of time, it is the return journey of A in an inertial frame which is different from the inertial frame in which it was at rest on the outward journey, that causes the difference in the ages of the twins. As only A has to jump from one inertial frame to another to return, all observers will agree that it is he who is in motion and that the clocks in the rocket are running slow. Consequently, it is A who would be the younger twin on his return to the earth.

Transformation Law for Velocity

We shall now find the relativistic transformation law for velocity, i.e., we should like to see how the velocity in one inertial frame would appear from another inertial frame of reference. Let us consider the motion of a particle with respect to two standard frames of references S and S$'$. If the space-time coordinates of the particle in the two frames are (x, y, z, t) and (x$'$, y$'$, z$'$, t$'$), then its velocities $V = (V_x, V_y, V_z)$ and $V' = (V'_x, V'_y, V'_z)$ as measured in these frames are given by

$$V = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$V' = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right).$$

We have proved earlier that

$$x = \gamma\,(x' + v\,t')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\,(t' + \frac{v}{c^2}\,x').$$

Differentiating with respect to t, we get

$$\frac{dx}{dt} = \gamma\left(\frac{dx'}{dt'} + v\right)\frac{dt'}{dt} \tag{2.24a}$$

$$\frac{dy}{dt} = \frac{dy'}{dt'}\,\frac{dt'}{dt} \tag{2.24b}$$

$$\frac{dz}{dt} = \frac{dz'}{dt'}\,\frac{dt'}{dt} \tag{2.24c}$$

$$1 = \gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)\frac{dt'}{dt}. \tag{2.24d}$$

Substituting the expression for dt'/dt from equation (2.24d) in equation (2.24a), we get

$$\frac{dx}{dt} = \left(\frac{dx'}{dt'} + v\right)/\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)$$

or
$$V_x = \frac{V'_x + v}{1 + \frac{v}{c^2}V'_x} \tag{2.25a}$$

Similarly, we can show that

$$V_y = \frac{V'_y}{\gamma\left(1 + \frac{v}{c^2}V'_x\right)} \tag{2.25b}$$

and
$$V_z = \frac{V'_z}{\gamma\left(1 + \frac{v}{c^2}V'_x\right)}. \tag{2.25c}$$

Equations (2.25) give the relativistic transformation laws for the components of velocity. Notice that in the case of velocity, in general, even the transverse components in the two frames differ from each other.

For very small values of v/c, equations (2.25) approximate to

$$V_x = V'_x + v \tag{2.26a}$$

$$V_y = V'_y \tag{2.26b}$$

$$V_z = V'_z \tag{2.26c}$$

which is the classical transformation law for the velocity. Since the transformation equations (2.26) give the classical law of addition of velocities $V' = (V'_x, V'_y, V'_z)$ and $v = (v, 0, 0)$, equations (2.25) may be regarded as representing the relativistic law of addition of velocities V' and v. However, ordinary mechanical velocities, being too small as compared with c, do not provide a suitable test of the relativistic law for the addition of velocities.

Show That

Transformation of L.C factor

$$\sqrt{1-\frac{u'^2}{c^2}} = \sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}} \Big/ \left(1-\frac{v}{c^2}u_x\right)^2$$

Results

(1) It is interesting to notice that the relativistic law of addition is such that if we add any velocity to the velocity of light it remains unchanged. For example, a light signal from a source at rest on the x′-axis in S′ will be observed in S, in accordance with equation (2.25a), to be moving along the x-axis with a velocity

$$V_x = \frac{c + v}{1 + \frac{v}{c^2}\, c} = c.$$

Using equations (2.25), we obtain the following relation

(2) $$c^2 - V^2 = \frac{c^2\,(c^2 - V'^2)(c^2 - v^2)}{(c^2 + V'\, v)^2}, \qquad (2.27)$$

Notes

where $V^2 = V_x^2 + V_y^2 + V_z^2$ and $V'^2 = V_x'^2 + V_y'^2 + V_z'^2$.

For $V' < c$ and $v < c$, the right hand side and consequently the left hand side of equation (2.27) is positive, i.e.,

$$c^2 - V^2 > 0.$$

Notes

Therefore $V < c$. This shows that the resultant of two velocities V' and v, each of which is less than c, is itself less than c. Hence it is impossible to attain the velocity of light c by adding velocities each one of which is less than c.

The inverse transformation law for velocities is obtained from equations (2.26) by interchanging primed and unprimed quantities and changing v to −v, so that

$$V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}\, V_x}, \qquad (2.28a)$$

$$V_y' = \frac{V_y}{\gamma\,(1 - \frac{v}{c^2}\, V_x)}, \qquad (2.28b)$$

$$V_z' = \frac{V_z}{\gamma\,(1 - \frac{v}{c^2}\, V_x)}. \qquad (2.28c)$$

Problem

Show that the magnitude of the resultant 3-velocity V' is given by

Transformation Law for Acceleration

Let us next find the transformation law for acceleration. This may be obtained from the transformation law for velocity by differentiation. Thus differentiating equation (2.28) with respect to t', we get

$$\frac{dV_x'}{dt'} = \left[\left(1 - V_x \frac{v}{c^2}\right)\frac{dV_x}{dt} - (V_x - v)\left(-\frac{v}{c^2}\frac{dV_x}{dt}\right)\right]\frac{dt}{dt'}$$

or

$$a_x' = \frac{1 - \frac{v^2}{c^2}}{\left(1 - V_x \frac{v}{c^2}\right)^2}\left(a_x \frac{dt}{dt'}\right),$$

where

$$a_x' = \frac{dV_x'}{dt'} \quad \text{and} \quad a_x = \frac{dV_x}{dt}. \tag{2.30}$$

But

$$t' = \gamma\left(t - \frac{v}{c^2}x\right),$$

which on differentiation with respect to t yields

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right) = \gamma\left(1 - \frac{v}{c^2}V_x\right).$$

Substituting this expression in equation (2.30) and simplifying, we obtain

$$a_x' = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 - V_x \frac{v}{c^2}\right)^3} a_x. \tag{2.31a}$$

Similarly, we can show that

$$a_y' = \frac{1 - \frac{v^2}{c^2}}{\left(1 - V_x \frac{v}{c^2}\right)^2}\left[a_y + \frac{V_y \frac{v}{c^2} a_x}{1 - V_x \frac{v}{c^2}}\right] \tag{2.31b}$$

and

$$a_z' = \frac{1 - \frac{v^2}{c^2}}{\left(1 - V_x \frac{v}{c^2}\right)^2}\left[a_z + \frac{V_z \frac{v}{c^2} a_x}{1 - V_x \frac{v}{c^2}}\right] \tag{2.31c}$$

Equations (2.31) give the transformation law for acceleration.