrated by composite transformation matrix:

$$\begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_{x1}*s_{x2} + 0*0 + 0*0 & s_{x1}*0 + 0*s_{y2} + 0*0 & s_{x1}*0 + 0*0 + 0*1 \\ 0*s_{x2} + s_{y1}*0 + 0*0 & 0*0 + s_{y1}*s_{y2} + 0*0 & 0*0 + s_{y1}*0 + 0*1 \\ 0*s_{x2} + 0*0 + 1*0 & 0*0 + 0*s_{y2} + 1*0 & 0*0 + 0*0 + 1*1 \end{bmatrix}$$

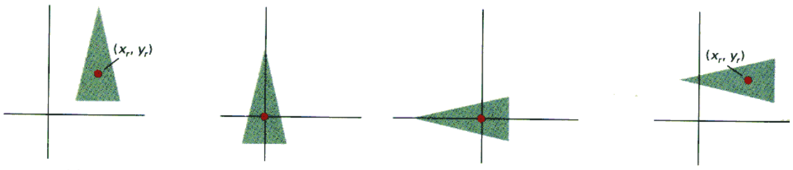
$$= \begin{bmatrix} s_{x1}*s_{x2} & 0 & 0 \\ 0 & s_{y1}*s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This demonstrates that 2 successive scalings with respect to the origin are multiplicative.

General Pivot-Point Rotation

Rotation about an arbitrary pivot point is not as simple as rotation about the origin. The procedure of rotation about an arbitrary pivot point is:

1. Translate the object so that the pivot-point position is moved to the origin.
2. Rotate the object about the origin.
3. Translate the object so that the pivot point is returned to its original position.



(a) Original Position of Object and Pivot Point
(b) Translation of Object so that Pivot Point (x_r, y_r) Is at Origin
(c) Rotation about Origin
(d) Translation of Object so that the Pivot Point Is Returned to Position (x_r, y_r)

The corresponding composite transformation matrix is:

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

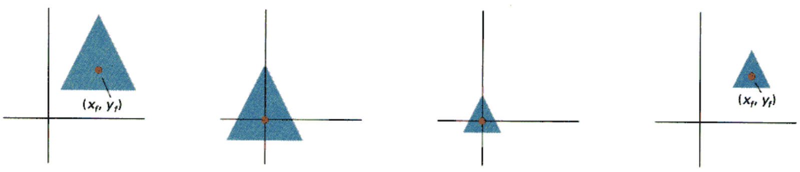
$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r \\ \sin\theta & \cos\theta & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & -x_r\cos\theta + y_r\sin\theta + x_r \\ \sin\theta & \cos\theta & -x_r\sin\theta - y_r\cos\theta + y_r \\ 0 & 0 & 1 \end{bmatrix}$$

General Fixed-Point Scaling

Scaling with respect to an arbitrary fixed point is not as simple as scaling with respect to the origin. The procedure of scaling with respect to an arbitrary fixed point is:

1. Translate the object so that the fixed point coincides with the origin.
2. Scale the object with respect to the origin.
3. Use the inverse translation of step 1 to return the object to its original position.



(a) Original Position of Object and Fixed Point
(b) Translate Object so that Fixed Point (x_f, y_f) Is at Origin
(c) Scale Object with Respect to Origin
(d) Translate Object so that the Fixed Point Is Returned to Position (x_f, y_f)

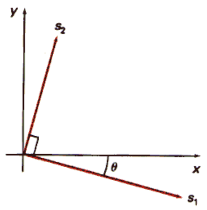
The corresponding composite transformation matrix is:

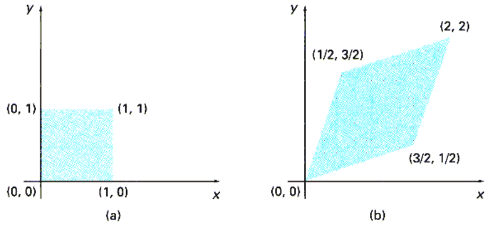
$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

<u>General Scaling Direction</u>

Scaling along an arbitrary direction is not as simple as scaling along the x-y axis. The procedure of scaling along and normal to an arbitrary direction (s_1 and s_2), with respect to the origin, is:

1. Rotate the object so that the directions for s_1 and s_2 coincide with the x and y axes respectively.
2. Scale the object with respect to the origin using (s_1, s_2).
3. Use an opposite rotation to return points to their original orientation.





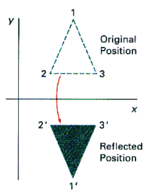
A square (a) is converted to a parallelogram (b) using the composite transformation matrix 5-35, with $s_1 = 1$, $s_2 = 2$, and $\theta = 45°$.

The corresponding composite transformation matrix is:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.4 Other Transformations

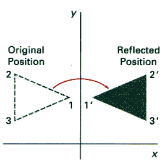
<u>Reflection</u>



Reflection about the x axis:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

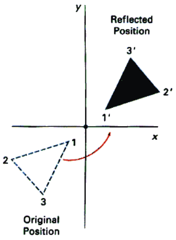
ie. x'=x; y'=-y



Reflection about the y axis:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

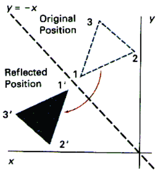
ie. x'=-x; y'=y



Flipping both x and y coordinates of a point relative to the origin:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

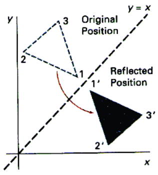
ie. x'=-x; y'=-y



Reflection about the diagonal line y=x:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

ie. x'=y; y'=x

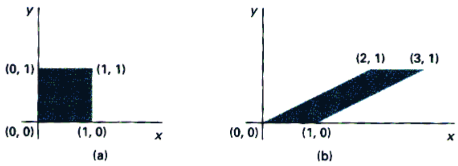


Reflection about the diagonal line y=-x:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

ie. x'=-y; y'=-x

<u>Shear</u>



X-direction shear, with a shearing parameter sh_x, relative to the x-axis:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

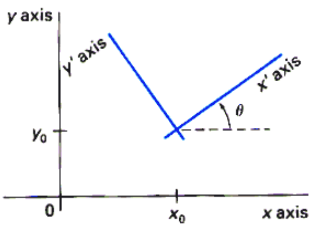
ie. $x'=x+y*sh_x$; y'=-x

Exercise: Think of a y-direction shear, with a shearing parameter sh_y, relative to the y-axis.

4.5 Transformation Between 2 Cartesian Systems

For modelling and design applications, individual objects may be defined in their own local Cartesian References. The local coordinates must then be transformed to position the objects within the overall scene coordinate system.

Suppose we want to transform object descriptions from the xy system to the x'y' system:



The composite transformation is:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & x_r \\ \sin(-\theta) & \cos(-\theta) & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$