The Poincaré Group

A linear transformation in space-time continuum, viz.,

$$x'_\mu = a_{\mu\nu} x_\nu + b_\mu, \quad \mu, \nu = 1, 2, 3, 4 \qquad (2.3$$

where the space coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$ are real, and $x_4 = i$

pure imaginary, while $a_{\mu\nu}$ and b_μ are constants, is said to be a Lorentz transformation if it satisfies the equation

$$dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 - c^2 dt'^2 \qquad (2.39)$$

Equation (2.39) may be written as

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 + dx_4'^2$$

or

$$dx_\mu \, dx_\mu = dx'_\mu \, dx'_\mu \qquad (2.40)$$

Taking differentials of both sides of equation (2.38), we get

$$dx'_\mu = a_{\mu\nu} \, dx_\nu$$

Substituting this expression for dx'_μ in equation (2.40), we have

$$dx_\mu \, dx_\mu = a_{\mu\nu} \, dx_\nu \, a_{\mu\sigma} \, dx_\sigma = a_{\mu\nu} \, a_{\mu\sigma} \, dx_\nu \, dx_\sigma \qquad (2.41)$$

where, in order to avoid any confusion which may arise if any letter index occurs more than twice, we have used different dummy indices on the right hand side of equation (2.41). Comparing the coefficients on the two sides of equation (2.41), we get

$$a_{\mu\nu} \, a_{\mu\sigma} = \delta_{\nu\sigma} \qquad (2.42)$$

Since x_1, x_2, x_3 are all real and x_4 is pure imaginary, the coefficients $a_{\mu\nu}$ and the constants b_μ should be such that a_{ij}, a_{44}, b_i are real while a_{j4}, a_{4j}, b_4 are pure imaginary. Equation (2.42) expresses the orthogonality of the transformation defined by equation (2.38).

Thus the linear transformation $x'_\mu = a_{\mu\nu} \, x_\nu + b_\mu$ is a Lorentz transformation provided that $a_{\mu\nu} \, a_{\mu\sigma} = \delta_{\nu\sigma}$ and a_{ij}, a_{44}, b_i are real while a_{j4}, a_{4j}, b_4 are pure imaginary.

Equations (2.38) and (2.39) may be written in matrix form as

$$X' = A\,X + b \qquad (2.43)$$

in the Minkowski-space where the transformation matrix A is orthogonal, $A^T A = I$, and the elements of A and b are such that a_{ij}, a_{44}, b_i are real while a_{j4}, a_{4i}, b_4 are pure imaginary.

It may be pointed out that the Lorentz transformations are a subset of orthogonal transformations in a 4-dimensional space, where x_1, x_2, x_3, x_4 are, in general, complex variables, so that the conditions that

a_{ij}, a_{44}, b_i are real

and a_{j4}, a_{4i}, b_4 are pure imaginary

are not imposed. Symbolically we can write

$$A = \begin{pmatrix} r & r & r & i \\ r & r & r & i \\ r & r & r & i \\ i & i & i & r \end{pmatrix}, \qquad b = \begin{pmatrix} r \\ r \\ r \\ i \end{pmatrix},$$

where r stands for a real and i for a pure imaginary element.

By taking the determinant of both sides of the relation $A^T A = I$, it follows that $|A^T A| = |I|$ or $|A^T|\,|A| = |A|\,|A| = |A|^2 = 1$, where we have made use of the fact that the determinant $|A|$ of a matrix A is unchanged by the interchange of its rows and columns. This yields $|A| = \pm 1$. A Lorentz transformation with $|A| = +1\,(-1)$ is called a proper (improper) Lorentz transformation.

It is also common to use the variable $x_0 = ct$. In this notation, a Lorentz transformation is defined as a linear transformation

$$x'_\mu = a_{\mu\nu}\, x_\nu + b_\mu, \qquad \mu, \nu = 0, 1, 2, 3$$

satisfying the equation

$$dx_1^2 + dx_2^2 + dx_3^2 - c^2\,dt^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 - c^2\,dt'^2. \qquad (2.39')$$

Equation (2.39) may then be written as

$$-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 = -dx_0'^2 + dx_1'^2 + dx_2'^2 + dx_3'^2. \qquad (2.47)$$

To express the definition in terms of matrices, we define the column X′ and X and the matrices g and A by the equations

$$X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X' = \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix},$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{diag}\,(1, -1, -1, -1),$$

$$A = [a_{\mu\nu}] = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then, in matrix form, equations (2.38) and (2.47) may be written as

$$X' = A\,X + b \tag{2.}$$

$$dX^T g\, dX = dX'^T g\, dX'. \tag{2.}$$

Substituting the expressions for dX'^T and dX' obtained by using eq (2.43) in equation (2.48), we get

$$dX^T g\, dX = dX^T A^T g\, A\, dX.$$

Since this relation must be valid for all dX, it would be possible only if

$$A^T g\, A = g.$$

Notice that now all the a_{μ} and the b_{μ} are real. Hence a Lorentz transformation may be defined as a linear transformation, $X' = AX + b$, in a space-time continuum $(x_0 = ct, x_1, x_2, x_3)$ such that $A^T g A = g$, where $g = \text{diag}(1, -1, -1, -1)$ and all the elements of A and b are real.

We will now show that the Lorentz transformations form a group under successive application of transformations, i.e., they possess the following four characteristics of a group.

(1) The product of any two Lorentz transformations

$$X' = AX + b, \quad A^T g A = g \tag{2.49}$$

$$X'' = CX' + d, \quad C^T g C = g \tag{2.50}$$

is again a Lorentz transformation. That is if the Lorentz transformation (2.49) takes us from reference system S to S′, and the Lorentz transformation (2.50) takes us from the reference system S′ to S″, then the successive applications of these transformations will give another Lorentz transformation which will take us from the reference system S to S″. This resultant Lorentz transformation is obtained by substituting the expression for X′ from the first relation in the second:

$$X'' = C(AX + b) + d = CAX + Cb + d = FX + h, \tag{2.51}$$

where $F = CA$ and $h = Cb + d$.

Since $F^T g F = (CA)^T g (CA) = A^T C^T g C A = A^T g A = g$, where the relations $A^T g A = g = C^T g C$ have been used, we conclude that the transformation (2.51) is a Lorentz transformation.

It may be noticed that these transformations are not commutative because in general $AC \neq CA$, i.e., we arrive at different results depending upon whether the transformation represented by A is applied after that represented by C, or vice versa.

(2) The multiplication is associative because of the linearity of the transformations.

(3) The Lorentz transformation

$$X' = X = IX, \quad I^T g I = g$$

serves as the identity transformation, i.e., as a transformation which leaves everything unchanged.

(4) The inverse of each Lorentz transformation exists and is itself a Lorentz transformation. Consider the Lorentz transformation

$$X' = A X + b, \qquad A^T g A = g. \tag{2.49'}$$

Since $|A| = \pm 1 \neq 0$, the matrix A is non-singular and its inverse exists. Multiplying both sides of equation (2.49′) by A^{-1} from the left, we get

$$A^{-1} X' = X + A^{-1} b$$

or
$$X = A^{-1} X' - A^{-1} b = A^{-1} X' + d, \quad \text{where } d = A^{-1} b. \tag{2.52}$$

To find out whether this linear transformation is a Lorentz transformation or not, we consider the matrix relation

$$A^T g A = g.$$

Multiplying from the left by $(A^T)^{-1}$ and from the right by A^{-1}, we get

$$(A^T)^{-1} A^T g A A^{-1} = (A^T)^{-1} g A^{-1}$$

or

$$g = (A^{-1})^T g A^{-1}.$$

This shows that the linear transformation (2.52) is a Lorentz transformation. This is the inverse of the Lorentz transformation (2.49′) and it takes us from S′ back to S.

Hence the set of all Lorentz transformations forms a group. This group is called the Poincaré group or the inhomogeneous Lorentz group or the general Lorentz group or the complete Lorentz group and is denoted by $\mathcal{L}$.