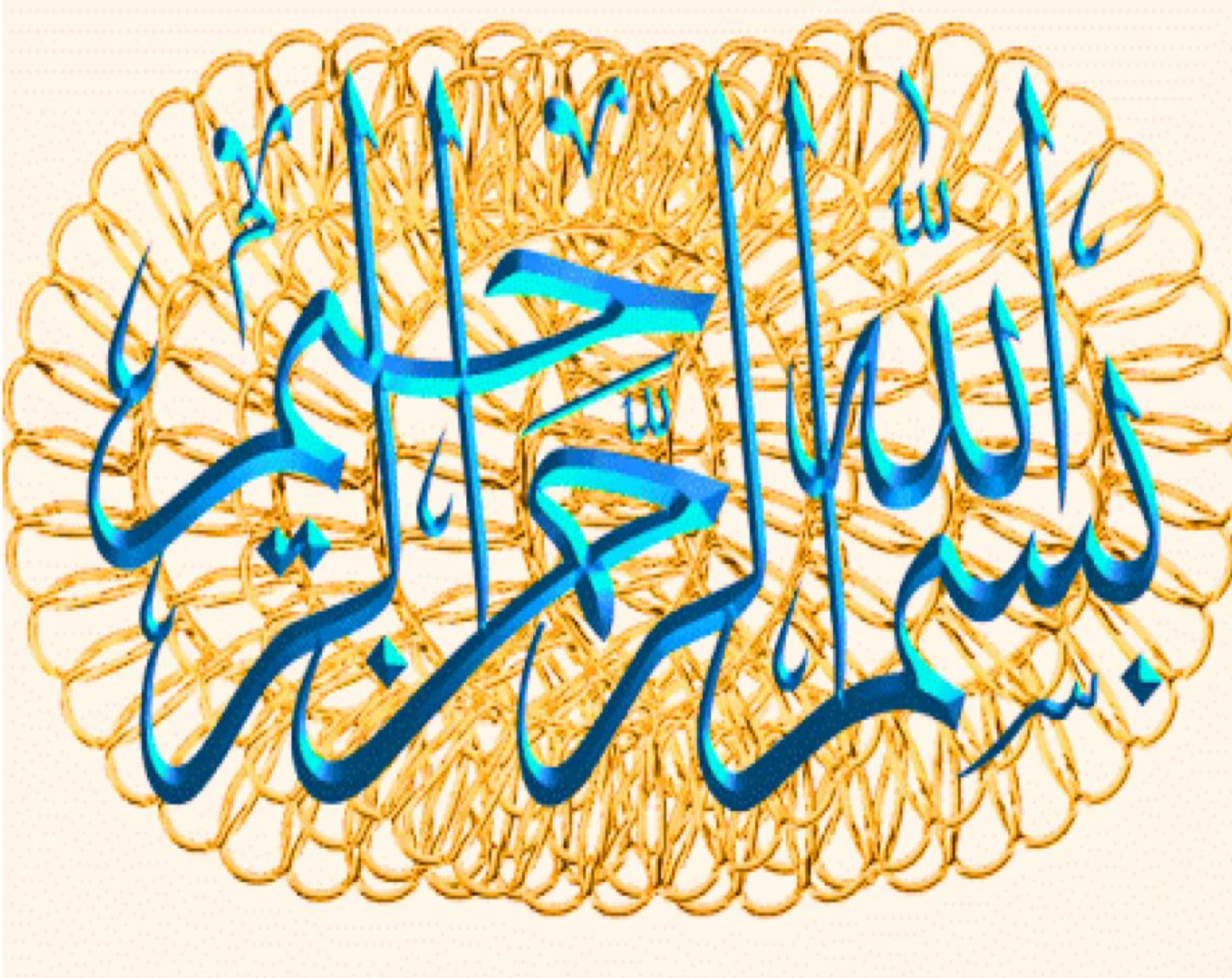

بسم الله الرحمن الرحيم


السلام عليكم
As-salamu alaykum

Online Lectures Series

By

MUHAMMAD SAFDAR

Course title: Mathematics-1

Course Code: URCM-5107

University of Sargodha, Sargodha

Lecture-7&8: Preliminaries

Matrices and Determinants

7.0 Matrices and its Types

7.1 Matrices and Systems of Linear Equations

7.2 The Algebra of Matrices

7.3 Inverses of Matrices and Matrix Equations

7.4 Determinants and Cramer's Rule

Introduction

While solving linear systems of equations, a new notation was introduce reduce the amount of writing. For this new notation the word *matrix* was first used the English mathematician James Sylvester (1814 – 1897). Arthur Cay (1821 – 1895) developed the theory of matrices and used them in the lin transformations. Now-a-days, matrices are used in high speed computers and als other various disciplines.

The concept of determinants was used by Chinese and Japanese but Japanese mathematician Seki Kowa (1642 – 1708) and the German Mathematic Gottfried Wilhelm Leibniz (1646 – 1716) are credited for the invention determinants. G. Cramer (1704 – 1752) employed the determinants successfully solving the systems of linear equations.

Matrix:

An arrangement of different elements in the form of rows and columns, within square brackets is called Matrix. It is always denoted by capital Alphabets.

e.g $A=\begin{bmatrix}1 & 7\\ 3 & 2\end{bmatrix}$, $B=\begin{bmatrix}5 & 3 & 7\\ 1 & 6 & 8\end{bmatrix}$

7.0 Types of Matrices

Order:

Order of Matrix tells us about no of rows and no of columns.

Order of Matrix = no of rows × no of columns.

If $A=\begin{bmatrix}1 & 1 & 3\\ 2 & 1 & -7\end{bmatrix}$ then order of $A = 2\times 3$

Row Matrix:

A matrix having single row is called row matrix.

e.g $A=\begin{bmatrix}1 & 3 & 7\end{bmatrix}$, $B=\begin{bmatrix}1 & 6 & 3\end{bmatrix}$

Column Matrix:

A matrix having single column is called Column Matrix.

e.g $A=\begin{bmatrix}5\\ 1\\ 4\end{bmatrix}$

Square Matrix:

A matrix in which no of rows equal to the no of columns is called square matrix.

e.g $A=\begin{bmatrix}2 & 3\\ 1 & 7\end{bmatrix}$, $B=\begin{bmatrix}1 & 1 & 7\\ -1 & 5 & 1\\ 7 & 4 & 6\end{bmatrix}$

Rectangular Matrix:

The matrix in which no of rows is not equal to the no of columns is called Rectangular Matrix.

e.g $A=\begin{bmatrix}1 & 5 & 2\\ 3 & 7 & 1\end{bmatrix}$

Diagonal Matrix:

A Square matrix having each of its elements equal to zero except at lea element in its diagonal is called diagonal matrix.

$$\text{e.g } A=\begin{bmatrix}3 & 0\\ 0 & 5\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 0 & 9\end{bmatrix},\qquad C=\begin{bmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -4\end{bmatrix}$$

Scalar Matrix:

A diagonal matrix having same elements in its diagonal is called a Scalar matri

$$\text{e.g } A=\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix},\quad B=\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

Identity Matrix:

A scalar matrix having 1 as its elements in the diagonal is called an identity ma

$$\text{e.g } A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Null Matrix:

A matrix in which all elements are equal to zero is called Null matrix or zero matrix.

e.g $A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

Equal Matrixes:

Two matrixes are said to be equal if they are of same order with the same correspondence elements.

e.g $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\ 4 & 7\end{bmatrix}$ $\text{If } A=B \text{ then } a=3, b=1, c=4, d=7$

Upper Triangular Matrix:

If all elements below the main diagonal of a square matrix are zero then it is called upper triangular matrix.

e.g $\begin{bmatrix}2 & 5 & 6\\ 0 & 4 & 3\\ 0 & 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 7\\ 0 & 2 & 3\\ 0 & 0 & 4\end{bmatrix}$

Lower Triangular Matrix:

If all elements above the main diagonal of a square matrix are zero then it is called Lower triangular matrix.

e.g $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 3 & 5 & 7 \end{bmatrix}$

Triangular Matrix:

A matrix which is either upper triangular or lower triangular is called a triangular matrix.

Symmetric Matrix:

Let "A" be a square matrix if $A' = A$ then "A" is called symmetric matrix.

Skew Symmetric Matrix:

Let "A" be a square matrix if $A' = -A$ then "A" is called skew symmetric matrix or Anti symmetric matrix.

Hermitian Matrix:

Let "A" be a square matrix if $(\bar{A})' = A$ then "A" is called Hermitian Matrix.

Skew Hermitian Matrix:

Let "A" be a square matrix if $(\bar{A})^t = -A$ then "A" is called Skew Hermitian Matrix or Anti Hermitian Matrix.

Leading Entry (L.E):

The first non zero entry in any non zero row of an matrix is called leading entry.

Echelon Form:

(i) First non zero element of each row should be 1.

(ii) All elements under this 1 should be zero

e.g $\begin{bmatrix} 0 & 1 & 7 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

Reduce Echelon Form:

First two conditions are same of echelon form

All elements above leading entry (1) should be zero.

$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ *etc*

Rank:

Number of non zero rows (not all elements zero) in the echelon form of a matrix is called Rank of the matrix.

Singular and Non-Singular Matrices

A square matrix A is ***singular*** if $|A| = 0$, otherwise it is a ***non-singular*** matrix.

In the above example, $|B| = 0 \Rightarrow B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ is a singular matrix

and $|A| = 10 \neq 0 \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ is a non-singular matrix.

Adjoint of a 2 × 2 Matrix

The adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $adj\ A$ and is defined as:

$$adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Transpose of a Matrix: If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A. It is denoted by A'. If $A = [a_{ij}]_{m \times n}$, then the transpose of A is defined as:

Example 1:

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$$

Taking transpose of A ar

$$A^t = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix}$$

3.7 Properties of Determinants which Help in their Evaluation

1. For a square matrix A, $|A| = |A^t|$
2. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.
3. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.
4. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.
5. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in \mathbb{R}$, then the determinant of the resulting matrix is $k|A|$.
6. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$|B| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

7. If to each entry of a row (or a column) of a square matrix A is added a non-zero multiple of the corresponding entry of another row (or column), then the determinant of the resulting matrix is $|A|$.

If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Now we prove the above mentioned properties of determinants.

Proporty 1: If the rows and columns of a determinant are interchanged, then the value of the determinant does not change. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \text{ (rows and columns are interchanged)}$$

Property 2: The value of a determinant changes sign if any two rows (columns) are interchanged. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and $\begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22} = -(a_{11}a_{22} - a_{12}a_{21})$ (columns are interchanged)

Property 3: If all the entries in any row (column) are zero, the value of the determinant is zero. For example,

$$\begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0 \text{ (expanding by } C_1\text{)}$$

Property 4: If any two rows (columns) of a determinant are identical, the value of the determinant is zero. For example,

$$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0, \text{ (it can be proved by expanding the determinant)}$$

Property 5: If any row (column) of a determinant is multiplied by a non-zero number k, the value of the new determinant becomes equal to k times the value of original determinant. For example,

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, multiplying first row by a non-zero number k, we get

$$\begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = ka_{11}\,a_{22} - ka_{12}\,a_{21} = k\,(a_{11}\,a_{22} - a_{12}\,a_{21}) = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Property 6: If any row (column) of a determinant consists of two terms, it can be written as the sum of two determinants as given below:

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix} \text{ (proof is left for the reader)}$$

Property 7: If any row (column) of a determinant is multiplied by a non-zero number k and the result is added to the corresponding entries of another row (column), the value of the determinant does not change. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} + ka_{11} \\ a_{21} & a_{22} + ka_{21} \end{vmatrix} \quad (k \text{ multiple of } C_1 \text{ is added to } C_2)$$

It can be proved by expanding both the sides. Proof is left for the reader.

Chapter Overview

A *matrix* is simply a rectangular array of numbers. Matrices are used to organize information into categories that correspond to the rows and columns of the matrix. For example, a scientist might organize information on a population of endangered whales as follows:

	Immature	Juvenile	Adult
Male	12	52	18
Female	15	42	11

This is a compact way of saying there are 12 immature males, 15 immature females, 18 adult males, and so on.

In this chapter we will represent a linear system by a matrix, called the *augmented matrix* of the system:

Linear System **Augmented Matrix**

$$\begin{cases} 2x - y = 5 \\ x + 4y = 7 \end{cases} \qquad \begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & 7 \end{bmatrix}$$

Equation 1
Equation 2
x
y

The augmented matrix contains the same information as the system, but in a simpler form. The operations we learned for systems of equations can now be performed on the augmented matrix. For example, multiplying an equation by a constant corresponds to multiplying a *row* of the matrix by a constant.

One of the important ideas in this chapter is to think of a matrix as a single object, so we denote a matrix by a single letter, such as A, B, etc. We can add, subtract, and multiply matrices, just as we do ordinary numbers. So for matrices of appropriate size, $A + B$, $A - B$, $2A$, and AB all have meaning. We will pay special attention to matrix multiplication—it's defined in a way that makes it possible to write a linear system as a single *matrix equation*

$$AX = B$$

where X is the unknown matrix. We will see that solving this matrix equation for the matrix X is analogous to solving the algebraic equation $ax = b$ for the number x.

Matrix multiplication has many applications. A modern application to computer graphics is discussed in *Focus on Modeling* (page 567). Matrix multiplication is

7.1 Matrices and Systems of Linear Equations

Matrices

We begin by defining the various elements that make up a matrix.

Definition of Matrix

An $m \times n$ **matrix** is a rectangular array of numbers with m **rows** and n **columns**.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \leftarrow \end{matrix} \Bigg\} \; m \text{ rows}$$

$$\underbrace{\uparrow \quad \uparrow \quad \uparrow \qquad\quad \uparrow}_{n \text{ columns}}$$

We say that the matrix has **dimension** $m \times n$. The numbers a_{ij} are the **entries** of the matrix. The subscript on the entry a_{ij} indicates that it is in the ith row and the jth column.

Here are some examples of matrices.

Matrix	Dimension	
$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$	2×3	2 rows by 3 columns
$\begin{bmatrix} 6 & -5 & 0 & 1 \end{bmatrix}$	1×4	1 row by 4 columns

The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the **augmented matrix** of the system, by writing only the coefficients and constants that appear in the equations. Here is an example.

Linear system

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

Example 1 Finding the Augmented Matrix of a Linear System

Write the augmented matrix of the system of equations.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

Solution

First we write the linear system with the variables lined up in columns.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ \phantom{6x - {}} 7y + z = 5 \end{cases}$$

The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

■

Elementary Row Operations

The operations that we used in Section 6.3 to solve linear systems correspond to operations on the rows of the augmented matrix of the system. For example, adding a multiple of one equation to another corresponds to adding a multiple of one row to another.

Elementary Row Operations

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Note that performing any of these operations on the augmented matrix of a system does not change its solution. We use the following notation to describe the elementary row operations:

Symbol	Description
$R_i + kR_j \to R_i$	Change the ith row by adding k times row j to it, and then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \leftrightarrow R_j$	Interchange the ith and jth rows.

In the next example we compare the two ways of writing systems of linear equations.

Example 2 Using Elementary Row Operations to Solve a Linear System

Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

Solution

Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison, we write both the system of equations and its augmented matrix.

	System		Augmented matrix
	$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$		$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
Add $(-1) \times$ Equation 1 to Equation 2. Add $(-3) \times$ Equation 1 to Equation 3.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - R_1 \to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$
Multiply Equation 3 by $\frac{1}{2}$.	$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{\frac{1}{2}R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Add $(-3) \times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).	$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases}$	$\xrightarrow{R_2 - 3R_3 \to R_2}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$
Interchange Equations 2 and 3.	$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases}$	$\xrightarrow{R_2 \leftrightarrow R_3}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Now we use back-substitution to find that $x = 2$, $y = 7$, and $z = 3$. The solution is $(2, 7, 3)$. ■

Gaussian Elimination

In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form. This form is described in the following box.

Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon form** if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

In the following matrices the first matrix is in reduced row-echelon form, but the second one is just in row-echelon form. The third matrix is not in row-echelon form. The entries in red are the leading entries.

Reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them.

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows.

Not in row-echelon form

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 7 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's do not shift to the right in successive rows.

Here is a systematic way to put a matrix in row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.
- At each stage make sure that every leading entry is to the right of the leading entry in the row above it—rearrange the rows if necessary.
- Continue this process until you arrive at a matrix in row-echelon form.

This is how the process might work for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \to \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \to \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution. This technique is called **Gaussian elimination**, in honor of its inventor, the German mathematician C. F. Gauss (see page 349).

Solving a System Using Gaussian Elimination

1. **Augmented Matrix.** Write the augmented matrix of the system.
2. **Row-Echelon Form.** Use elementary row operations to change the augmented matrix to row-echelon form.
3. **Back-Substitution.** Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.

Example 3 Solving a System Using Row-Echelon Form

Solve the system of linear equations using Gaussian elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution

We first write the augmented matrix of the system, and then use elementary row operations to put it in row-echelon form.

Need a 1 here.

$$\left[\begin{array}{rrrr} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need 0's here.

$$\xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{rrrr} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need a 1 here.

$$\xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{rrrr} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 0 here.

$$\xrightarrow{R_3 - 5R_2 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{array}\right]$$

Need a 1 here.

$$\xrightarrow{-\frac{1}{10}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We now have an equivalent matrix in row-echelon form, and the corresponding system of equations is

$$\begin{cases} x + 2y - z = 1 \\ y + 4z = -7 \\ z = -2 \end{cases}$$

We use back-substitution to solve the system.

$$\begin{aligned} y + 2(-2) &= -3 && \text{Back-substitute } z = -2 \text{ into Equation 2} \\ y &= 1 && \text{Solve for } y \\ x + 2(1) - (-2) &= 1 && \text{Back-substitute } y = 1 \text{ and } z = -2 \text{ into Equation 1} \\ x &= -3 && \text{Solve for } x \end{aligned}$$

So the solution of the system is $(-3, 1, -2)$. ■

Gauss-Jordan Elimination

If we put the augmented matrix of a linear system in *reduced* row-echelon form, then we don't need to back-substitute to solve the system. To put a matrix in reduced row-echelon form, we use the following steps.

- Use the elementary row operations to put the matrix in row-echelon form.
- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work up.

Here is how the process works for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Using the reduced row-echelon form to solve a system is called **Gauss-Jordan elimination**. The process is illustrated in the next example.

Example 4 Solving a System Using Reduced Row-Echelon Form

Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Solution

In Example 3 we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form. We continue using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Need 0's here.

$$\xrightarrow[R_1 + R_3 \to R_1]{R_2 - 4R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Need a 0 here.

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We now have an equivalent matrix in reduced row-echelon form, and the corresponding system of equations is

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

Hence we immediately arrive at the solution $(-3, 1, -2)$.

Inconsistent and Dependent Systems

The systems of linear equations that we considered in Examples 1–4 had exactly one solution. But as we know from Section 6.3 a linear system may have one solution, no solution, or infinitely many solutions. Fortunately, the row-echelon form of a system allows us to determine which of these cases applies, as described in the following box.

First we need some terminology. A **leading variable** in a linear system is one that corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

The Solutions of a Linear System in Row-Echelon Form

Suppose the augmented matrix of a system of linear equations has been transformed by Gaussian elimination into row-echelon form. Then exactly one of the following is true.

1. **No solution.** If the row-echelon form contains a row that represents the equation $0 = c$ where c is not zero, then the system has no solution. A system with no solution is called **inconsistent**.

2. **One solution.** If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which we find using back-substitution or Gauss-Jordan elimination.

3. **Infinitely many solutions.** If the variables in the row-echelon form are not all leading variables, and if the system is not inconsistent, then it has infinitely many solutions. In this case, the system is called **dependent**. We solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

The matrices below, all in row-echelon form, illustrate the three cases described in the box.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says $0 = 1$.

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable.

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable.

Example 5 A System with No Solution

Solve the system.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

Solution

We transform the system into row-echelon form.

$$\begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{bmatrix} \xrightarrow{\frac{1}{18}R_3} \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This last matrix is in row-echelon form, so we can stop the Gaussian elimination process. Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we pick for x, y, and z, the last equation will never be a true statement. This means the system *has no solution*. ■

Figure 3 shows the row-echelon form produced by a TI-83 calculator for the augmented matrix in Example 5. You should check that this gives the same solution.

Example 6 A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x \qquad\quad + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

Solution

We transform the system into reduced row-echelon form.

$$\begin{bmatrix} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{bmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. Since the equation adds no new information about the variables, we can drop it from the system. So the last matrix corresponds to the system

$$\begin{cases} x \qquad\quad - 7z = -5 & \text{Equation 1} \\ \qquad y - 3z = \;\;\, 1 & \text{Equation 2} \end{cases}$$

Leading variables

Now we solve for the leading variables x and y in terms of the nonleading variable z:

$$x = 7z - 5 \qquad \text{Solve for x in Equation 1}$$

$$y = 3z + 1 \qquad \text{Solve for y in Equation 2}$$

To obtain the complete solution, we let t represent any real number, and we express x, y, and z in terms of t:

$$x = 7t - 5$$

$$y = 3t + 1$$

$$z = t$$

We can also write the solution as the ordered triple $(7t - 5, 3t + 1, t)$, where t is any real number. ■

In Example 6, to get specific solutions we give a specific value to t. For example, if $t = 1$, then

$$x = 7(1) - 5 = 2$$
$$y = 3(1) + 1 = 4$$
$$z = 1$$

Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	Solution $(7t - 5, 3t + 1, t)$
-1	$(-12, -2, -1)$
0	$(-5, 1, 0)$
2	$(9, 7, 2)$
5	$(30, 16, 5)$

7.1 Exercises

1–6 ■ State the dimension of the matrix.

1. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 5 & 4 & 0 \\ 0 & 2 & 11 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 12 \\ 35 \end{bmatrix}$

4. $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7–14 ■ A matrix is given.

(a) Determine whether the matrix is in row-echelon form.

(b) Determine whether the matrix is in reduced row-echelon form.

(c) Write the system of equations for which the given matrix is the augmented matrix.

7. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

15–24 ■ The system of linear equations has a unique solution. Find the solution using Gaussian elimination or Gauss-Jordan elimination.

15. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$

16. $\begin{cases} x + y + 6z = 3 \\ x + y + 3z = 3 \\ x + 2y + 4z = 7 \end{cases}$

17. $\begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$

18. $\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y = -7 \end{cases}$

19. $\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$

20. $\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$

21. $\begin{cases} x_1 + 2x_2 - x_3 = 9 \\ 2x_1 - x_3 = -2 \\ 3x_1 + 5x_2 + 2x_3 = 22 \end{cases}$

22. $\begin{cases} 2x_1 + x_2 = 7 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$

23. $\begin{cases} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{cases}$

24. $\begin{cases} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{cases}$

25–34 ■ Determine whether the system of linear equations is inconsistent or dependent. If it is dependent, find the complete solution.

25. $$\begin{cases} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{cases}$$

26. $$\begin{cases} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{cases}$$

27. $$\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$$

28. $$\begin{cases} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y + 20z = -26 \end{cases}$$

29. $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$

30. $\begin{cases} -2x + 6y - 2z = -12 \\ x - 3y + 2z = 10 \\ -x + 3y + 2z = 6 \end{cases}$

31. $\begin{cases} x + 4y - 2z = -3 \\ 2x - y + 5z = 12 \\ 8x + 5y + 11z = 30 \end{cases}$

32. $\begin{cases} 3r + 2s - 3t = 10 \\ r - s - t = -5 \\ r + 4s - t = 20 \end{cases}$

33. $\begin{cases} 2x + y - 2z = 12 \\ -x - \frac{1}{2}y + z = -6 \\ 3x + \frac{3}{2}y - 3z = 18 \end{cases}$

34. $\begin{cases} y - 5z = 7 \\ 3x + 2y = 12 \\ 3x + 10z = 80 \end{cases}$

35–46 ■ Solve the system of linear equations.

35. $\begin{cases} 4x - 3y + z = -8 \\ -2x + y - 3z = -4 \\ x - y + 2z = 3 \end{cases}$

36. $\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$

37. $\begin{cases} x + 2y - 3z = -5 \\ -2x - 4y - 6z = 10 \\ 3x + 7y - 2z = -13 \end{cases}$

38. $\begin{cases} 3x - y + 2z = -1 \\ 4x - 2y + z = -7 \\ -x + 3y - 2z = -1 \end{cases}$

7.2 The Algebra of Matrices

Equality of Matrices

Two matrices are equal if they have the same entries in the same positions.

Equality of Matrices

The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they have the same dimension $m \times n$, and corresponding entries are equal, that is,

$$a_{ij} = b_{ij}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example 1 Equal Matrices

Find a, b, c, and d, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

Solution

Since the two matrices are equal, corresponding entries must be the same. So we must have $a = 1$, $b = 3$, $c = 5$, and $d = 2$. ■

Addition, Subtraction, and Scalar Multiplication of Matrices

Two matrices can be added or subtracted if they have the same dimension. (Otherwise, their sum or difference is undefined.) We add or subtract the matrices by adding or subtracting corresponding entries. To multiply a matrix by a number, we multiply every element of the matrix by that number. This is called the *scalar product*.

Sum, Difference, and Scalar Product of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension $m \times n$, and let c be any real number.

1. The **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B.

$$A + B = [a_{ij} + b_{ij}]$$

2. The **difference** $A - B$ is the $m \times n$ matrix obtained by subtracting corresponding entries of A and B.

$$A - B = [a_{ij} - b_{ij}]$$

3. The **scalar product** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c.

$$cA = [ca_{ij}]$$

Example 2 Performing Algebraic Operations on Matrices

Let

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

Carry out each indicated operation, or explain why it cannot be performed

(a) $A + B$ (b) $C - D$ (c) $C + A$ (d) $5A$

Solution

(a) $A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$

(b) $C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$

(c) $C + A$ is undefined because we can't add matrices of different dimensions.

(d) $5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$ ■

Properties of Addition and Scalar Multiplication of Matrices

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = cdA$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$ $c(A + B) = cA + cB$	Distributive Properties of Scalar Multiplication

Example 3 Solving a Matrix Equation

Solve the matrix equation

$$2X - A = B$$

for the unknown matrix X, where

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

Solution

We use the properties of matrices to solve for X.

$$2X - A = B \qquad \text{Given equation}$$

$$2X = B + A \qquad \text{Add the matrix } A \text{ to each side}$$

$$X = \tfrac{1}{2}(B + A) \qquad \text{Multiply each side by the scalar } \tfrac{1}{2}$$

So

$$X = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}\right) \qquad \text{Substitute the matrices } A \text{ and } B$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 2 \\ -4 & 4 \end{bmatrix} \qquad \text{Add matrices}$$

$$= \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} \qquad \text{Multiply by the scalar } \tfrac{1}{2}$$

■

Multiplication of Matrices

Multiplying two matrices is more difficult to describe than other matrix operations. In later examples we will see why taking the matrix product involves a rather complex procedure, which we now describe.

First, the product AB (or $A \cdot B$) of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. This means that if we write their dimensions side by side, the two inner numbers must match:

Matrices	A	B
Dimensions	$m \times n$	$n \times k$

Columns in A   Rows in B

If the dimensions of A and B match in this fashion, then the product AB is a matrix of dimension $m \times k$. Before describing the procedure for obtaining the elements of AB, we define the *inner product* of a row of A and a column of B.

If $[a_1 \quad a_2 \quad \cdots \quad a_n]$ is a row of A, and if $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a column of B, then

their **inner product** is the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. For example, taking

the inner product of $[2 \quad -1 \quad 0 \quad 4]$ and $\begin{bmatrix} 5 \\ 4 \\ -3 \\ \frac{1}{2} \end{bmatrix}$ gives

$$2 \cdot 5 + (-1) \cdot 4 + 0 \cdot (-3) + 4 \cdot \tfrac{1}{2} = 8$$

We now define the **product** AB of two matrices.

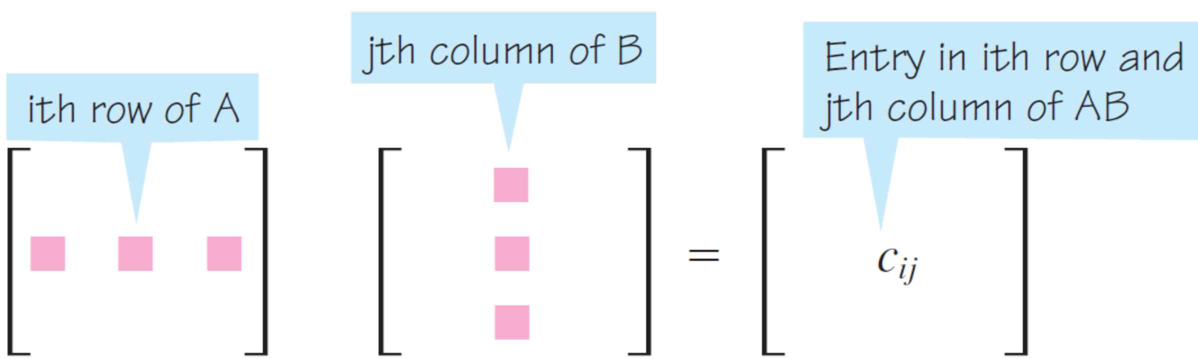
Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times k$ matrix, then their product is the $m \times k$ matrix

$$C = [c_{ij}]$$

where c_{ij} is the inner product of the ith row of A and the jth column of B. We write the product as

This definition of matrix product says that each entry in the matrix AB is obtained from a *row* of A and a *column* of B as follows: The entry c_{ij} in the ith row and jth column of the matrix AB is obtained by multiplying the entries in the ith row of A with the corresponding entries in the jth column of B and adding the results.



Example 4 Multiplying Matrices

Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

Calculate, if possible, the products AB and BA.

Solution

Since A has dimension 2×2 and B has dimension 2×3, the product AB is defined and has dimension 2×3. We can thus write

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in using the rule defining the product of two matrices. If we define $C = AB = [c_{ij}]$, then the entry c_{11} is the inner product of the first row of A and the first column of B:

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1 \cdot (-1) + 3 \cdot 0 = -1$$

Similarly, we calculate the remaining entries of the product as follows.

Entry	Inner product of:	Value	Product matrix
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 5 + 3 \cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 2 + 3 \cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot (-1) + 0 \cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 5 + 0 \cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 2 + 0 \cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Inner numbers match, so product is defined.

$2 \times 2 \quad 2 \times 3$

Outer numbers give dimension of product: 2 X 3.

Thus, we have $$AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$$

The product BA is not defined, however, because the dimensions of B and A are

$$2 \times 3 \qquad \text{and} \qquad 2 \times 2$$

The inner two numbers are not the same, so the rows and columns won't match up when we try to calculate the product. ■

Not equal, so product not defined.

$2 \times 3 \quad 2 \times 2$

Figure 1.

Properties of Matrix Multiplication

Although matrix multiplication is not commutative, it does obey the Associative and Distributive Properties.

Properties of Matrix Multiplication

Let A, B, and C be matrices for which the following products are defined. Then

$A(BC) = (AB)C$	Associative Property
$A(B + C) = AB + AC$ $(B + C)A = BA + CA$	Distributive Property

Example 5 Matrix Multiplication Is Not Commutative

Let $$A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$$

Calculate the products AB and BA.

Solution

Since both matrices A and B have dimension 2×2, both products AB and BA are defined, and each product is also a 2×2 matrix.

$$AB = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 7 \cdot 9 & 5 \cdot 2 + 7 \cdot (-1) \\ (-3) \cdot 1 + 0 \cdot 9 & (-3) \cdot 2 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot (-3) & 1 \cdot 7 + 2 \cdot 0 \\ 9 \cdot 5 + (-1) \cdot (-3) & 9 \cdot 7 + (-1) \cdot 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix}$$

This shows that, in general, $AB \neq BA$. In fact, in this example AB and BA don't even have an entry in common. ■

Applications of Matrix Multiplication

Example 6 Writing a Linear System as a Matrix Equation

Show that the following matrix equation is equivalent to the system of equations in Example 2 of Section 7.1.

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Solution

If we perform matrix multiplication on the left side of the equation, we get

$$\begin{bmatrix} x - y + 3z \\ x + 2y - 2z \\ 3x - y + 5z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Because two matrices are equal only if their corresponding entries are equal, we equate entries to get

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

7.2 Exercises

1–2 ■ Determine whether the matrices A and B are equal.

1. $A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} \frac{1}{4} & \ln 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 & 0 \\ \sqrt{4} & \frac{6}{2} \end{bmatrix}$

3–10 ■ Perform the matrix operation, or if it is impossible, explain why.

3. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

5. $3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$

6. $2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

11–16 ■ Solve the matrix equation for the unknown matrix or explain why no solution exists.

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

11. $2X + A = B$

12. $3X - B = C$

13. $2(B - X) = D$

14. $5(X - C) = D$

15. $\frac{1}{5}(X + D) = C$

16. $2A = B - 3X$

17–38 ■ The matrices A, B, C, D, E, F, and G are defined as follows.

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix}$$

17. $B + C$

19. $C - B$

21. $3B + 2C$

23. $2C - 6B$

25. AD

27. BF

18. $B + F$

20. $5A$

22. $C - 5A$

24. DA

26. BC

29. $(DA)B$

31. GE

33. A^3

35. B^2

37. $BF + FE$

28. GF

30. $D(AB)$

32. A^2

34. $DB + DC$

36. F^2

39–42 ■ Solve for x and y.

39. $\begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$

40. $3\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 6 \end{bmatrix}$

41. $2\begin{bmatrix} x & y \\ x + y & x - y \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

42. $\begin{bmatrix} x & y \\ -y & x \end{bmatrix} - \begin{bmatrix} y & x \\ x & -y \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -6 & 6 \end{bmatrix}$

43–46 ■ Write the system of equations as a matrix equation (see Example 6).

43. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$

44. $\begin{cases} 6x - y + z = 12 \\ 2x + z = 7 \\ y - 2z = 4 \end{cases}$

7.3 Inverses of Matrices and Matrix Equations

The Inverse of a Matrix

First, we define *identity matrices*, which play the same role for matrix multiplication as the number 1 does for ordinary multiplication of numbers; that is, $1 \cdot a = a \cdot 1 = a$ for all numbers a. In the following definition the term **main diagonal** refers to the entries of a square matrix whose row and column numbers are the same. These entries stretch diagonally down the matrix, from top left to bottom right.

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus, the 2×2, 3×3, and 4×4 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \qquad \text{and} \qquad I_n \cdot B = B$$

whenever these products are defined.

Example 1 Identity Matrices

The following matrix products show how multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$ ■

If A and B are $n \times n$ matrices, and if $AB = BA = I_n$, then we say that B is the *inverse* of A, and we write $B = A^{-1}$. The concept of the inverse of a matrix is analogous to that of the reciprocal of a real number.

Inverse of a Matrix

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the **inverse** of A.

Example 2 Verifying That a Matrix Is an Inverse

Verify that B is the inverse of A, where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Solution

We perform the matrix multiplications to show that $AB = I$ and $BA = I$:

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1(-5) & 2(-1) + 1 \cdot 2 \\ 5 \cdot 3 + 3(-5) & 5(-1) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + (-1)5 & 3 \cdot 1 + (-1)3 \\ (-5)2 + 2 \cdot 5 & (-5)1 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■

Finding the Inverse of a 2 × 2 Matrix

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

Inverse of a 2 × 2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If $ad - bc = 0$, then A has no inverse.

Example 3 Finding the Inverse of a 2 × 2 Matrix

Let

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Find A^{-1} and verify that $AA^{-1} = A^{-1}A = I_2$.

Solution

Using the rule for the inverse of a 2×2 matrix, we get

$$A^{-1} = \frac{1}{4 \cdot 3 - 5 \cdot 2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

To verify that this is indeed the inverse of A, we calculate AA^{-1} and $A^{-1}A$:

$$AA^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot \frac{3}{2} + 5(-1) & 4\left(-\frac{5}{2}\right) + 5 \cdot 2 \\ 2 \cdot \frac{3}{2} + 3(-1) & 2\left(-\frac{5}{2}\right) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \cdot 4 + \left(-\frac{5}{2}\right)2 & \frac{3}{2} \cdot 5 + \left(-\frac{5}{2}\right)3 \\ (-1)4 + 2 \cdot 2 & (-1)5 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ■

The quantity $ad - bc$ that appears in the rule for calculating the inverse of a 2×2 matrix is called the **determinant** of the matrix. If the determinant is 0, then the matrix does not have an inverse (since we cannot divide by 0).

Finding the Inverse of an $n \times n$ Matrix

For 3×3 and larger square matrices, the following technique provides the most efficient way to calculate their inverses. If A is an $n \times n$ matrix, we first construct the $n \times 2n$ matrix that has the entries of A on the left and of the identity matrix I_n on the right:

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

We then use the elementary row operations on this new large matrix to change the left side into the identity matrix. (This means that we are changing the large matrix to reduced row-echelon form.) The right side is transformed automatically into A^{-1}. (We omit the proof of this fact.)

Example 4 Finding the Inverse of a 3 × 3 Matrix

Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Verify that $AA^{-1} = A^{-1}A = I_3$.

Solution

(a) We begin with the 3×6 matrix whose left half is A and whose right half is the identity matrix.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

We then transform the left half of this new matrix into the identity matrix by performing the following sequence of elementary row operations on the *entire* new matrix:

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

We have now transformed the left half of this matrix into an identity matrix. (This means we've put the entire matrix in reduced row-echelon form.) Note that to do this in as systematic a fashion as possible, we first changed the elements below the main diagonal to zeros, just as we would if we were using Gaussian elimination. We then changed each main diagonal element to a 1 by multiplying by the appropriate constant(s). Finally, we completed the process by changing the remaining entries on the left side to zeros.

The right half is now A^{-1}.

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) We calculate AA^{-1} and $A^{-1}A$, and verify that both products give the identity matrix I_3.

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 5 A Matrix That Does Not Have an Inverse

Find the inverse of the matrix.

$$\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution

We proceed as follows.

$$\left[\begin{array}{ccc|ccc} 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & -7 & -21 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

Matrix Equations

We saw in Example 6 in Section 7.2 that a system of linear equations can be written as a single matrix equation. For example, the system

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}_{B}$$

If we let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

then this matrix equation can be written as

$$AX = B$$

The matrix A is called the **coefficient matrix**.

We solve this matrix equation by multiplying each side by the inverse of A (provided this inverse exists):

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \text{Multiply on left by } A^{-1}$$

$$(A^{-1}A)X = A^{-1}B \qquad \text{Associative Property}$$

$$I_3X = A^{-1}B \qquad \text{Property of inverses}$$

$$X = A^{-1}B \qquad \text{Property of identity matrix}$$

Solving the matrix equation $AX = B$ is very similar to solving the simple real-number equation

$$3x = 12$$

which we do by multiplying each side by the reciprocal (or inverse) of 3:

$$\tfrac{1}{3}(3x) = \tfrac{1}{3}(12)$$

$$x = 4$$

In Example 4 we showed that

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

So, from $X = A^{-1}B$ we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -11 \\ -23 \\ 7 \end{bmatrix}$$

$$X \quad = \quad A^{-1} \quad B$$

Thus, $x = -11$, $y = -23$, $z = 7$ is the solution of the original system.

We have proved that the matrix equation $AX = B$ can be solved by the following

Solving a Matrix Equation

If A is a square $n \times n$ matrix that has an inverse A^{-1}, and if X is a variable matrix and B a known matrix, both with n rows, then the solution of the matrix equation

$$AX = B$$

is given by

$$X = A^{-1}B$$

Example 6 Solving a System Using a Matrix Inverse

(a) Write the system of equations as a matrix equation.

(b) Solve the system by solving the matrix equation.

$$\begin{cases} 2x - 5y = 15 \\ 3x - 6y = 36 \end{cases}$$

$$3x - 6y = 36$$

Solution

(a) We write the system as a matrix equation of the form $AX = B$:

$$\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 36 \end{bmatrix}$$

A X = B

(b) Using the rule for finding the inverse of a 2×2 matrix, we get

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}^{-1} = \frac{1}{2(-6) - (-5)3} \begin{bmatrix} -6 & -(-5) \\ -3 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}$$

Multiplying each side of the matrix equation by this inverse matrix, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 36 \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \end{bmatrix}$$

X = A^{-1} B

So $x = 30$ and $y = 9$.

7.3 Exercises

1–4 ■ Calculate the products AB and BA to verify that B is the inverse of A.

1. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

5–6 ■ Find the inverse of the matrix and verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$.

5. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

6. $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$

7–22 ■ Find the inverse of the matrix if it exists.

7. $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

10. $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

11. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

12. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

23–30 ■ Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix, as in Example 6. Use the inverses from Exercises 7–10, 15, 16, 19, and 21.

23. $\begin{cases} 5x + 3y = 4 \\ 3x + 2y = 0 \end{cases}$

24. $\begin{cases} 3x + 4y = 10 \\ 7x + 9y = 20 \end{cases}$

25. $\begin{cases} 2x + 5y = 2 \\ -5x - 13y = 20 \end{cases}$

26. $\begin{cases} -7x + 4y = 0 \\ 8x - 5y = 100 \end{cases}$

27. $\begin{cases} 2x + 4y + z = 7 \\ -x + y - z = 0 \\ x + 4y = -2 \end{cases}$

28. $\begin{cases} 5x + 7y + 4z = 1 \\ 3x - y + 3z = 1 \\ 6x + 7y + 5z = 1 \end{cases}$

7.4 Determinants and Cramer's Rule

If a matrix is **square** (that is, if it has the same number of rows as columns), then
we can assign to it a number called its *determinant*.

Determinant of a 2 × 2 Matrix

We denote the determinant of a square matrix A by the symbol det(A) or $|A|$.

Determinant of a 2 × 2 Matrix

The **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1 Determinant of a 2 × 2 Matrix

Evaluate $|A|$ for $A = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$.

Solution

$$\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix} = 6 \cdot 3 - (-3)2 = 18 - (-6) = 24$$

We will use both notations, $\det(A)$ and $|A|$, for the determinant of A. Although the symbol $|A|$ looks like the absolute value symbol, it will be clear from the context which meaning is intended.

To evaluate a 2×2 determinant, we take the product of the diagonal from top left to bottom right, and subtract the product from top right to bottom left, as indicated by the arrows.

Determinant of an $n \times n$ Matrix

To define the concept of determinant for an arbitrary $n \times n$ matrix, we need the following terminology.

Let A be an $n \times n$ matrix.

1. The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

2. The **cofactor** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

then the minor M_{12} is the determinant of the matrix obtained by deleting the first row and second column from A. Thus

$$M_{12} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} = 0(6) - 4(-2) = 8$$

So, the cofactor $A_{12} = (-1)^{1+2} M_{12} = -8$. Similarly

$$M_{33} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 0 = 4$$

So, $A_{33} = (-1)^{3+3} M_{33} = 4$.

Note that the cofactor of a_{ij} is simply the minor of a_{ij} multiplied by either 1 or -1, depending on whether $i + j$ is even or odd. Thus, in a 3×3 matrix we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The Determinant of a Square Matrix

If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor, and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Example 2 Determinant of a 3 × 3 Matrix

Evaluate the determinant of the matrix.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$$= 2(2 \cdot 6 - 4 \cdot 5) - 3[0 \cdot 6 - 4(-2)] - [0 \cdot 5 - 2(-2)]$$

$$= -16 - 24 - 4$$

$$= -44$$

■

In our definition of the determinant we used the cofactors of elements in the first row only. This is called **expanding the determinant by the first row**

Example 3 Expanding a Determinant about a Row and a Column

Let A be the matrix of Example 2. Evaluate the determinant of A by expanding

(a) by the second row

(b) by the third column

Verify that each expansion gives the same value.

Solution

(a) Expanding by the second row, we get

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & -4 \\ -2 & 5 & 6 \end{vmatrix} = -0 \begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 0 + 2[2 \cdot 6 - (-1)(-2)] - 4[2 \cdot 5 - 3(-2)]$$

$$= 0 + 20 - 64 = -44$$

Invertibility Criterion

If A is a square matrix, then A has an inverse if and only if $\det(A) \neq 0$.

Row and Column Transformations

The preceding example shows that if we expand a determinant about a row or column that contains many zeros, our work is reduced considerably because we don't have to evaluate the cofactors of the elements that are zero. The following principle often simplifies the process of finding a determinant by introducing zeros into it without changing its value.

Row and Column Transformations of a Determinant

If A is a square matrix, and if the matrix B is obtained from A by adding a multiple of one row to another, or a multiple of one column to another, then $\det(A) = \det(B)$.

Example 5 Using Row and Column Transformations to Calculate a Determinant

Find the determinant of the matrix A. Does it have an inverse?

$$A = \begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 24 & 6 & 1 & -12 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

Solution

If we add -3 times row 1 to row 3, we change all but one element of row 3 to zeros:

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

This new matrix has the same determinant as A, and if we expand its determinant by the third row, we get

$$\det(A) = 4\begin{vmatrix} 8 & 2 & -4 \\ 3 & 5 & 11 \\ 2 & 2 & -1 \end{vmatrix}$$

Now, adding 2 times column 3 to column 1 in this determinant gives us

$$\det(A) = 4\begin{vmatrix} 0 & 2 & -4 \\ 25 & 5 & 11 \\ 0 & 2 & -1 \end{vmatrix} \qquad \text{Expand this by column 1}$$

$$= 4(-25)\begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix}$$

$$= 4(-25)[2(-1) - (-4)2] = -600$$

Since the determinant of A is not zero, A does have an inverse. ■

Cramer's Rule

The solutions of linear equations can sometimes be expressed using determinants. To illustrate, let's solve the following pair of linear equations for the variable x.

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

To eliminate the variable y, we multiply the first equation by d and the second by b, and subtract.

$$\begin{array}{r} adx + bdy = rd \\ bcx + bdy = bs \\ \hline adx - bcx = rd - bs \end{array}$$

Factoring the left-hand side, we get $(ad - bc)x = rd - bs$. Assuming that $ad - bc \neq 0$, we can now solve this equation for x:

$$x = \frac{rd - bs}{ad - bc}$$

Similarly, we find

$$y = \frac{as - cr}{ad - bc}$$

The numerator and denominator of the fractions for x and y are determinants of 2×2 matrices. So we can express the solution of the system using determinants as follows.

Cramer's Rule for Systems in Two Variables

The linear system

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

has the solution

$$x = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$

Using the notation

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad D_x = \begin{bmatrix} r & b \\ s & d \end{bmatrix} \qquad D_y = \begin{bmatrix} a & r \\ c & s \end{bmatrix}$$

Coefficient matrix

Replace first column of D by r and s.

Replace second column of D by r and s.

we can write the solution of the system as

$$x = \frac{|D_x|}{|D|} \quad \text{and} \quad y = \frac{|D_y|}{|D|}$$

Example 6 Using Cramer's Rule to Solve a System with Two Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$$

Solution

For this system we have

$$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 \cdot 8 - 6 \cdot 1 = 10$$

$$|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)8 - 6 \cdot 2 = -20$$

$$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)1 = 5$$

The solution is

$$x = \frac{|D_x|}{|D|} = \frac{-20}{10} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{5}{10} = \frac{1}{2}$$

■

Cramer's Rule can be extended to apply to any system of n linear equations in n variables in which the determinant of the coefficient matrix is not zero. As we saw in the preceding section, any such system can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

By analogy with our derivation of Cramer's Rule in the case of two equations in two unknowns, we let D be the coefficient matrix in this system, and D_{x_i} be the matrix obtained by replacing the ith column of D by the numbers $b_1, b_2, \ldots, b_n$ that appear to the right of the equal sign. The solution of the system is then given by the following rule.

Cramer's Rule

If a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$, and if $|D| \neq 0$, then its solutions are

$$x_1 = \frac{|D_{x_1}|}{|D|}, \quad x_2 = \frac{|D_{x_2}|}{|D|}, \quad \ldots, \quad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the $n \times 1$ matrix B.

Example 7 Using Cramer's Rule to Solve a System with Three Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - 3y + 4z = 1 \\ x \qquad\quad + 6z = 0 \\ 3x - 2y \qquad\quad = 5 \end{cases}$$

Solution

First, we evaluate the determinants that appear in Cramer's Rule. Note that D is the coefficient matrix and that D_x, D_y, and D_z are obtained by replacing the first, second, and third columns of D by the constant terms.

$$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38 \qquad |D_x| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$$

$$|D_y| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22 \qquad |D_z| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$$

Now we use Cramer's Rule to get the solution:

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19}$$

$$y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19}$$

$$z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

7.4 Exercises

1–8 ■ Find the determinant of the matrix, if it exists.

1. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 1 & \frac{1}{2} \end{bmatrix}$

8. $\begin{bmatrix} 2.2 & -1.4 \\ 0.5 & 1.0 \end{bmatrix}$

9–14 ■ Evaluate the minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

9. M_{11}, A_{11}

10. M_{33}, A_{33}

11. M_{12}, A_{12}

12. M_{13}, A_{13}

13. M_{23}, A_{23}

14. M_{32}, A_{32}

15–22 ■ Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

15. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -1 \\ 0 & 2 & 6 \end{bmatrix}$

18. $\begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 30 & 0 & 20 \\ 0 & -10 & -20 \\ 40 & 0 & 10 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

28. Consider the system

$$\begin{cases} x + 2y + 6z = 5 \\ -3x - 6y + 5z = 8 \\ 2x + 6y + 9z = 7 \end{cases}$$

(a) Verify that $x = -1$, $y = 0$, $z = 1$ is a solution of the system.

(b) Find the determinant of the coefficient matrix.

(c) Without solving the system, determine whether there are any other solutions.

(d) Can Cramer's Rule be used to solve this system? Why or why not?

29–44 ■ Use Cramer's Rule to solve the system.

29. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

30. $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$

31. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$

32. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{6}y = -\frac{3}{2} \end{cases}$

33. $\begin{cases} 0.4x + 1.2y = 0.4 \\ 1.2x + 1.6y = 3.2 \end{cases}$

34. $\begin{cases} 10x - 17y = 21 \\ 20x - 31y = 39 \end{cases}$

35. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$

36. $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

37. $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_2 + x_3 = 8 \end{cases}$

38. $\begin{cases} -2a + c = 2 \\ a + 2b - c = 9 \\ 3a + 5b + 2c = 22 \end{cases}$

39. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y + \frac{1}{2}z = \frac{7}{10} \\ -\frac{2}{3}x + \frac{2}{5}y + \frac{3}{2}z = \frac{11}{10} \\ x - \frac{4}{5}y + z = \frac{9}{5} \end{cases}$

40. $\begin{cases} 2x - y = 5 \\ 5x + 3z = 19 \\ 4y + 7z = 17 \end{cases}$

41. $\begin{cases} 3y + 5z = 4 \\ 2x - z = 10 \\ 4x + 7y = 0 \end{cases}$

42. $\begin{cases} 2x - 5y = 4 \\ x + y - z = 8 \\ 3x + 5z = 0 \end{cases}$

EXAMPLE 1 Deciding Whether an Ordered Pair Is a Solution

Decide whether the given ordered pair is a solution of the system.

(a) $\begin{aligned} x + y &= 6 \\ 4x - y &= 14 \end{aligned}$; $(4, 2)$

Replace x with 4 and y with 2 in each equation of the system.

$$\begin{aligned} x + y &= 6 \\ 4 + 2 &= 6 \quad ? \\ 6 &= 6 \quad \text{True} \end{aligned} \qquad\qquad \begin{aligned} 4x - y &= 14 \\ 4(4) - 2 &= 14 \quad ? \\ 14 &= 14 \quad \text{True} \end{aligned}$$

(b) $\begin{aligned} 3x + 2y &= 11 \\ x + 5y &= 36 \end{aligned}$; $(-1, 7)$

$$\begin{aligned} 3x + 2y &= 11 \\ 3(-1) + 2(7) &= 11 \quad ? \\ -3 + 14 &= 11 \\ 11 &= 11 \quad \text{True} \end{aligned} \qquad\qquad \begin{aligned} x + 5y &= 36 \\ -1 + 5(7) &= 36 \quad ? \\ -1 + 35 &= 36 \\ 34 &= 36 \quad \text{False} \end{aligned}$$

The ordered pair $(-1, 7)$ is not a solution of the system, since it does not make *both* equations true.

EXAMPLE 2 Solving a System by Graphing

Solve the system of equations by graphing.

$$x + y = 5 \qquad (1)$$
$$2x - y = 4 \qquad (2)$$

When we graph these linear equations as shown in Figure 2, the graph suggests that the point of intersection is the ordered pair (3, 2).

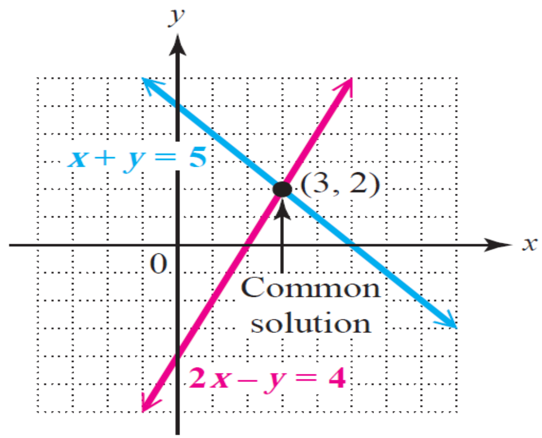


FIGURE 2

To be sure that (3, 2) is a solution of *both* equations, we check by substituting 3 for x and 2 for y in each equation.

$x + y = 5$ (1)	$2x - y = 4$ (2)
$3 + 2 = 5$?	$2(3) - 2 = 4$?
$5 = 5$ True	$6 - 2 = 4$?
	$4 = 4$ True

Since (3, 2) makes both equations true, {(3, 2)} is the solution set of the system.

There are three possibilities for the solution set of a linear system in two variables.

Graphs of Linear Systems in Two Variables

1. The two graphs intersect in a single point. The coordinates of this point give the only solution of the system. In this case the system is **consistent,** and the equations are **independent.** This is the most common case. See Figure 3(a).
2. The graphs are parallel lines. In this case the system is **inconsistent;** that is, there is no solution common to both equations of the system, and the solution set is ∅. See Figure 3(b).
3. The graphs are the same line. In this case the equations are **dependent,** since any solution of one equation of the system is also a solution of the other. The solution set is an infinite set of ordered pairs representing the points on the line. See Figure 3(c).

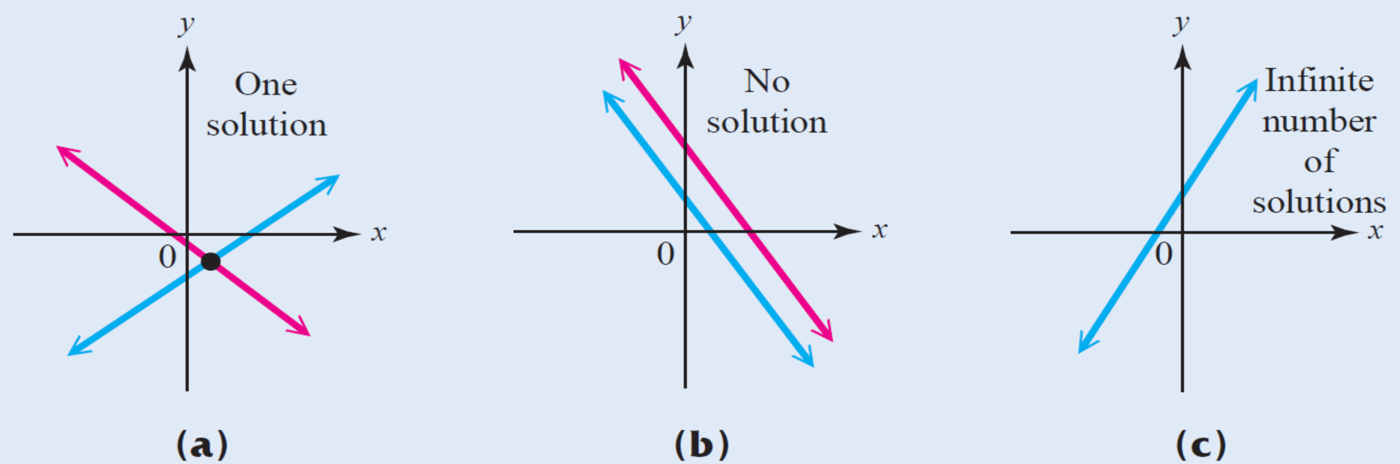


FIGURE 3

OBJECTIVE 3 Solve linear systems (with two equations and two variables) by elimination. While it is possible to find the solution of a system of equations by graphing, it can be difficult to read exact coordinates, especially if they are not integers, from a graph. Because of this, we usually use algebraic methods to solve systems. One such method, called the **elimination method,** involves combining the two equations of the system so that one variable is *eliminated*. This is done using the following fact.

$$\text{If} \quad a = b \text{ and } c = d, \quad \text{then} \quad a + c = b + d.$$

The general method of solving a system by the elimination method is summarized as follows.

Solving a Linear System by Elimination

Step 1 **Write both equations in standard form $Ax + By = C$.**

Step 2 **Make the coefficients of one pair of variable terms opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the x- or y-terms is 0.

Step 3 **Add** the new equations to eliminate a variable. The sum should be an equation with just one variable.

Step 4 **Solve** the equation from Step 3 for the remaining variable.

Step 5 **Find the other value.** Substitute the result of Step 4 into either of the original equations and solve for the other variable.

Step 6 **Check** the solution in both of the original equations. Then write the solution set.

EXAMPLE 3 Solving a System by Elimination

Solve the system

$$5x - 2y = 4 \qquad (1)$$
$$2x + 3y = 13. \qquad (2)$$

Step 1 Both equations are in standard form.

Step 2 Suppose that you wish to eliminate the variable x. One way to do this is to multiply equation (1) by 2 and equation (2) by -5.

$$10x - 4y = 8 \qquad \text{2 times each side of equation (1)}$$
$$-10x - 15y = -65 \qquad \text{−5 times each side of equation (2)}$$

Step 3 Now add.

$$\begin{array}{rcl} 10x - 4y &=& 8 \\ -10x - 15y &=& -65 \\ \hline -19y &=& -57 \end{array}$$

Step 4 Solve for y. $y = 3$ Divide by -19.

Step 5 To find x, substitute 3 for y in either equation (1) or (2). Substituting in equation (2) gives

$$2x + 3y = 13 \qquad (2)$$
$$2x + 3(3) = 13 \qquad \text{Let } y = 3.$$
$$2x + 9 = 13$$
$$2x = 4 \qquad \text{Subtract 9.}$$
$$x = 2. \qquad \text{Divide by 2.}$$

Step 6 The solution is (2, 3). To check, substitute 2 for x and 3 for y in both equations (1) and (2).

$5x - 2y = 4$	(1)	$2x + 3y = 13$	(2)
$5(2) - 2(3) = 4$	?	$2(2) + 3(3) = 13$	?
$10 - 6 = 4$	?	$4 + 9 = 13$	?
$4 = 4$	True	$13 = 13$	True

The solution set is $\{(2, 3)\}$.

EXAMPLE 4 Solving a System with Fractional Coefficients

Solve the system

$$5x - 2y = 4 \qquad (1)$$

$$\frac{1}{2}x + \frac{3}{4}y = \frac{13}{4}. \qquad (2)$$

If an equation in a system has fractional coefficients, as in equation (2), first multiply by the least common denominator to clear the fractions.

$$4\left(\frac{1}{2}x + \frac{3}{4}y\right) = 4 \cdot \frac{13}{4} \qquad \text{Multiply equation (2) by the LCD, 4.}$$

$$4 \cdot \frac{1}{2}x + 4 \cdot \frac{3}{4}y = 4 \cdot \frac{13}{4} \qquad \text{Distributive property}$$

$$2x + 3y = 13 \qquad \text{Equivalent to equation (2)}$$

The system of equations becomes

$$5x - 2y = 4 \qquad (1)$$

$$2x + 3y = 13, \qquad \text{Equation (2) with fractions cleared}$$

which is identical to the system we solved in Example 3. The solution set is $\{(2, 3)\}$. To confirm this, check the solution in both equations (1) and (2).

OBJECTIVE 4 Solve special systems. As we saw in Figures 3(b) and (c), some systems of linear equations have no solution or an infinite number of solutions. Examples 5 and 6 show how to recognize these systems when solving algebraically.

EXAMPLE 5 Solving a System of Dependent Equations

Solve the system

$$\begin{aligned} 2x - y &= 3 \qquad (1) \\ 6x - 3y &= 9. \qquad (2) \end{aligned}$$

We multiply equation (1) by -3, and then add the result to equation (2).

$$\begin{aligned} -6x + 3y &= -9 \qquad -3 \text{ times each side of equation (1)} \\ 6x - 3y &= 9 \qquad (2) \\ \hline 0 &= 0 \qquad \text{True} \end{aligned}$$

Adding these equations gives the true statement $0 = 0$. In the original system, we could get equation (2) from equation (1) by multiplying equation (1) by 3. Because of this, equations (1) and (2) are equivalent and have the same graph, as shown in Figure 4. The equations are dependent. The solution set is the set of all points on the line with equation $2x - y = 3$, written

and read "the set of all ordered pairs (x, y), such that $2x - y = 3$."

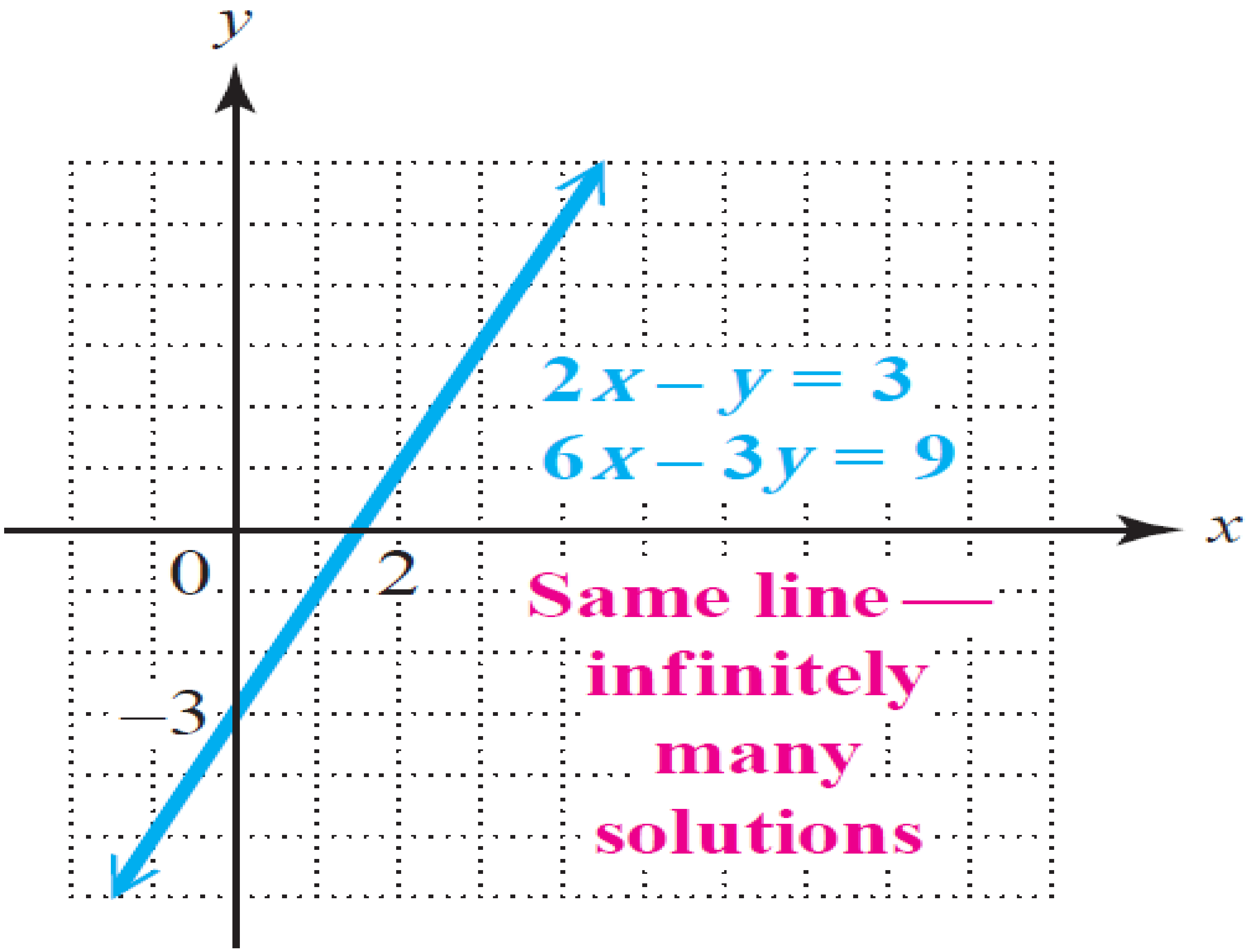


FIGURE 4

EXAMPLE 6 Solving an Inconsistent System

Solve the system

$$x + 3y = 4 \qquad (1)$$
$$-2x - 6y = 3. \qquad (2)$$

Multiply equation (1) by 2, and then add the result to equation (2).

$$\begin{aligned} 2x + 6y &= 8 &&\text{Equation (1) multiplied by 2} \\ -2x - 6y &= 3 &&(2) \\ \hline 0 &= 11 &&\text{False} \end{aligned}$$

The result of the addition step is a false statement, which indicates that the system is inconsistent. As shown in Figure 5, the graphs of the equations of the system are parallel lines. There are no ordered pairs that satisfy both equations, so there is no solution for the svstem: the solution set is Ø.

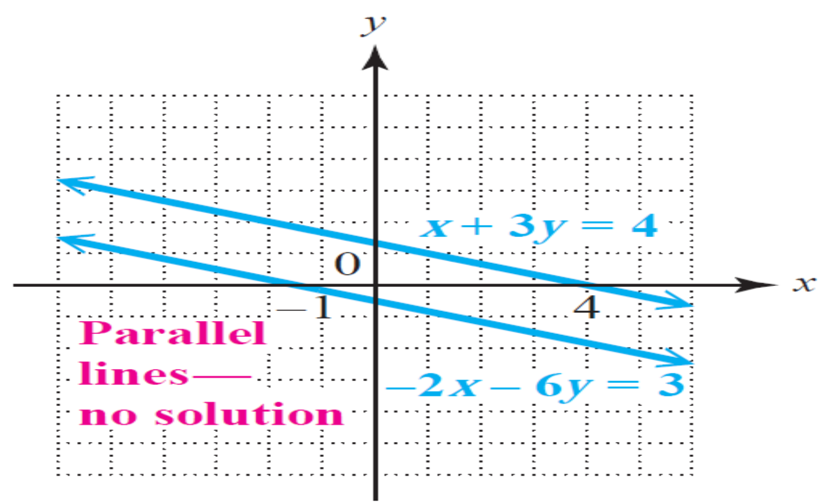


FIGURE 5

Special Cases of Linear Systems

If both variables are eliminated when a system of linear equations is solved,

1. there are infinitely many solutions if the resulting statement is *true;*

2. there is no solution if the resulting statement is *false.*

Slopes and y-intercepts can be used to decide if the graphs of a system of equations are parallel lines or if they coincide. In Example 5, writing each equation in slope-intercept form shows that both lines have slope 2 and y-intercept $(0, -3)$, so the graphs are the same line and the system has an infinite solution set.

In Example 6, both equations have slope $-\frac{1}{3}$ but y-intercepts $\left(0, \frac{4}{3}\right)$ and $\left(0, -\frac{1}{2}\right)$, showing that the graphs are two distinct parallel lines. Thus, the system has no solution

solution.

OBJECTIVE 5 Solve linear systems (with two equations and two variables) by substitution. Linear systems can also be solved algebraically by the **substitution method.** This method is most useful for solving linear systems in which one variable has coefficient 1 or -1.

The substitution method is summarized as follows.

Solving a Linear System by Substitution

Step 1 **Solve one of the equations for either variable.** If one of the variable terms has coefficient 1 or -1, choose it since the substitution method is usually easier this way.

Step 2 **Substitute** for that variable in the other equation. The result should be an equation with just one variable.

Step 3 **Solve** the equation from Step 2.

Step 4 **Find the other value.** Substitute the result from Step 3 into the equation from Step 1 to find the value of the other variable.

Step 5 **Check** the solution in both of the original equations. Then write the solution set.

EXAMPLE 7 Solving a System by Substitution

Solve the system

$$3x + 2y = 13 \qquad (1)$$
$$4x - y = -1. \qquad (2)$$

Step 1 To use the substitution method, first solve one of the equations for either x or y. Since the coefficient of y in equation (2) is -1, it is easiest to solve for y in equation (2).

$$4x - y = -1 \qquad (2)$$
$$-y = -1 - 4x \qquad \text{Subtract } 4x.$$
$$y = 1 + 4x \qquad \text{Multiply by } -1.$$

Step 2 Substitute $1 + 4x$ for y in equation (1).

$$3x + 2y = 13 \qquad (1)$$
$$3x + 2(1 + 4x) = 13 \qquad \text{Let } y = 1 + 4x.$$

Step 3 Solve for x.

$$3x + 2 + 8x = 13 \qquad \text{Distributive property}$$
$$11x = 11 \qquad \text{Combine terms; subtract 2.}$$
$$x = 1 \qquad \text{Divide by 11.}$$

Step 4 Now solve for y. Since $y = 1 + 4x$,

$$y = 1 + 4(1) = 5. \qquad \text{Let } x = 1.$$

Step 5 Check the solution (1, 5) in both equations (1) and (2).

$3x + 2y = 13$	(1)	$4x - y = -1$	(2)
$3(1) + 2(5) = 13$	?	$4(1) - 5 = -1$	?
$3 + 10 = 13$	?	$4 - 5 =$	
$13 = 13$	True	$-1 =$	

The solution set is $\{(1, 5)\}$.

EXAMPLE 8 Solving a System by Substitution

Solve the system

$$\frac{2}{3}x - \frac{1}{2}y = \frac{7}{6} \qquad (1)$$

$$3x - 2y = 6. \qquad (2)$$

This system will be easier to solve if you clear the fractions in equation (1). Multiply by the LCD, 6.

$$6 \cdot \frac{2}{3}x - 6 \cdot \frac{1}{2}y = 6 \cdot \frac{7}{6}$$

$$4x - 3y = 7 \qquad (3)$$

Now the system consists of equations (2) and (3). To use the substitution method, one equation must be solved for one of the two variables. Solve equation (2) for x.

$$3x = 2y + 6$$

$$x = \frac{2y + 6}{3}$$

Substitute $\frac{2y + 6}{3}$ for x in equation (3).

$$4x - 3y = 7 \qquad (3)$$

$$4\left(\frac{2y + 6}{3}\right) - 3y = 7 \qquad \text{Let } x = \frac{2y+6}{3}.$$

$$3\left[4\left(\frac{2y + 6}{3}\right)\right] - 3(3y) = 3(7) \qquad \text{Multiply by 3 to clear the fraction.}$$

$$4(2y + 6) - 9y = 21$$

$$8y + 24 - 9y = 21 \qquad \text{Distributive property}$$

$$24 - y = 21 \qquad \text{Combine terms.}$$

$$-y = -3 \qquad \text{Subtract 24.}$$

$$y = 3 \qquad \text{Multiply by } -1.$$

Since $x = \frac{2y + 6}{3}$ and $y = 3$,

$$x = \frac{2(3) + 6}{3} = \frac{6 + 6}{3} = 4.$$

A check verifies that the solution set is $\{(4, 3)\}$.

4.2 Systems of Linear Equations in Three Variables

OBJECTIVES

1 Understand the geometry of systems of three equations in three variables.

A solution of an equation in three variables, such as

$$2x + 3y - z = 4,$$

is called an **ordered triple** and is written (x, y, z). For example, the ordered triple $(0, 1, -1)$ is a solution of the equation, because

$$2(0) + 3(1) - (-1) = 0 + 3 + 1 = 4.$$

Verify that another solution of this equation is $(10, -3, 7)$.

In the rest of this chapter, the term *linear equation* is extended to equations of the form

$$Ax + By + Cz + \cdots + Dw = K,$$

where not all the coefficients $A, B, C, \ldots, D$ equal 0. For example,

$$2x + 3y - 5z = 7 \qquad \text{and} \qquad x - 2y - z + 3u - 2w = 8$$

are linear equations, the first with three variables and the second with five variables.

OBJECTIVE 2 Solve linear systems (with three equations and three variables) by elimination. Since graphing to find the solution set of a system of three equations in three variables is impractical, these systems are solved with an extension of the elimination method, summarized as follows.

Solving a Linear System in Three Variables

Step 1 **Eliminate a variable.** Use the elimination method to eliminate any variable from any two of the original equations. The result is an equation in two variables.

Step 2 **Eliminate the same variable again.** Eliminate the *same* variable from any *other* two equations. The result is an equation in the same two variables as in Step 1.

Step 3 **Eliminate a different variable and solve.** Use the elimination method to eliminate a second variable from the two equations in two variables that result from Steps 1 and 2. The result is an equation in one variable that gives the value of that variable.

Step 4 **Find a second value.** Substitute the value of the variable found in Step 3 into either of the equations in two variables to find the value of the second variable.

Step 5 **Find a third value.** Use the values of the two variables from Steps 3 and 4 to find the value of the third variable by substituting into an appropriate equation.

Step 6 **Check** the solution in all of the original equations. Then write the solution set.

EXAMPLE 1 Solving a System in Three Variables

Solve the system

$$\begin{aligned} 4x + 8y + z &= 2 & (1) \\ x + 7y - 3z &= -14 & (2) \\ 2x - 3y + 2z &= 3. & (3) \end{aligned}$$

Step 1 As before, the elimination method involves eliminating a variable from the sum of two equations. The choice of which variable to eliminate is arbitrary.

Suppose we decide to begin by eliminating z. We multiply equation (1) by 3 and then add the result to equation (2).

$$\begin{aligned} 12x + 24y + 3z &= 6 && \text{Multiply each side of (1) by 3.} \\ x + 7y - 3z &= -14 && (2) \\ \hline 13x + 31y &= -8 && \text{Add.} \quad (4) \end{aligned}$$

Step 2 Equation (4) has only two variables. To get another equation without z, we multiply equation (1) by -2 and add the result to equation (3). It is essential at this point to *eliminate the same variable, z*.

$$\begin{aligned} -8x - 16y - 2z &= -4 && \text{Multiply each side of (1) by } -2. \\ 2x - 3y + 2z &= 3 && (3) \\ \hline -6x - 19y &= -1 && \text{Add.} \quad (5) \end{aligned}$$

Step 3 Now we solve the system of equations (4) and (5) for x and y. This step is possible only if the *same* variable is eliminated in Steps 1 and 2.

$$78x + 186y = -48 \qquad \text{Multiply each side of (4) by 6.}$$
$$-78x - 247y = -13 \qquad \text{Multiply each side of (5) by 13.}$$
$$-61y = -61 \qquad \text{Add.}$$
$$y = 1$$

Step 4 Now we substitute 1 for y in either equation (4) or (5). Choosing (5) gives

$$-6x - 19y = -1 \qquad (5)$$
$$-6x - 19(1) = -1 \qquad \text{Let } y = 1.$$
$$-6x - 19 = -1$$
$$-6x = 18$$
$$x = -3.$$

Step 5 We substitute -3 for x and 1 for y in any one of the three original equations to find z. Choosing (1) gives

$$4x + 8y + z = 2 \qquad (1)$$
$$4(-3) + 8(1) + z = 2 \qquad \text{Let } x = -3 \text{ and } y = 1.$$
$$-4 + z = 2$$
$$z = 6.$$

Step 6 It appears that the ordered triple $(-3, 1, 6)$ is the only solution of the system. We must check that the solution satisfies all three equations of the system. For equation (1),

$$4x + 8y + z = 2 \qquad (1)$$
$$4(-3) + 8(1) + 6 = 2 \qquad ?$$
$$-12 + 8 + 6 = 2 \qquad ?$$
$$2 = 2. \qquad \text{True}$$

Because $(-3, 1, 6)$ also satisfies equations (2) and (3), the solution set is $\{(-3, 1, 6)\}$.

OBJECTIVE 3 **Solve linear systems (with three equations and three variables) where some of the equations have missing terms.** When this happens, one elimination step can be omitted.

EXAMPLE 2 Solving a System of Equations with Missing Terms

Solve the system

$$\begin{aligned} 6x - 12y &= -5 & (1)\\ 8y + z &= 0 & (2)\\ 9x - z &= 12. & (3) \end{aligned}$$

Since equation (3) is missing the variable y, eliminate y using equations (1) and (2).

$$\begin{aligned} 12x - 24y \qquad &= -10 && \text{Multiply each side of (1) by 2.}\\ 24y + 3z &= 0 && \text{Multiply each side of (2) by 3.}\\ \hline 12x \qquad + 3z &= -10 && \text{Add. (4)} \end{aligned}$$

Use this result, together with equation (3), to eliminate z. Multiply equation (3) by 3.

$$\begin{aligned} 27x - 3z &= 36 && \text{Multiply each side of (3) by 3.}\\ 12x + 3z &= -10 && (4)\\ \hline 39x \qquad &= 26 && \text{Add.}\\ x = \frac{26}{39} &= \frac{2}{3} \end{aligned}$$

Substituting into equation (3) gives

$$\begin{aligned} 9x - z &= 12 && (3)\\ 9\left(\frac{2}{3}\right) - z &= 12 && \text{Let } x = \tfrac{2}{3}.\\ 6 - z &= 12\\ z &= -6. \end{aligned}$$

Substituting -6 for z in equation (2) gives

$$\begin{aligned} 8y + z &= 0 && (2)\\ 8y - 6 &= 0 && \text{Let } z = -6.\\ 8y &= 6\\ y &= \frac{3}{4}. \end{aligned}$$

Check in each of the original equations of the system to verify that the solution set of the system is $\left\{\left(\frac{2}{3}, \frac{3}{4}, -6\right)\right\}$.

OBJECTIVE 4 Solve special systems (with three equations and three variables). Linear systems with three variables may be inconsistent or may include dependent equations. The next examples illustrate these cases.

EXAMPLE 3 Solving an Inconsistent System with Three Variables

Solve the system

$$\begin{aligned} 2x - 4y + 6z &= 5 \quad &(1)\\ -x + 3y - 2z &= -1 \quad &(2)\\ x - 2y + 3z &= 1. \quad &(3) \end{aligned}$$

Eliminate x by adding equations (2) and (3) to get the equation

$$y + z = 0.$$

Now, *eliminate x again,* using equations (1) and (3).

$$\begin{aligned} -2x + 4y - 6z &= -2 \quad &\text{Multiply each side of (3) by } -2.\\ 2x - 4y + 6z &= 5 \quad &(1)\\ \hline 0 &= 3 \quad &\text{False} \end{aligned}$$

The resulting false statement indicates that equations (1) and (3) have no common solution. Thus, the system is inconsistent and the solution set is $\emptyset$. The graph of this system would show these two planes parallel to one another.

OBJECTIVE 1 Define a matrix. An ordered array of numbers such as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 2 \end{bmatrix}$$

Columns

Rows

is called a **matrix.** The numbers are called **elements** of the matrix. Matrices (the plural of *matrix*) are named according to the number of **rows** and **columns** they contain. The rows are read horizontally, and the columns are read vertically. For example, the first row in the preceding matrix is 2 3 5 and the first column is $\begin{matrix} 2 \\ 7 \end{matrix}$. This matrix is a 2×3 (read "two by three") matrix because it has 2 rows and 3 columns. The number of rows is given first, and then the number of columns. Two other examples follow.

$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \quad 2 \times 2 \text{ matrix} \qquad \begin{bmatrix} 8 & -1 & -3 \\ 2 & 1 & 6 \\ 0 & 5 & -3 \\ 5 & 9 & 7 \end{bmatrix} \quad 4 \times 3 \text{ matrix}$$

A **square matrix** is one that has the same number of rows as columns. The 2×2 matrix is a square matrix.

Figure 10 shows how a graphing calculator displays the preceding two matrices. Work with matrices is made much easier by using technology when available. Consult your owner's manual for details.

In this section, we discuss a matrix method of solving linear systems that is really just a very structured way of using the elimination method. The advantage of this new method is that it can be done by a graphing calculator or a computer, allowing large systems of equations to be solved easily.

OBJECTIVE 2 Write the augmented matrix for a system. To begin, we write an *augmented matrix* for the system. An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups. For example, to solve the system

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5, \end{aligned}$$

start with the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right].$$

Place the coefficients of the variables to the left of the bar, and the constants to the right. The bar separates the coefficients from the constants. The matrix is just a shorthand way of writing the system of equations, so the rows of the augmented matrix can be treated the same as the equations of a system of equations.

We know that exchanging the position of two equations in a system does not change the system. Also, multiplying any equation in a system by a nonzero number does not change the system. Comparable changes to the augmented matrix of a system of equations produce new matrices that correspond to systems with the same solutions as the original system.

The following **row operations** produce new matrices that lead to systems having the same solutions as the original system.

Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The elements in any row may be multiplied by any nonzero real number.
3. Any row may be changed by adding to the elements of the row the product of a real number and the corresponding elements of another row.

Examples of these row operations follow.

Row operation 1:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 1 & 0 & 7 \\ 4 & 8 & -3 \\ 2 & 3 & 9 \end{bmatrix}.$$

Interchange row 1 and row 3.

Row operation 2:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 6 & 9 & 27 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Multiply the numbers in row 1 by 3.

Row operation 3:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 0 & 3 & -5 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Multiply the numbers in row 3 by -2; add them to the corresponding numbers in row 1.

The third row operation corresponds to the way we eliminated a variable from a pair of equations in the previous sections.

OBJECTIVE 3 Use row operations to solve a system with two equations. Row operations can be used to rewrite a matrix until it is the matrix of a system where the solution is easy to find. The goal is a matrix in the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

for systems with two or three equations, respectively. Notice that there are 1s down the diagonal from upper left to lower right and 0s below the 1s. A matrix written this way is said to be in **row echelon form.** When these matrices are rewritten as systems of equations, the value of one variable is known, and the rest can be found by substitution. The following examples illustrate this method.

EXAMPLE 1 Using Row Operations to Solve a System with Two Variables

Use row operations to solve the system

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5. \end{aligned}$$

We start with the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Now we use the various row operations to change this matrix into one that leads to a system that is easier to solve.

It is best to work by columns. We start with the first column and make sure that there is a 1 in the first row, first column position. There is already a 1 in this position. Next, we get 0 in every position below the first. To get a 0 in row two, column one, we use the third row operation and add to the numbers in row two the result of multiplying each number in row one by -2. (We abbreviate this as $-2R_1 + R_2$.) Row one remains unchanged.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 + 1(-2) & 1 + -3(-2) & -5 + 1(-2) \end{array}\right]$$

Original number from row two ↑   ↑ -2 times number from row one

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right] \quad -2R_1 + R_2$$

The matrix now has a 1 in the first position of column one, with 0 in every position below the first.

Now we go to column two. A 1 is needed in row two, column two. We get this 1 by using the second row operation, multiplying each number of row two by $\frac{1}{7}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -1 \end{array}\right] \quad \tfrac{1}{7}R_2$$

This augmented matrix leads to the system of equations

$$\begin{aligned} 1x - 3y &= 1 \\ 0x + 1y &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} x - 3y &= 1 \\ y &= -1. \end{aligned}$$

From the second equation, $y = -1$. We substitute -1 for y in the first equation to get

$$\begin{aligned} x - 3y &= 1 \\ x - 3(-1) &= 1 \\ x + 3 &= 1 \\ x &= -2. \end{aligned}$$

The solution set of the system is $\{(-2, -1)\}$. Check this solution by substitution in both equations of the system.

If the augmented matrix of the system in Example 1 is entered as matrix [A] in a graphing calculator (Figure 11(a)) and the row echelon form of the matrix is found (Figure 11(b)), then the system becomes

$$x + \frac{1}{2}y = -\frac{5}{2}$$
$$y = -1.$$

While this system looks different from the one we obtained in Example 1, it is equivalent, since its solution set is also $\{(-2, -1)\}$.

OBJECTIVE 4 Use row operations to solve a system with three equations. A linea system with three equations is solved in a similar way. We use row operations to ge 1s down the diagonal from left to right and all 0s below each 1.

EXAMPLE 2 Using Row Operations to Solve a System with Three Variables

Use row operations to solve the system

$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5. \end{aligned}$$

Start by writing the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

This matrix already has 1 in row one, column one. Next get 0s in the rest of colum one. First, add to row two the results of multiplying each number of row one by −3 This gives the matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 1 & 3 & 2 & 5 \end{array}\right]. \qquad -3R_1 + R_2$$

Now add to the numbers in row three the results of multiplying each number of ro one by −1.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \qquad -1R_1 + R_3$$

Introduce 1 in row two, column two by multiplying each number in row two by $\frac{1}{6}$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \quad \tfrac{1}{6}R_2$$

To obtain 0 in row three, column two, add to row three the results of multiplying each number in row two by -4.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \quad -4R_2 + R_3$$

Finally, obtain 1 in row three, column three by multiplying each number in row three by $\frac{3}{23}$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \tfrac{3}{23}R_3$$

This final matrix gives the system of equations

$$\begin{aligned} x - y + 5z &= -6 \\ y - \frac{8}{3}z &= \frac{14}{3} \\ z &= -1. \end{aligned}$$

Substitute -1 for z in the second equation, $y - \frac{8}{3}z = \frac{14}{3}$, to find that $y = 2$. Finally, substitute 2 for y and -1 for z in the first equation, $x - y + 5z = -6$, to determine that $x = 1$. The solution set of the original system is $\{(1, 2, -1)\}$. Check by substitution.

OBJECTIVE 5 Use row operations to solve special systems. In the final example we show how to recognize inconsistent systems or systems with dependent equations when solving these systems with row operations.

EXAMPLE 3 Recognizing Inconsistent Systems or Dependent Equations

Use row operations to solve each system.

(a) $$\begin{aligned} 2x - 3y &= 8 \\ -6x + 9y &= 4 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right]$$ Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -6 & 9 & 4 \end{array}\right]$$ $\frac{1}{2}R_1$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 28 \end{array}\right]$$ $6R_1 + R_2$

The corresponding system of equations is

$$x - \frac{3}{2}y = 4$$

$$0 = 28, \quad \text{False}$$

which has no solution and is inconsistent. The solution set is $\emptyset$.

(b) $-10x + 12y = 30$
$5x - 6y = -15$

$$\left[\begin{array}{rr|r} -10 & 12 & 30 \\ 5 & -6 & -15 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{rr|r} 1 & -\frac{6}{5} & -3 \\ 5 & -6 & -15 \end{array}\right] \quad -\tfrac{1}{10}R_1$$

$$\left[\begin{array}{rr|r} 1 & -\frac{6}{5} & -3 \\ 0 & 0 & 0 \end{array}\right] \quad -5R_1 + R_2$$

The corresponding system is

$$x - \frac{6}{5}y = -3$$

$$0 = 0, \qquad \text{True}$$

which has dependent equations. Using the second equation of the original system, we write the solution set as

$$\{(x, y) \mid 5x - 6y = -15\}.$$

CONNECTIONS

An extension of the matrix method described in this section involves transforming an augmented matrix into **reduced row echelon form.** This form has 1s down the main diagonal and 0s above and below this diagonal. For example, the matrix for the system in Example 2 could be transformed into the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \text{which gives the equivalent system} \quad \begin{aligned} x &= 1 \\ y &= 2 \\ z &= -1. \end{aligned}$$

The calculator screens in Figure 12 indicate how easily this transformation can be obtained using technology.

For Discussion or Writing

1. Write the reduced row echelon form for the matrix of the system in Example 1.
2. If transforming to reduced row echelon form leads to all 0s in the final row, what kind of system is represented?

ASSIGNMENTS #1

A

3.6.1 Minor and Cofactor of an Element of a Matrix or its Determinant

Minor of an Element: Let us consider a square matrix A of order 3. Then the minor of an element a_{ij}, denoted by M_{ij} is the determinant of the $(3-1)\times(3-1)$ matrix formed by deleting the ith row and the jth column of $A(\text{or}|A|)$.

For example, if

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$, then the matrix obtained by deleting the first row

and the second column of A is $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ (see adjoining figure) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

and its determinant is the minor of a_{12}, that is, $M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of an Element: The cofactor of an element a_{ij} denoted by A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} \times M_{ij}$$

where M_{ij} is the minor of the element a_{ij} of A or $|A|$.

For example, $A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

3.6.2 Determinant of a Square Matrix of Order $n \geq 3$:

The determinant of a square matrix of order n is the sum of the products of each element of row (or column) and its cofactor.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$, then

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + ... + a_{in}A_{in} \quad \text{for } i = 1, 2, 3, ..., n$$

$$\text{or } |A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + ... + a_{nj}A_{nj} \quad \text{for } j = 1, 2, 3, ..., n$$

Putting $i = 1$, we have

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + ... + a_{1n}A_{1n}$ which is called the expansion of $|A|$ by the first row.

If A is a matrix of order 3, that is, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} \quad \text{for } i = 1, 2, 3 \qquad (1)$$

$$\text{or } |A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} \quad \text{for } j = 1, 2, 3 \qquad (2)$$

For example, for $i = 1, j = 1$ and $j = 2$, we have

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \qquad \text{(i)}$$

$$\text{or } |A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \qquad \text{(ii)}$$

$$\text{or } |A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \qquad \text{(iii)}$$

(iii)

(iii) can be written as: $|A| = a_{12}(-1)^{1+2}M_{12} + a_{22}(-1)^{2+2}M_{22} + a_{32}(-1)^{3+2}M_{32}$

i.e., $|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$ (iv)

Similarly (i) can be written as $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ (v)

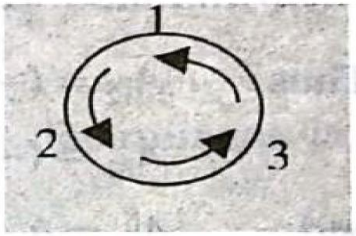
Putting the values of M_{11}, M_{12} and M_{13} in (v), we obtain

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$ (vi)

or $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$ (vi)′

The second scripts of positive terms are in circular order of anti-clockwise direction i.e., these are as 123, 231, 312 (adjoining figure) while the second scripts of negative terms are such as 132, 213, 321.



An alternative way to remember the expansion of the determinant $|A|$ given in (vi)′ is shown in the figure below.

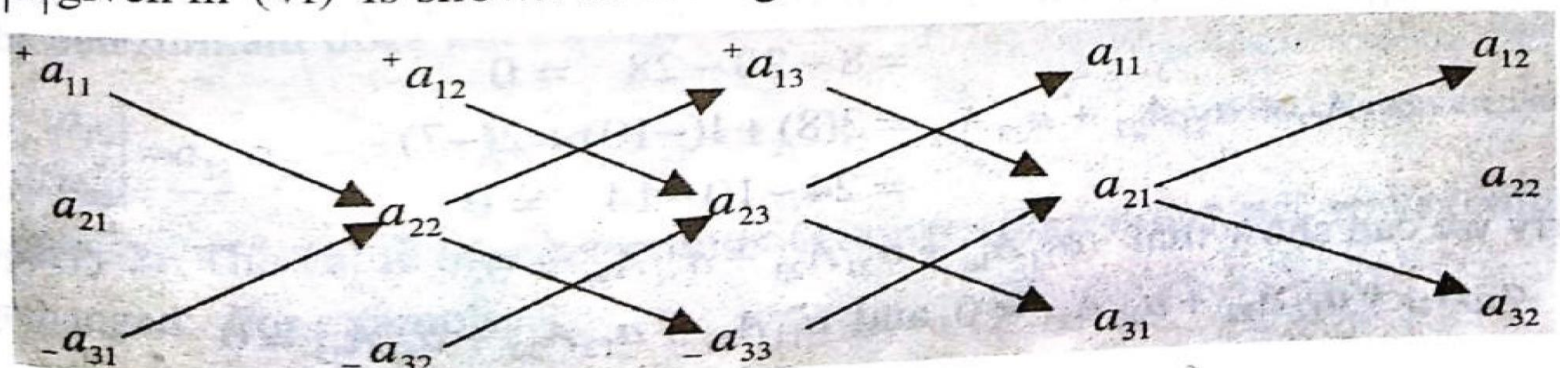


Fig.

Example 1: Evaluate the determinant of $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Solution: $|A| = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{vmatrix}$

Using the result (v) of the Art.3.6.2, that is,

$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, we get,

$$|A| = 1\begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

$$= 1[6 - 1(-3)] + 2[(-2).2 - 1.4] + 3[(-2)(-3) - 12]$$

$$= (6 + 3) + 2(-4 - 4) + 3(6 - 12)$$

$$= 9 - 16 - 18 = -25$$

$= 9 - 16 - 18 = -25$

Example 2: Find the cofactors A_{12}, A_{22} and A_{32} if $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ and find $|A|$.

Solution: We first find M_{12}, M_{22} and M_{32}.

$$M_{12} = \begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} = -4 - 4 = -8; \qquad M_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10$$

and $$M_{32} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 - (-6) = 7$$

Thus $$A_{12} = (-1)^{1+2} M_{12} = (-1)(-8) = 8; \qquad A_{22} = (-1)^{2+2} M_{22} = 1(-10) = -10$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)(7) = -7$$

and $$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = (-2)8 + 3(-10) + (-3)(-7)$$
$$= -16 - 30 + 21 = -25$$

Note that $$a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 1(8) + (-2)(-10) + 4(-7)$$
$$= 8 + 20 - 28 = 0$$

and $$a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = 3(8) + 1(-10) + 2(-7)$$
$$= 24 - 10 - 14 = 0$$

Similarly we can show that $a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 0$;

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0 \text{ and } a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

3.8 Adjoint and Inverse of a Square Matrix of Order $n \geq 3$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the matrix of co-factors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$,

and adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Inverse of a Square Matrix of Order $n \geq 3$: Let A be a non singular square matrix of order n. If there exists a matrix B such that $AB = BA = I_n$, then B is called the multiplicative inverse of A and is denoted by A^{-1}. It is obvious that the order of A^{-1} is $n \times n$.

Thus $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

If A is a non singular matrix, then

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

Example 6: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution: We first find the cofactors of the elements of A.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 1.(2+1) = 3,\quad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1) = 1$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 1.(0-2) = -2,\quad A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = (-1)(0+2) = -2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1.(1-2) = -1,\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = (-1)(-1-0) = 1$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 1.(0-4) = -4,\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1-0) = -1$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1.(2-0) = 2$

Thus $[A_{ij}]_{3\times3} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

and $\text{adj} A = [A'_{ij}]_{3\times3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ $(\because A'_{ij} = A_{ji} \text{ for } i, j = 1, 2, 3)$

Since

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(3) + 0(1) + 2(-2)$$
$$= 3 + 0 - 4 = -1,$$

So

$$A^{-1} = \frac{1}{|A|}\text{adj} A = \frac{1}{-1}\begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

1.

If A and B are square matrices of the same order, then explain why in general

i) $(A+B)^2 \neq A^2 + 2AB + B^2$

ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

iii) $(A+B)(A-B) \neq A^2 - B^2$

2. Find the inverse by using the Cofactor method.

15. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

16. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & -1 \\ 0 & 2 & 6 \end{bmatrix}$

18. $\begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 30 & 0 & 20 \\ 0 & -10 & -20 \\ 40 & 0 & 10 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

B

$$4x_1 + 3x_2 = 5$$
$$3x_1 - x_2 = 7$$

Sol. In matrix form

$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$A \; X = B \Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = -4 - 9 = -13 \neq 0$$

$$adj\; A = \begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{adj A}{|A|} = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

Now $X = A^{-1}B$

$$X = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$X = \frac{-1}{13}\begin{bmatrix} -5-21 \\ -15+28 \end{bmatrix}$$

$$X = \frac{-1}{13}\begin{bmatrix} -26 \\ 13 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

19.
$$4x + 2y - 3z = 6$$
$$x - 4y + z = -4$$
$$-x + 2z = 2$$

14.
$$x + y = 4$$
$$2x - y = 2$$

15.
$$x + y = -5$$
$$-2x + y = 1$$

20.
$$2x + 3y - 4z = 4$$
$$x - 6y + z = -16$$
$$-x + 3z = 8$$

C

1. If $A=\begin{bmatrix}1 & -2 & 5\\ -2 & 3 & -1\\ 5 & -1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 1 & -2\\ 1 & 0 & -1\\ -2 & -1 & 2\end{bmatrix}$, then show that A + B is symmetric: Multan 2008

Sol. $A+B=\begin{bmatrix}1 & -2 & 5\\ -2 & 3 & -1\\ 5 & -1 & 0\end{bmatrix}+\begin{bmatrix}-3 & 1 & -2\\ 1 & 0 & -1\\ -2 & -1 & 2\end{bmatrix}$

$$=\begin{bmatrix}1-3 & -2+1 & 5-2\\ -2+1 & 3+0 & -1-1\\ 5-2 & -1-1 & 0+2\end{bmatrix}=\begin{bmatrix}-2 & -1 & 3\\ -1 & 3 & -2\\ 3 & -2 & 2\end{bmatrix}$$

Now $(A+B)'=\begin{bmatrix}-2 & -1 & 3\\ -1 & 3 & -2\\ 3 & -2 & 2\end{bmatrix}'$

$$=\begin{bmatrix}-2 & -1 & 3\\ -1 & 3 & -2\\ 3 & -2 & 2\end{bmatrix}=(A+B)$$

Hence $(A+B)$ is symmetric

2. Prove that if possible

i $A + A'$ is symmetric

$A - A'$ is skew symmetric

ii $B + B'$ is symmetric

$B - B^t$ is symmetric

3

If A is any square matrix of order 3, show that:

$A + A'$ is symmetric and skew symmetric

6. If $A=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, Show that

i. $A+(\bar{A})'$ is hermitian

Sol. Let $A=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$

$$\bar{A}=\begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$$

$$(\bar{A})'=\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A+(\bar{A})' = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$= \begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -i+i \end{bmatrix} = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\overline{A+(\bar{A})'} = \begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})'}\right)' = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})'}\right)' = A+(\bar{A})'$$

So $A+(\bar{A})'$ is Hermitian

ii. $A-(\bar{A})'$ is skew-hermitain Multan 2007

Let $A=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} \Rightarrow \bar{A}=\begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix} \Rightarrow (\bar{A})'=\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

Sol. Now $A-(\bar{A})'=\begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}-\begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}=\begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i-i \end{bmatrix}$

$=\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix} \Rightarrow \overline{(A-(\bar{A})')}=\begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} \Rightarrow \left(\overline{(A-(\bar{A})')}\right)'=\begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}=-\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$

$\left(\overline{(A-(\bar{A})')}\right)'=-\left(A-(\bar{A})'\right)$ Hence $A-(\bar{A})'$ is skew hermitaian.

iii

If $A=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ show that $A - A'$ is Hermitian

$A + A'$ is skew Hermitian

Q # 1. Select the Correct Option

i. A square matrix $A=[a_{ij}]$ with complex entries is skew hermetian if $(\overline{A})' = ?$

a) A b) $-A$ c) $|A|$ d) $|A|$

ii. The matrix $\begin{bmatrix} 5 & 1 \\ 15 & 3 \end{bmatrix}$ is:

a) Singular b) Non Singular c) Symmetric d) Skew symmetric

iii. For trivial solution $|A|$ is:

a) 1 b) -1 c) Zero d) Not defined

iv. (0,0,0) is ________ solution of homogeneous system of linear equation is

a) Trivial b) Non trivial c) unique d) Non

v. If $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ then A_{22} is equal to:

a) 10 b) -10 c) -18 d) -11

vi. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is

a) Diagonal matrix b) Zero matrix c) Scalar matrix d) Identity matrix

vii. If $\begin{bmatrix} -1 & 3 \\ x & 1 \end{bmatrix} = 0$ then value of x is

a) -3 b) $\frac{1}{3}$ c) $-\frac{1}{3}$ d) 3

viii. If A is a square matrix of order 2×2 then $|KA|$ equals:

a) $K|A|$ b) $\frac{1}{K}|A|$ c) $2K|A|$ d) $K^2|A|$

ix. If $A=[a_{ij}]$ is a square matrix of order n if $a_{ij}=0 \quad \forall i \neq j$ and $a_{ij}=1 \quad, \forall \quad i=j$ then A is matrix

a) Unit b) Null c) Symmetric d) Skew Symmetric

x. If A and B are confirmable for multiplication if $(AB)' = ?$

a) AB b) BA c) $A'B'$ d) $B'A'$

(2 X 20 = 40)

10. **Find the rank of the following matrices:**

i. $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ **(Federal)**

Sol. $R \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix}$ by $R_2 - 2R_1$, $R_3 - 3R_1$ $\Rightarrow R \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8 & -2 & -6 \end{bmatrix}$ by $(-1/4)$

$R \begin{bmatrix} 1 & 0 & 7/4 & 5/4 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$ by $R_1 + R_2$, $R_3 - 8R_2$ $\Rightarrow R \begin{bmatrix} 1 & 0 & 7/4 & 5/4 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ by $\frac{-1}{8} R_3$

$R \begin{bmatrix} 1 & 0 & 7/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ by $R_1 - 5/4R_3$, $R_2 - \frac{1}{4} R_3$

Hence Rank = 3

Find the Rank of the following matrices

7. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ASSIGNMENTS #2

1.0 Read to : Unit 3# from FSC Book (Matrices and determinant)

1.1 Read to **4.1 Systems of Linear Equations in Two Variables**

By Lial.M,L, Hornsby, J

1.3 Unit#6 Systems of Equations and Inequalities Unit# 7 Matrices and Determinants by James Stewart

ONLY READING

ASSIGNMENTS #3

Find the inverse of the following matrices. Also find their inverse by using row and column operations.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Method of Column Operation

$$\left[\begin{array}{ccc}1 & -3 & 2\\ 2 & 1 & 0\\ 0 & -1 & 1\\ \hdashline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{ccc}1 & 0 & 0\\ 2 & 7 & -4\\ 0 & -1 & 1\\ \hdashline 1 & 3 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] C_2+3C_1, \quad C_3-2C_1$$

$\frac{1}{7}C_2$ $\qquad\qquad$ C_1-2C_2, C_3+4C_2

$$\underset{\sim}{C}\left[\begin{array}{ccc}1 & 0 & 0\\ 2 & 1 & -4\\ 0 & -1/7 & 1\\ \hdashline 1 & 3/7 & -2\\ 0 & 1/7 & 0\\ 0 & 0 & 1\end{array}\right] \Rightarrow \underset{\sim}{C}\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 2/7 & -1/7 & 3/7\\ \hdashline 1/7 & 3/7 & -2/7\\ -2/7 & 1/7 & 4/7\\ 0 & 0 & 1\end{array}\right]$$

$\frac{3}{7}C_3$ $\qquad\qquad$ $C_2+\frac{1}{7}C_3, \& \; C_1-\frac{2}{7}C_3$

$$\begin{array}{c} \\ C \\ \\ \end{array}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/7 & -1/7 & 1 \\ \hline 1/7 & 3/7 & -2/3 \\ -2/7 & 1/7 & 4/3 \\ 0 & 0 & 7/3 \end{array}\right] \Rightarrow C\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \\ -2/3 & 1/3 & 7/3 \end{array}\right]$$

Hence $A^{-1}=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$

13. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$

14. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

Find the inverse by using column operation.

THE END