Chapter

8 Four-Vectors

The theory of special relativity can be developed in a very concise and yet elegant way by using the so called concept of four-vectors. It is a conceptual extension of ordinary three-dimensional vectors to four-dimensional vectors.

8.1 SPACE-TIME CONTINUUM (MINKOWSKI SPACE)

Lorentz Transformation (L.T.) and its consequences suggest that the space coordinates (x, y, z) and time (t) should be treated uniformly in the theory of relativity. Minkowski suggested that the ordinary three-dimensional space plus the time be considered as four-dimensional continuum or Minkowski space.

Just as the ordinary three dimensional space can be labelled by three coordinates x, y and z; it is convenient to think of a four dimensional space-time labelled by the four coordinates x, y, z and t. For dimensional uniformity, the fourth coordinate may be taken as ct or preferably as ict $(i = \sqrt{-1})$ to remind us that space and time are not quite the same or equivalent.

Instead of variables x, y, z and t in frame S, we now introduce the variables X_μ ($\mu = 1, 2, 3, 4$) given by

$$X_1 = x;\ X_2 = y;\ X_3 = z \text{ and } X_4 = ict \qquad \text{... (1a)}$$

The corresponding quantities in frame S' are X'_μ ($\mu = 1, 2, 3, 4$); that is

$$X'_1 = x';\ X'_2 = y';\ X'_3 = z' \text{ and } X'_4 = ict' \qquad \text{... (1b)}$$

From L.T.; since $x' = \Gamma(x - vt)$ . . . etc.,

$$X'_1 = \Gamma\left(X_1 - \frac{vX_4}{ic}\right) = \Gamma\left(X_1 + \frac{iv}{c}X_4\right) \qquad \text{... (2)}$$

$$X'_2 = X_2;\ X'_3 = X_3 \qquad \text{... (3)}$$

Also since $$t' = \Gamma\left(t - \frac{vx}{c^2}\right)$$

$$ict' = \Gamma\left(ict - \frac{ivx}{c}\right)$$

$$\therefore \quad X'_4 = \Gamma\left(X_4 - \frac{ivX_1}{c}\right) \qquad \text{... (4)}$$

The inverse relations are easily written down by changing the sign of v. Thus

$$X_1 = \Gamma\left(X_1' - \frac{ivX_4'}{c}\right) \quad \text{... (5a)}$$

$$X_2 = X_2' \,;\; X_3 = X_3' \quad \text{... (5b)}$$

And
$$X_4 = \Gamma\left(X_4' + \frac{ivX_1'}{c}\right) \quad \text{... (5c)}$$

Note that the above equations are just Lorentz transformations for X_1, X_2, X_3, X_4 just as equations (6) and (7) of chapter 2 are L.T. for coordinates x, y, z and t.

8.2 FOUR-VECTORS

A four-vector A_μ is a set of four quantities (A_1, A_2, A_3, A_4) which transform under a Lorentz Transformation in the same way as the X_1, X_2, X_3, X_4 coordinates of a point in the four dimensional space-time continuum. Thus

$$\left.\begin{aligned} A_1' &= \Gamma\left(A_1 + \frac{ivA_4}{c}\right);\; A_2' = A_2;\; A_3' = A_3 \\ \text{and} \quad A_4' &= \Gamma\left(A_4 - \frac{ivA_1}{c}\right) \end{aligned}\right\} \quad \text{... (6)}$$

The first three components $(A_1, A_2\, A_3)$ are the components of an ordinary three-dimensional vector $\vec{A}$ just as x, y, z are the components of $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$

The sum of two four-vectors is also a four-vector so that $C_\mu = (A + B)_\mu = A_\mu + B_\mu$.

Thus $\quad C_1 = (A_1 + B_1);\; C_2 = (A_2 + B_2);\; C_3 = (A_3 + B_3);\; C_4 = (A_4 + B_4)$

The scalar product of two four-vectors is defined by extending the scalar product $(\vec{A}\cdot\vec{B})$ of two ordinary vectors $\vec{A}$ and $\vec{B}$. Thus scalar product of two four-vectors A and B

$$= \Sigma_\mu A_\mu B_\mu = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4.$$

The square of the 'length' of a four-vector is defined as the scalar product of the four-vector with itself.

Thus $\quad (\text{length})^2 = \Sigma_\mu A_\mu^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2$

$$= \Gamma^2\left(A_1' - \frac{ivA_4'}{c}\right)^2 + A_2'^2 + A_3'^2 + \Gamma^2\left(A_4' - \frac{ivA_1'}{c}\right)^2$$

$$= A_1'^2 + A_2'^2 + A_3'^2 + A_4'^2 = \Sigma_\mu A_\mu'^2 \text{ because } \Gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1.$$

We now see that the length of a four-vector is unchanged by a L.T. It is an invariant. An invariant is a quantity whose value does not change in a L.T. For example, the dot product of two four-vectors $(\Sigma_\mu A_\mu B_\mu)$ is an invariant. 'Length' of a four-vector is a special case of dot product or scalar product of two vectors.

If all the components of a four-vector are multiplied by a scalar (α) an invariant, then we get a new four-vector of length α times the original length.

Question:
Express Lorent'z Transformation in term of Minkowski space coordinates.

OR

Express Lorent'z Transformation in term of rotation.

Solution:

Since Lorent'z transformation are

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \;, \; z' = z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}\right\}$$

By using Minkowski space coordinates

$$x_1 = x$$
$$x_2 = y$$
$$x_3 = z$$
$$x_4 = ict \Rightarrow t = \frac{x_4}{ic} \quad \& \quad t' = \frac{x_4'}{ic}$$

$\because$ In L.T $c' = c$.

Then we get

$$\because x_1' = \gamma(x - vt)$$
$$= \gamma\left[x_1 - v\frac{x_4}{ic}\right]$$
$$= \gamma x_1 + \frac{i^2 \gamma v}{ic} x_4$$

$\because \cos\theta = \gamma$ & $\sin\theta = \frac{i\gamma v}{c}$

$$= \gamma x_1 + \frac{i\gamma v}{c} x_4$$
$$x_1' = \cos\theta\, x_1 + \sin\theta\, x_4$$
$$\boxed{x_1' = x_1\cos\theta + x_4\sin\theta}$$

$$\boxed{x_2' = x_2} \quad \& \quad \boxed{x_3' = x_3}$$

$$\frac{x_4'}{ic} = \gamma\left(\frac{x_4}{ic} - \frac{v}{c^2}x_1\right)$$

Multiplying ic on both sides:

$$x_4' = \gamma x_4 - \frac{i\gamma v}{c}x_1$$

(since $\cos\theta = \gamma$, $\sin\theta = \frac{i\gamma v}{c}$)

$$x_4' = \cos\theta\ x_4 - \sin\theta\ x_1$$

$$\boxed{x_4' = x_4\cos\theta - x_1\sin\theta}$$

Hence

$$\left.\begin{aligned} x_1' &= x_1\cos\theta + x_4\sin\theta \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= x_4\cos\theta - x_1\sin\theta \end{aligned}\right\}$$

These are called Lorentz transformation in terms of Minkowski space or in terms of rotation.

In Matrix Form:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix};$$

where

$$A = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{bmatrix}$$

Question: **Prove that $AA^T = I$**

OR

set of L.T Equation in terms of minkowski space system is Orthognal.

Solution: Since

$$A^T = \begin{bmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} \cos\theta & 0 & 0 & +\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+\sin^2\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cos^2\theta+\sin^2\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \Rightarrow \boxed{AA^T = I}$$

This shows that system is orthognal.