Interval and Light Cone

Let us now examine the relationship which various events may bear to each other. Consider two events occurring at points (x_1, y_1, z_1) and (x_2, y_2, z_2) at times t_1 and t_2 respectively in a frame of reference S. In the 4-dimensional space-time continuum these two events will be represented by the points (x_1, y_1, z_1, ict_1) and (x_2, y_2, z_2, ict_2) respectively. The distance Δs between these two points in 4-dimensional space is called the **interval** between the two events, and is given by

$$(\Delta s)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2 (t_2 - t_1)^2 .$$

If we denote $(x_2 - x_1)^2$, etc. by Δx^2 ($\equiv (\Delta x)^2$) etc., this equation takes the form

$$(\Delta s)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = \Delta r^2 - c^2 \Delta t^2, \qquad (2.32)$$

where $\Delta r = + (\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}$ is the spatial distance and Δt is the time interval between the two events. Equation (2.32) shows that, depending upon the spatial and temporal separations, the intervals between two events can be classified into the following three categories:

(1) $(\Delta s)^2 > 0$. Then $\Delta r^2 > c^2 \Delta t^2$, i.e., the spatial distance Δr between the two events is greater than the distance $c\,\Delta t$ which a ray of light can cover during the time interval Δt between the events. In this case, the two events are

either so far apart in coordinate space and/or occur in such rapid succession that one of them occurs even before a light signal from the other can reach it. Since no signal can be propagated with a velocity greater than that of light, the two events cannot be connected causally, i.e., one cannot be the cause of the other. The interval Δs between two such events is said to be a **space-like interval.**

(2) $(\Delta s)^2 < 0$. Then $\Delta r^2 < c^2 \Delta t^2$. In this case, the spatial distance between the two events can be covered by a light signal during the time interval between them. Therefore, one of the events can be the cause of the other, i.e., the two events can be connected causally. The interval Δs between two such events is said to be a **time-like interval.**

(3) $(\Delta s)^2 = 0$. Then $\Delta r^2 = c^2 \Delta t^2$ so that the spatial distance Δr between the two events can be just covered by a light signal during the time Δt and a causal relationship between the two events can occur. The interval Δs between two such events is said to be a **null or light-like interval.** It may be remarked that the path of a ray of light in space-time continuum is always represented by a null vector.

Since the length of a 4-vector in the world space is an invariant quantity, the interval Δs between any two given events is also invariant, i.e., Δs is independent of the frame of reference. Thus although both space and time intervals change under a Lorentz transformation, the space-time interval is still invariant. That is, the character of an interval between two events will not change under a Lorentz transformation. For instance, no Lorentz transformation can transform a space-like interval into a time-like interval, or vice versa. It may be noted that the most significant difference between the metrics of space and space-time, viz.,

$$(\Delta r)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$(\Delta s)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

is that of *signature*. The space metric, irrespective of the choice of the coordinate system, is *positive definite* i.e., all the metric coefficients have positive signs which we represent by (+ + +). In contrast, the signature of the Minkowski space-time is (+ + + −) or, as we shall ... signature of that kind is call... between two ...

positive, negative or even zero. Such a metric is sometimes called a **pseudo-metric.**

We shall now give a geometrical representation of the relation among the interval Δs between two events and the corresponding spatial and temporal intervals. According to equation (2.32), this relationship is given by

$$\Delta s^2 + c^2 \Delta t^2 = \Delta r^2.$$

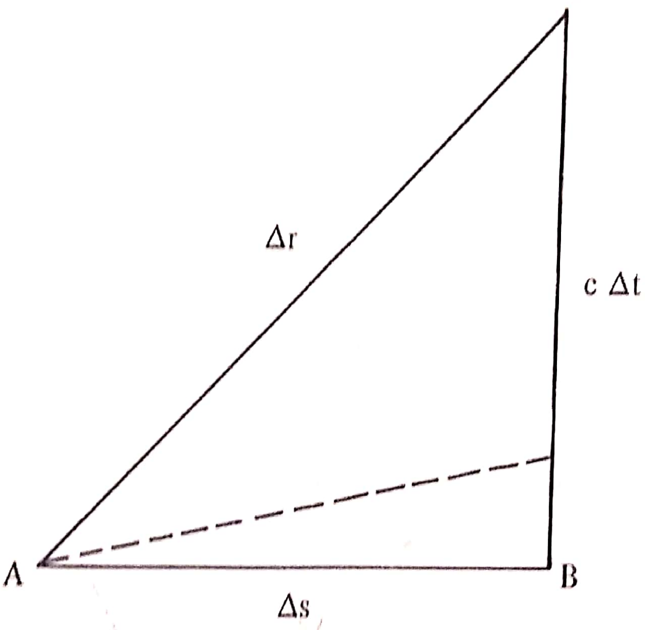

Fig. 2.6. Geometrical representation of Δs

The terms Δr^2 and $c^2 \Delta t^2$ are always positive, so that, if the interval Δs is space-like, all the terms in this relation would be positive. Then the form of this relation suggests that Δr may be regarded as the hypotenuse of a right-angled triangle with Δs and $c\,\Delta t$ as its sides. This is shown in Fig. 2.6. Since Δs is invariant under a Lorentz transformation, the base of the triangle ABC would remain fixed in all inertial frames while its height and hypotenuse vary. The triangle ABC shows that for a fixed Δs, the quantities Δr and $c\,\Delta t$ can either increase together or decrease together; it is not possible to increase one of them and decrease the other, or to change only one of them and keep the other constant.

A similar geometrical interpretation can be given for a time-like interval.

For a null interval, the right-angled triangle collapses into a single vertical straight line equal to both Δr and $c\,\Delta t$.

We shall now illustrate the meaning of these terms by an example. Suppose that two events E_1 and E_2 occur in a laboratory at two points which are 9 m apart, the event E_2 occurring 10^{-8} s later than the event E_1. The spatial distance Δr between these two events is 9 m. The time interval Δt between these two events is 10^{-8} s and during this period light can travel a distance $c\,\Delta t = 3 \times 10^8 \times 10^{-8}$ m $= 3$ m only. This means that even a light signal emitted by the event E_1 at the time of its occurrence would reach the position of E_2 after the latter event has occurred. Since no signal can be propagated with a velocity greater than that of light and the event E_2 occurs before a light signal from E_1 reaches E_2, the two events E_1 and E_2 cannot be connected causally, i.e., E_1 cannot be the cause of E_2. Moreover, as $\Delta r = 9$ m is greater than $c\,\Delta t = 3$ m, the interval between the events E_1 and E_2 is space-like.

If the two events occur with a time interval of 10^{-6} s, then $\Delta r = 9$ m would be less than $c\,\Delta t = 3 \times 10^8 \times 10^{-6}$ m $= 300$ m. This interval would, therefore, be time-like. Since, in this case, signals from E_1 can reach E_2 before the event E_2 occurs, the two events can be connected causally.

If the time interval between the two events is 3×10^{-8} s, the interval Δs is null and the two events are just connected by a light signal.