4-Momentum and Variation of Mass with Velocity

Let us consider a particle in motion. Let V_μ be the 4-velocity of the particle. Then the product of the velocity 4-vector with a 4-scalar m_0 will again be a 4-vector which we denote by p_μ:

$$p_\mu = m_0 V_\mu .$$

This 4-vector is said to represent the 4-momentum of the particle. By virtue of equation (3.2e), we can write it as

$$p_\mu = m_0 V_\mu = \frac{m_0}{\sqrt{1 - V^2/c^2}} (V, ic) \tag{3.8}$$

so that

$$p_1 = m_0 V_1 = \frac{m_0}{\sqrt{1 - V^2/c^2}} V_x \tag{3.9a}$$

$$p_2 = m_0 V_2 = \frac{m_0}{\sqrt{1 - V^2/c^2}} V_y \tag{3.9b}$$

$$p_3 = m_0 V_3 = \frac{m_0}{\sqrt{1 - V^2/c^2}} V_z \tag{3.9c}$$

$$p_4 = m_0 V_4 = \frac{m_0}{\sqrt{1 - V^2/c^2}} ic\,. \tag{3.9d}$$

The first three of these equations show that if we want the space components of p_μ, viz., p_1, p_2, p_3, to be respectively the components p_x, p_y, p_z of the 3-momentum vector **p** for all velocities, then we must interpret $m_0/(1 - V^2/c^2)^{1/2}$ as the mass of the particle when it is moving with a velocity V. When the particle is at rest, $V = 0$ and its mass is m_0. This is known as the **rest mass** or the **proper mass** of the particle; it is the mass of the particle measured in an inertial frame of reference in which it is at rest. Hence according to the special theory of relativity, the mass of a particle is not a constant quantity. It depends upon the velocity of the particle and is given by the formula

$$m = \frac{m_0}{\sqrt{1 - V^2/c^2}}\,. \tag{3.10}$$

This relation shows that the mass m of a particle measured by an observer depends on its speed relative to the observer, i.e., on the reference frame from which it is measured, and is therefore an extrinsic property of the particle. In fact, the greater the speed of a particle relative to a given frame, the larger is its mass in that frame.

Equation (3.10) appears to be against everyday experience. In daily life one never observes a variation in mass with speed. This is because at ordinary speed the variation in mass with speed is extremely small. In fact, the relativistic increase in mass becomes significant only at speeds very close to that of light. Thus even at a tremendous speed such as 32000 km s^{-1}., the increase in mass is only about 0.5%. However, the mass is more than doubled when the speed is about 0.9c.) It must be emphasized that relation (3.10) cannot be deduced logically; it is only a consequence of the way we relate the space components of p_μ with p_x, p_y, p_z. However, since its first verification made in 1908 by Bucherer by using electrons from a radioactive source, it has been confirmed experimentally a number of times.

With the aforementioned interpretation of mass, we may write

$$p_\mu = (\mathbf{p}, imc). \tag{3.11}$$

Accordingly, the 3-momentum and ic times the mass form the four components of 4-momentum just as the space coordinates and ict form the four components of the position 4-vector. Similar results are met throughout the special theory of relativity. For instance, force and power, and electric current density vector and charge density, occur together as 4-vectors. This unification of different quantities as the components of a 4-vector is a great beauty of the 4-dimensional approach to the theory.

We shall now derive an important relation between the momentum and mass of a particle. Consider two frames of reference such that in one of them the particle is at rest. Then the components of the 4-momentum in the two frames are

$$p_\mu = (p_1, p_2, p_3, p_4) = (p_x, p_y, p_z, imc) = (\mathbf{p}, imc)$$

$$p'_\mu = (p'_1, p'_2, p'_3, p'_4) = (0, 0, 0, im_0c) = (0, im_0c). \tag{3.12}$$

Since the magnitude of a 4-vector must remain invariant, we must have

$$p^2 - m^2c^2 = -m_0^2c^2$$

or

$$p^2 + m_0^2c^2 = m^2c^2. \tag{3.13}$$

It is interesting to note that in a closed system, with no energy or matter entering or leaving, this relation connects the principle of conservation of energy with that of conservation of momentum. This is a relation which is frequently used in modern physics.

To find the transformation law for 4-momentum, we note that it is a 4-vector. Therefore, we must have

$$p'_1 = \gamma\left(p_1 + i\frac{v}{c}p_4\right)$$

$$p'_2 = p_2 \tag{3.14a}$$

$$p_3' = p_3 \tag{3.14c}$$

$$p_4' = \gamma\left(p_4 - i\frac{v}{c}p_1\right). \tag{3.14d}$$

But
$$p_1 = \frac{m_0}{\sqrt{1 - V^2/c^2}}V_x = m\,V_x = p_x\,,$$

$$p_2 = p_y, \qquad p_3 = p_z$$

and
$$p_4 = imc. \tag{3.11$'$}$$

Substituting these and similar expressions for primed quantities in equations (3.14), we get

$$p_x' = \gamma\,(p_x - m\,v) \tag{3.15a}$$

$$p_y' = p_y \tag{3.15b}$$

$$p_z' = p_z \tag{3.15c}$$

$$m' = \gamma\left(m - \frac{v}{c^2}p_x\right). \tag{3.15d}$$

The last equation gives us the transformation law for the mass of a particle.