
بسم الله الرحمن الرحيم

السلام عليكم

As-salamu alaykum

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By

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Lecture-5&6: Preliminaries

3.1 What Is a Function?
3.2 Graphs of Functions
3.3 Increasing and Decreasing Functions; Average Rate of Change
3.4 Transformations of Functions
[redacted]
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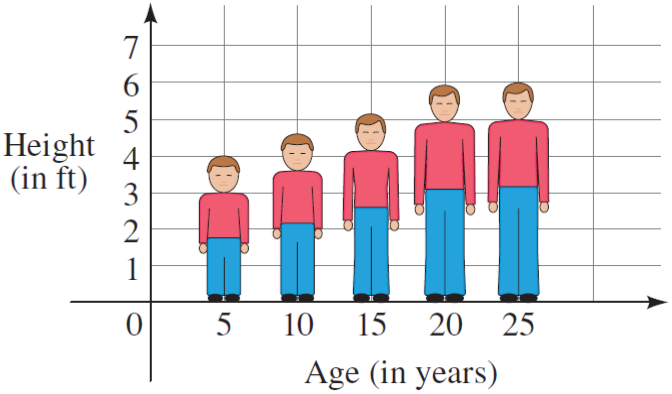
3.1 What Is a Function?

Functions All Around Us

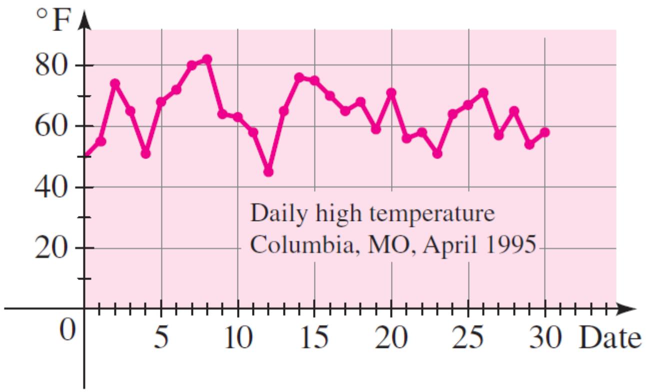
In nearly every physical phenomenon we observe that one quantity depends on another. For example, your height depends on your age, the temperature depends on the date, the cost of mailing a package depends on its weight (see Figure 1). We use the term *function* to describe this dependence of one quantity on another. That is, we say the following:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.

The U.S. Post Office uses a simple rule to determine the cost of mailing a package based on its weight. But it's not so easy to describe the rule that relates height to age or temperature to date.



Height is a function of age.



Temperature is a function of date.

w (ounces)	Postage (dollars)
$0 < w \le 1$	0.37
$1 < w \le 2$	0.60
$2 < w \le 3$	0.83
$3 < w \le 4$	1.06
$4 < w \le 5$	1.29
$5 < w \le 6$	1.52

Postage is a function of weight.

Can you think of other functions? Here are some more examples:

- The area of a circle is a function of its radius.
- The number of bacteria in a culture is a function of time.
- The weight of an astronaut is a function of her elevation.
- The price of a commodity is a function of the demand for that commodity.

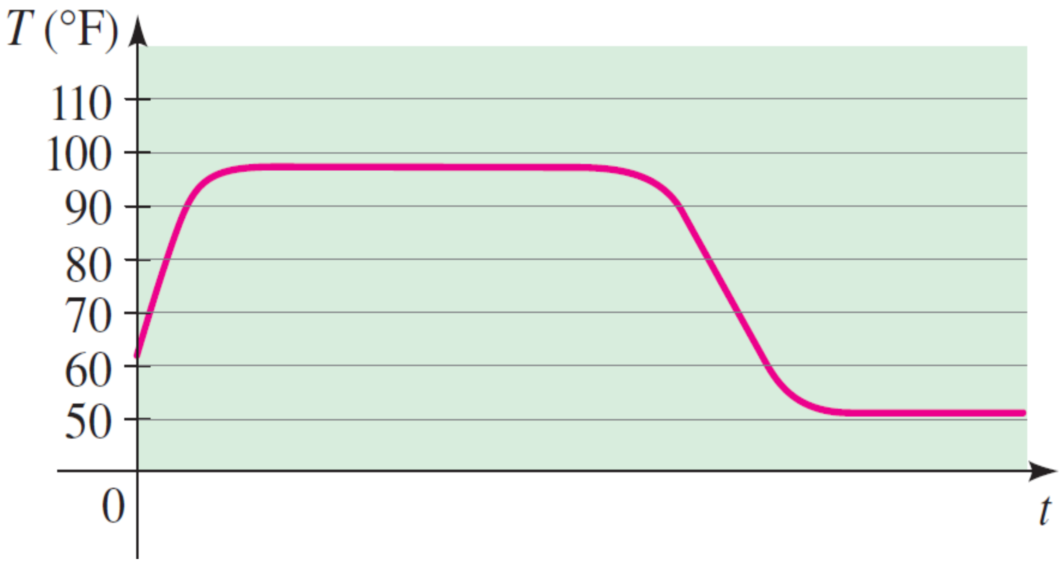
The area A of a circle depends on its radius r. The rule that describes this dependence is given by the formula $A = \pi r^2$. The number N of bacteria in a culture depends on the period of time t the culture has been growing. The rule that connects N and t in this case is given by the formula $N = 50 \cdot 2^t$. The weight w of an astronaut depends on her elevation h. Physicists use the rule $w = w_0 R^2/(R + h)^2$, where w_0 is the astronaut's weight at sea level and R is the radius of the earth. The price p of a commodity (wheat, for instance) depends on the demand y for that commodity.

Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph. For example, when you turn on a hot water faucet, the temperature of the water depends on how long the water has been running. So we can say

In the *Focus on Modeling*, pages 295–307, we learn how to find functions that model real-life data.

- Temperature of water from the faucet is a function of time.

Figure 2 shows a rough graph of the temperature *T* of the water as a function of the time *t* that has elapsed since the faucet was turned on. The graph shows that the initial temperature of the water is close to room temperature. When the water from the hot water tank reaches the faucet, the water's temperature *T* increases quickly. In the next phase, *T* is constant at the temperature of the water in the tank. When the tank is drained, *T* decreases to the temperature of the cold water supply.



perature *T* as a

Definition of Function

A function is a rule. In order to talk about a function, we need to give it a name. We will use letters such as f, g, h, . . . to represent functions. For example, we can use the letter f to represent a rule as follows:

"f" is the rule "square the number"

When we write $f(2)$, we mean "apply the rule f to the number 2." Applying the rule gives $f(2) = 2^2 = 4$. Similarly, $f(3) = 3^2 = 9$, $f(4) = 4^2 = 16$, and in general $f(x) = x^2$.

Definition of Function

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is,

$$\text{range of } f = \{f(x) \mid x \in A\}$$

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. For instance, in the bacteria example, t is the independent variable and N is the dependent variable.

It's helpful to think of a function as a **machine** (see Figure 3). If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

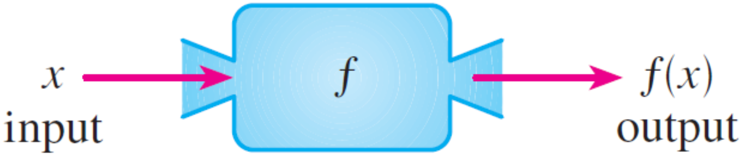


Figure 3

Machine diagram of f

as

Another way to picture a function is by an **arrow diagram** as in Figure 4. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

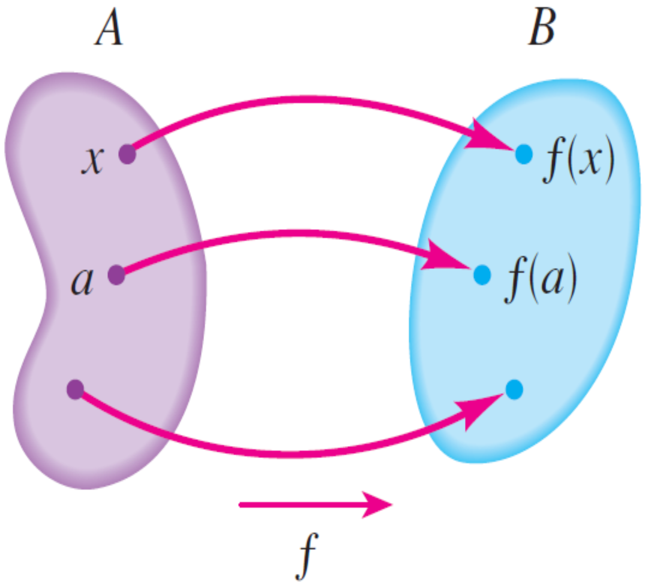


Figure 4

Arrow diagram of f

Example 1 The Squaring Function

The squaring function assigns to each real number x its square x^2. It is defined by

$$f(x) = x^2$$

(a) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.
(b) Find the domain and range of f.
(c) Draw a machine diagram for f.

Solution

(a) The values of f are found by substituting for x in $f(x) = x^2$.

$$f(3) = 3^2 = 9 \qquad f(-2) = (-2)^2 = 4 \qquad f(\sqrt{5}) = (\sqrt{5})^2 = 5$$

(b) The domain of f is the set $\mathbb{R}$ of all real numbers. The range of f consists of all values of $f(x)$, that is, all numbers of the form x^2. Since $x^2 \geq 0$ for all real numbers x, we can see that the range of f is $\{y \mid y \geq 0\} = [0, \infty)$.

(c) A machine diagram for this function is shown in Figure 5. ■

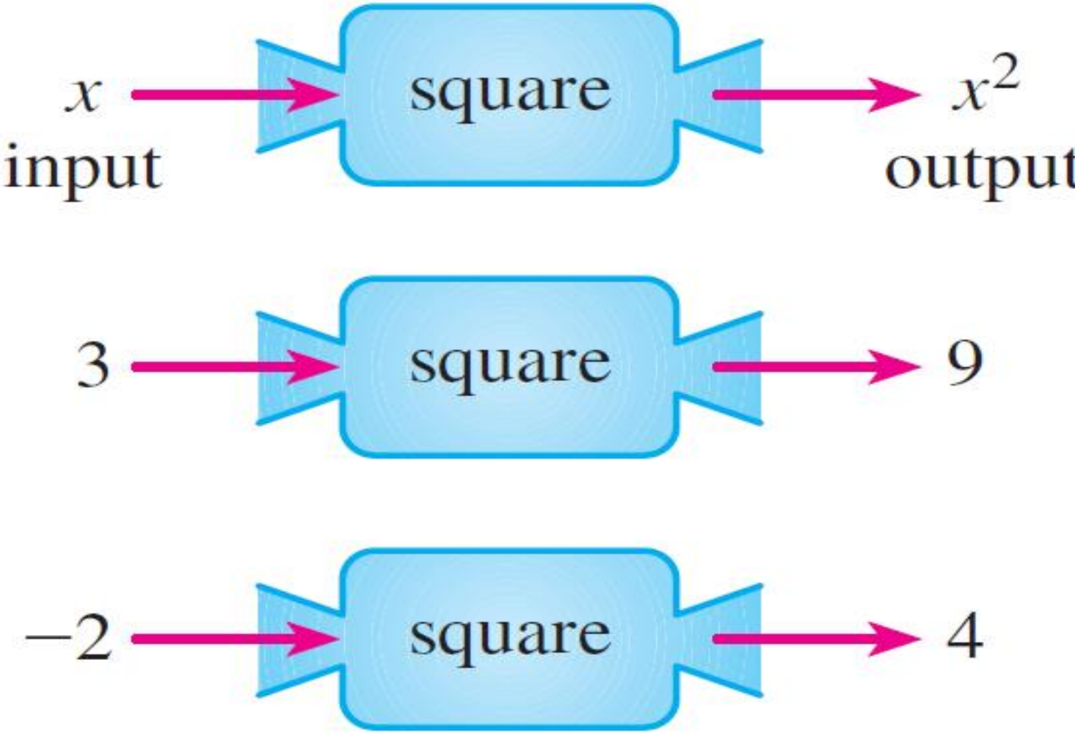


Figure 5

Machine diagram

Evaluating a Function

In the definition of a function the independent variable x plays the role of a "placeholder." For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as

$$f(\square) = 3 \cdot \square^2 + \square - 5$$

To evaluate f at a number, we substitute the number for the placeholder.

Example 2 Evaluating a Function

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ (b) $f(0)$ (c) $f(4)$ (d) $f\left(\frac{1}{2}\right)$

Solution

To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot 4^2 + 4 - 5 = 47$

(d) $f\left(\frac{1}{2}\right) = 3 \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} - 5 = -\frac{15}{4}$

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Here are some other examples.

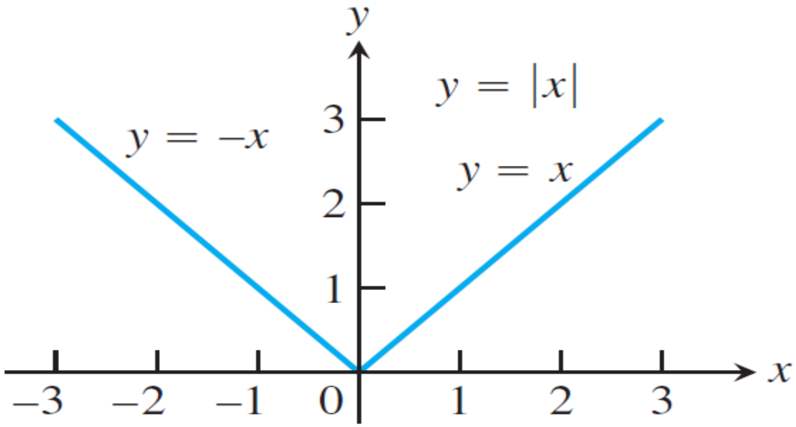


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Example 3 A Piecewise Defined Function

A cell phone plan costs \$39 a month. The plan includes 400 free minutes and charges 20¢ for each additional minute of usage. The monthly charges are a function of the number of minutes used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \le x \le 400 \\ 39 + 0.2(x - 400) & \text{if } x > 400 \end{cases}$$

Find $C(100)$, $C(400)$, and $C(480)$.

Solution

Remember that a function is a rule. Here is how we apply the rule for this function. First we look at the value of the input x. If $0 \le x \le 400$, then the value of $C(x)$ is 39. On the other hand, if $x > 400$, then the value of $C(x)$ is $39 + 0.2(x - 400)$.

Since $100 \le 400$, we have $C(100) = 39$.

Since $400 \le 400$, we have $C(400) = 39$.

Since $480 > 400$, we have $C(480) = 39 + 0.2(480 - 400) = 55$.

Thus, the plan charges \$39 for 100 minutes, \$39 for 400 minutes, and \$55 for 480 minutes. ■

Example 4 Evaluating a Function

If $f(x) = 2x^2 + 3x - 1$, evaluate the following.

(a) $f(a)$

(b) $f(-a)$

(c) $f(a + h)$

(d) $\dfrac{f(a + h) - f(a)}{h}, \quad h \neq 0$

Solution

(a) $f(a) = 2a^2 + 3a - 1$

(b) $f(-a) = 2(-a)^2 + 3(-a) - 1 = 2a^2 - 3a - 1$

(c)
$$\begin{aligned} f(a+h) &= 2(a+h)^2 + 3(a+h) - 1 \\ &= 2(a^2 + 2ah + h^2) + 3(a+h) - 1 \\ &= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1 \end{aligned}$$

(d) Using the results from parts (c) and (a), we have

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 + 3a + 3h - 1) - (2a^2 + 3a - 1)}{h} \\ &= \frac{4ah + 2h^2 + 3h}{} = 4a + 2h + 3 \end{aligned}$$

■

Example 5 The Weight of an Astronaut

If an astronaut weighs 130 pounds on the surface of the earth, then her weight when she is h miles above the earth is given by the function

$$w(h) = 130\left(\frac{3960}{3960 + h}\right)^2$$

(a) What is her weight when she is 100 mi above the earth?

(b) Construct a table of values for the function w that gives her weight at heights from 0 to 500 mi. What do you conclude from the table?

Solution

(a) We want the value of the function w when $h = 100$; that is, we must calculate $w(100)$.

$$w(100) = 130\left(\frac{3960}{3960 + 100}\right)^2 \approx 123.67$$

So at a height of 100 mi, she weighs about 124 lb.

(b) The table gives the astronaut's weight, rounded to the nearest pound, at 100-mile increments. The values in the table are calculated as in part (a).

h	$w(h)$
0	130
100	124
200	118
300	112
400	107
500	102

The table indicates that the higher the astronaut travels, the less she weighs. ■

The Domain of a Function

Recall that the *domain* of a function is the set of all inputs for the function. The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x^2, \qquad 0 \leq x \leq 5$$

then the domain is the set of all real numbers x for which $0 \leq x \leq 5$. If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain is the set of all real numbers for which the expression is defined as a real number.* For example, the function

$$f(x) = \frac{1}{x - 4}$$

is not defined at $x = 4$, so its domain is $\{x \mid x \neq 4\}$. The function

$$f(x) = \sqrt{x}$$

is not defined for negative x, so its domain is $\{x \mid x \geq 0\}$.

The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 13–16, we will encounter functions for which the elements of the sets are points in the coordinate plane or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r (so r, interpreted as a length, can only be positive in this formula). When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix 1), the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**.

Example 6 Finding Domains of Functions

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x^2 - x}$ (b) $g(x) = \sqrt{9 - x^2}$ (c) $h(t) = \dfrac{t}{\sqrt{t + 1}}$

Solution

(a) The function is not defined when the denominator is 0. Since

$$f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

we see that $f(x)$ is not defined when $x = 0$ or $x = 1$. Thus, the domain of f is

$$\{x \mid x \neq 0, x \neq 1\}$$

The domain may also be written in interval notation as

$$(\infty, 0) \cup (0, 1) \cup (1, \infty)$$

(b) We can't take the square root of a negative number, so we must have $9 - x^2 \geq 0$. Using the methods of Section 1.6, we can solve this inequality to find that $-3 \leq x \leq 3$. Thus, the domain of g is

$$\{x \mid -3 \leq x \leq 3\} = [-3, 3]$$

(c) We can't take the square root of a negative number, and we can't divide by 0, so we must have $t + 1 > 0$, that is, $t > -1$. So the domain of h is

$$\{t \mid t > -1\} = (-1, \infty)$$

■

The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite. The range of a function is not always easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the $\sqrt{x}$ key.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates an element of the domain D with a unique or single element in the set Y. In Figure 1.2, the arrows indicate that $f(a)$ is associated with a, $f(x)$ is associated with x, and so on. Notice that a function can have the same *value* at two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

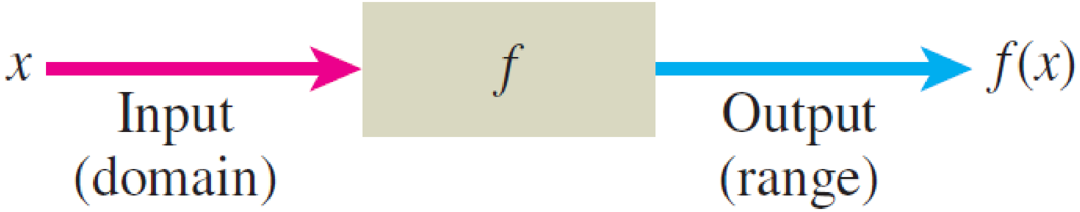


FIGURE 1.1 A diagram showing a function as a kind of machine.

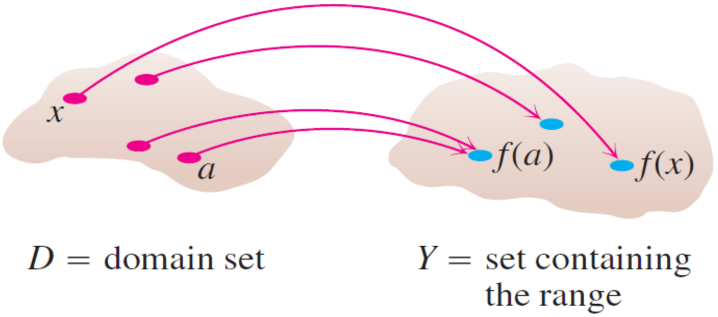


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

EXAMPLE 1 Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

3.1 Exercises

1–4 ■ Express the rule in function notation. (For example, the rule "square, then subtract 5" is expressed as the function $f(x) = x^2 - 5$.)

1. Multiply by 7, then add 2

2. Add 14, then divide by 7

3. Subtract 4, then square

4. Square, add 9, then take the square root

11–12 ■ Complete the table.

11. $f(x) = 2x^2 + 1$

x	$f(x)$
-1	
0	
1	
2	
3	

12. $g(x) = |2x - 3|$

x	$g(x)$
-2	
0	
1	
3	
5	

13–20 ■ Evaluate the function at the indicated values.

13. $f(x) = 2x + 1;$

$f(1), f(-2), f\left(\frac{1}{2}\right), f(a), f(-a), f(a + b)$

14. $f(x) = x^2 + 2x;$

$f(0), f(3), f(-3), f(a), f(-x), f\left(\frac{1}{a}\right)$

15. $g(x) = \frac{1 - x}{1 + x};$

$g(2), g(-2), g\left(\frac{1}{2}\right), g(a), g(a - 1), g(-1)$

16. $h(t) = t + \frac{1}{t};$

$h(1), h(-1), h(2), h\left(\frac{1}{2}\right), h(x), h\left(\frac{1}{x}\right)$

17. $f(x) = 2x^2 + 3x - 4;$

$f(0), f(2), f(-2), f(\sqrt{2}), f(x + 1), f(-x)$

18. $f(x) = x^3 - 4x^2;$

$f(0), f(1), f(-1), f\left(\frac{3}{2}\right), f\left(\dfrac{x}{2}\right), f(x^2)$

19. $f(x) = 2|x - 1|;$

$f(-2), f(0), f\left(\frac{1}{2}\right), f(2), f(x + 1), f(x^2 + 2)$

20. $f(x) = \dfrac{|x|}{x};$

$f(-2), f(-1), f(0), f(5), f(x^2), f\left(\dfrac{1}{x}\right)$

21–24 ■ Evaluate the piecewise defined function at the indicated values.

21. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$

$f(-2), f(-1), f(0), f(1), f(2)$

22. $f(x) = \begin{cases} 5 & \text{if } x \leq 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$

$f(-3), f(0), f(2), f(3), f(5)$

23. $f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq -1 \\ x & \text{if } x > -1 \end{cases}$

$f(-4), f\left(-\frac{3}{2}\right), f(-1), f(0), f(1)$

24. $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ x + 1 & \text{if } 0 \leq x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

$f(-5), f(0), f(1), f(2), f(5)$

25–28 ■ Use the function to evaluate the indicated expressions and simplify.

25. $f(x) = x^2 + 1;\quad f(x + 2),\ f(x) + f(2)$

26. $f(x) = 3x - 1;\quad f(2x),\ 2f(x)$

27. $f(x) = x + 4;\quad f(x^2),\ (f(x))^2$

28. $f(x) = 6x - 18;\quad f\left(\dfrac{x}{3}\right),\ \dfrac{f(x)}{3}$

29–34 ■ Find $f(a)$, $f(a + h)$, and $\dfrac{f(a + h) - f(a)}{h}$, where $h \neq 0$.

29. $f(x) = 3x + 2$

30. $f(x) = x^2 + 1$

31. $f(x) = 5$

32. $f(x) = \dfrac{1}{x + 1}$

33. $f(x) = 3 - 5x + 4x^2$

34. $f(x) = x^3$

35–56 ■ Find the domain of the function.

35. $f(x) = 2x$

36. $f(x) = x^2 + 1$

37. $f(x) = 2x, \quad -1 \leq x \leq 5$

38. $f(x) = x^2 + 1, \quad 0 \leq x \leq 5$

39. $f(x) = \dfrac{1}{x - 3}$

40. $f(x) = \dfrac{1}{3x - 6}$

41. $f(x) = \dfrac{x + 2}{x^2 - 1}$

42. $f(x) = \dfrac{x^4}{x^2 + x - 6}$

43. $f(x) = \sqrt{x - 5}$

44. $f(x) = \sqrt[4]{x + 9}$

45. $f(t) = \sqrt[3]{t - 1}$

46. $g(x) = \sqrt{7 - 3x}$

47. $h(x) = \sqrt{2x - 5}$

48. $G(x) = \sqrt{x^2 - 9}$

48. $G(x) = \sqrt{x^2 - 9}$

49. $g(x) = \dfrac{\sqrt{2 + x}}{3 - x}$

50. $g(x) = \dfrac{\sqrt{x}}{2x^2 + x - 1}$

51. $g(x) = \sqrt[4]{x^2 - 6x}$

52. $g(x) = \sqrt{x^2 - 2x - 8}$

52. $g(x) = \sqrt{x^2 - 2x - 8}$

53. $f(x) = \dfrac{3}{\sqrt{x - 4}}$

54. $f(x) = \dfrac{x^2}{\sqrt{6 - x}}$

55. $f(x) = \dfrac{(x + 1)^2}{\sqrt{2x - 1}}$

56. $f(x) = \dfrac{x}{\sqrt[4]{9 - x^2}}$

57. Production Cost The cost C in dollars of producing x yards of a certain fabric is given by the function

$$C(x) = 1500 + 3x + 0.02x^2 + 0.0001x^3$$

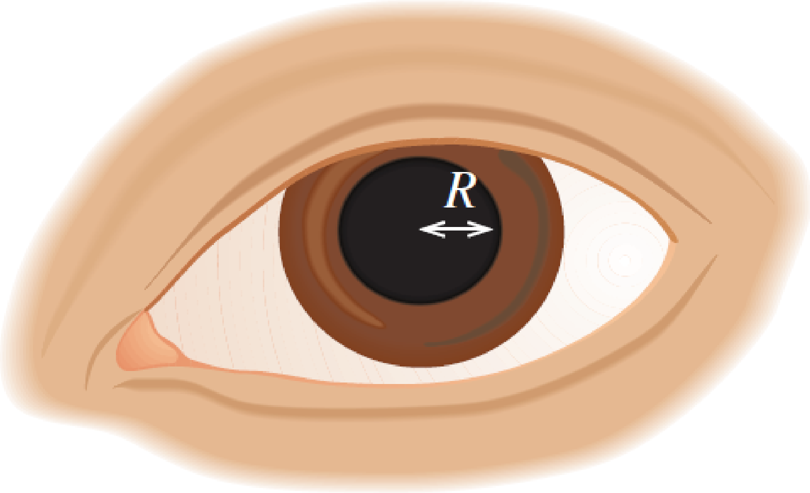
- **(a)** Find $C(10)$ and $C(100)$.
- **(b)** What do your answers in part (a) represent?
- **(c)** Find $C(0)$. (This number represents the *fixed costs.*)

62. Pupil Size When the brightness x of a light source is increased, the eye reacts by decreasing the radius R of the pupil. The dependence of R on x is given by the function

$$R(x) = \sqrt{\frac{13 + 7x^{0.4}}{1 + 4x^{0.4}}}$$

(a) Find $R(1)$, $R(10)$, and $R(100)$.

(b) Make a table of values of $R(x)$.



63. Relativity According to the Theory of Relativity, the length L of an object is a function of its velocity v with respect to an observer. For an object whose length at rest is 10 m, the function is given by

$$L(v) = 10\sqrt{1 - \frac{v^2}{c^2}}$$

where c is the speed of light.

(a) Find $L(0.5c)$, $L(0.75c)$, and $L(0.9c)$.

(b) How does the length of an object change as its velocity increases?

64. Income Tax In a certain country, income tax T is assessed according to the following function of income x:

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10{,}000 \\ 0.08x & \text{if } 10{,}000 < x \le 20{,}000 \\ 1600 + 0.15x & \text{if } 20{,}000 < x \end{cases}$$

(a) Find $T(5{,}000)$, $T(12{,}000)$, and $T(25{,}000)$.

(b) What do your answers in part (a) represent?

3.2 Graphs of Functions

Graphing Functions

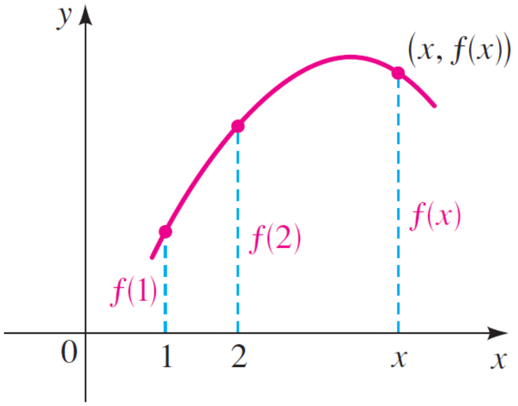


Figure 1

The height of the graph above the point x is the value of $f(x)$.

The Graph of a Function

If f is a function with domain A, then the **graph** of f is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

The graph of a function f gives a picture of the behavior or "life history" of the function. We can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 1).

A function f of the form $f(x) = mx + b$ is called a **linear function** because its graph is the graph of the equation $y = mx + b$, which represents a line with slope m and y-intercept b. A special case of a linear function occurs when the slope is $m = 0$. The function $f(x) = b$, where b is a given number, is called a **constant function** because all its values are the same number, namely, b. Its graph is the horizontal line $y = b$. Figure 2 shows the graphs of the constant function $f(x) = 3$ and the linear function $f(x) = 2x + 1$.

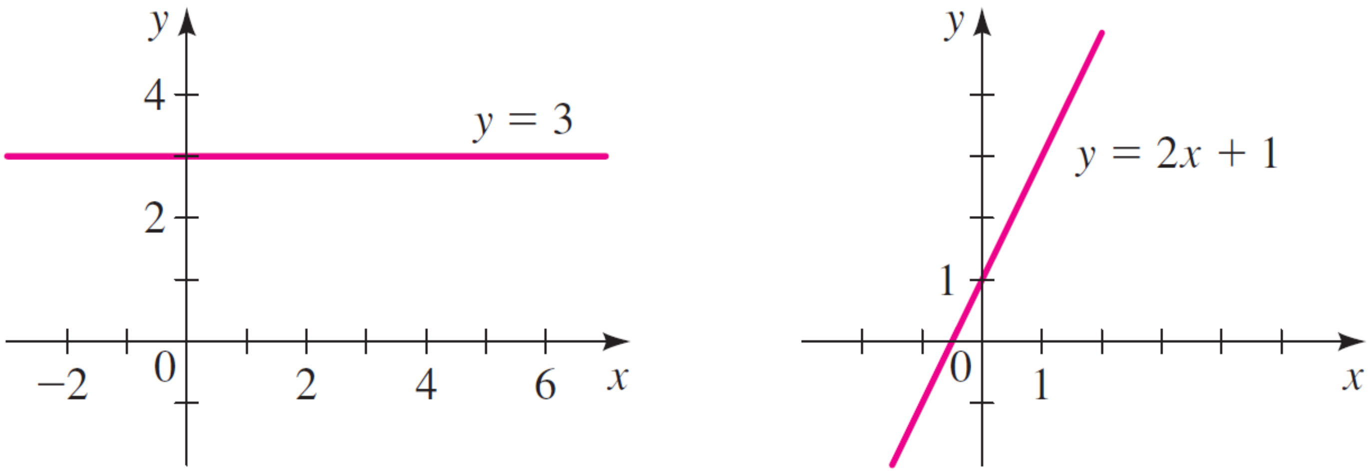


Figure 2 The constant function $f(x) = 3$ The linear function $f(x) = 2x + 1$

Example 1 Graphing Functions

Sketch the graphs of the following functions.

(a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$

Solution

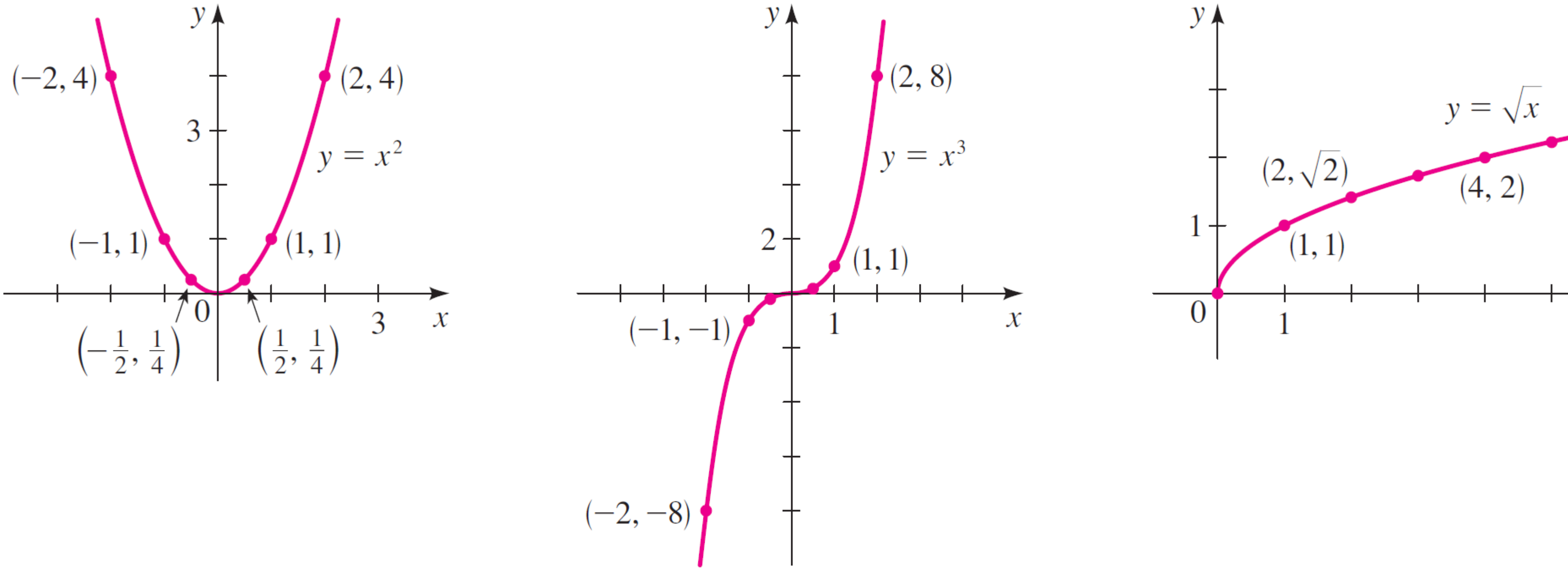
We first make a table of values. Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched in Figure 3.

them by a smooth curve to obtain the graph. The graphs are sketched in Fi

x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$



(a) $f(x) = x^2$

(b) $g(x) = x^3$

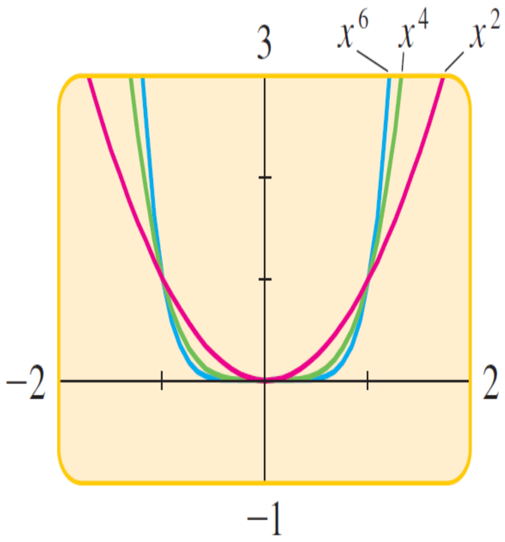
(c) $h(x) = \sqrt{x}$

Example 2 A Family of Power Functions

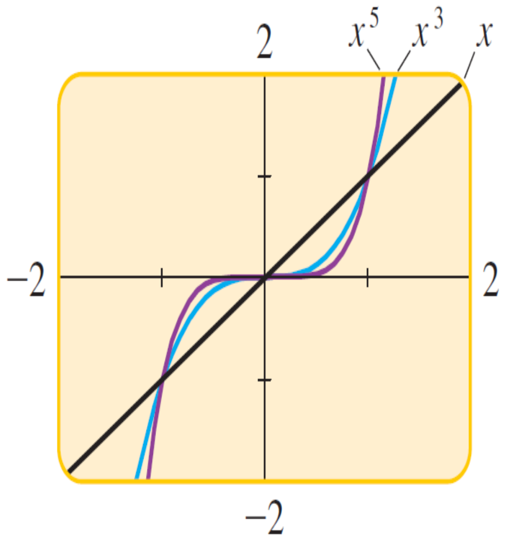
(a) Graph the functions $f(x) = x^n$ for $n = 2, 4$, and 6 in the viewing rectangle $[-2, 2]$ by $[-1, 3]$.

(b) Graph the functions $f(x) = x^n$ for $n = 1, 3$, and 5 in the viewing rectangle $[-2, 2]$ by $[-2, 2]$.

(c) What conclusions can you draw from these graphs?



(a) Even powers of x



(b) Odd powers of x

Solution

The graphs for parts (a) and (b) are shown in Figure 4.

(c) We see that the general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, the graph of $f(x) = x^n$ is similar to the parabola $y = x^2$.

If n is odd, the graph of $f(x) = x^n$ is similar to that of $y = x^3$. ■

Notice from Figure 4 that as n increases the graph of $y = x^n$ becomes flatter near 0 and steeper when $x > 1$. When $0 < x < 1$, the lower powers of x are the "bigger" functions. But when $x > 1$, the higher powers of x are the dominant functions.

Getting Information from the Graph of a Function

The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.

Example 3 Find the Values of a Function from a Graph

The function T graphed in Figure 5 gives the temperature between noon and 6 P.M. at a certain weather station.

(a) Find $T(1)$, $T(3)$, and $T(5)$.

(b) Which is larger, $T(2)$ or $T(4)$?

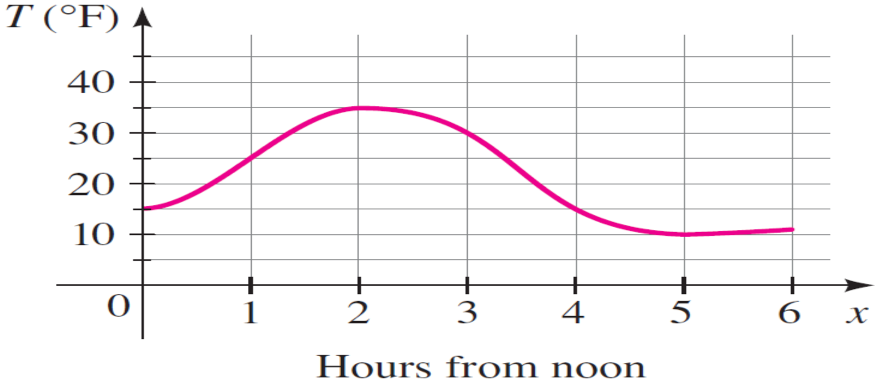


Figure 5
Temperature function

Solution

(a) $T(1)$ is the temperature at 1:00 P.M. It is represented by the height of the graph above the x-axis at $x = 1$. Thus, $T(1) = 25$. Similarly, $T(3) = 30$ and $T(5) = 10$.

(b) Since the graph is higher at $x = 2$ than at $x = 4$, it follows that $T(2)$ is larger than $T(4)$. ■

The graph of a function helps us picture the domain and range of the function on the *x*-axis and *y*-axis as shown in Figure 6.

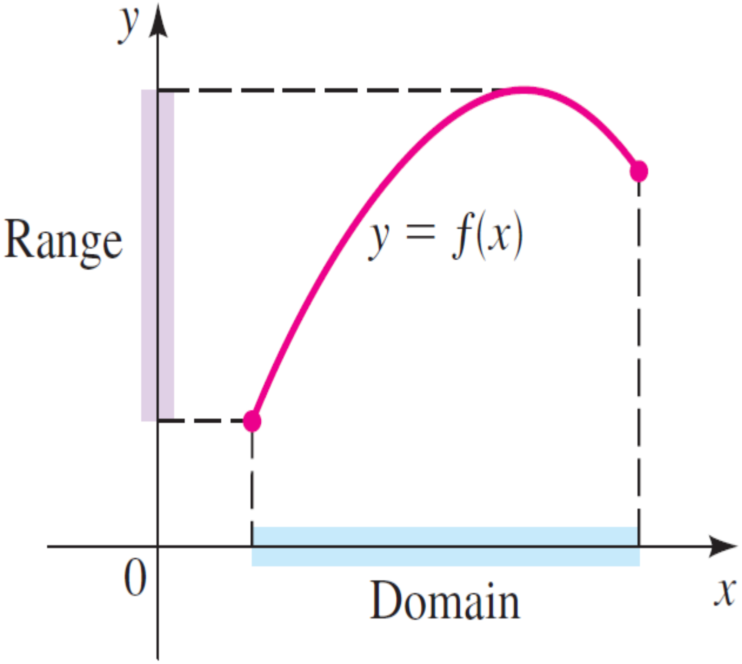


Figure 6

Domain and range of *f*

Example 4 Finding the Domain and Range from a Graph

(a) Use a graphing calculator to draw the graph of $f(x) = \sqrt{4 - x^2}$.
(b) Find the domain and range of f.

Solution

(a) The graph is shown in Figure 7.

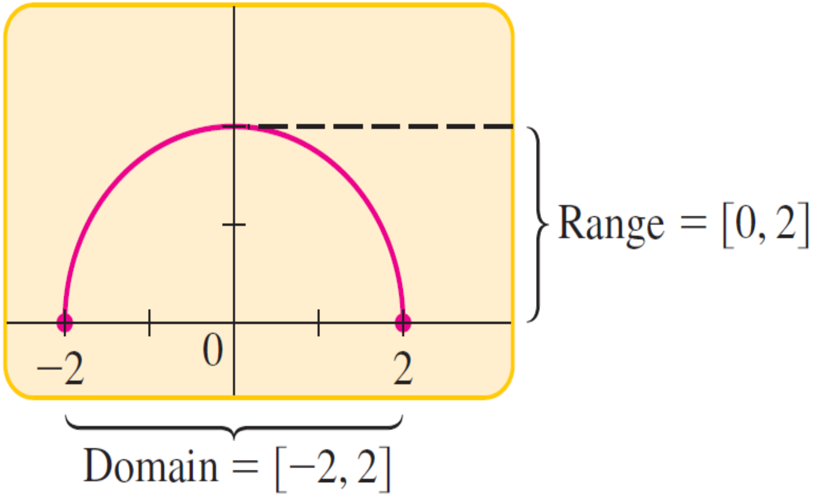


Figure 7

Graph of $f(x) = \sqrt{4 - x^2}$

(b) From the graph in Figure 7 we see that the domain is $[-2, 2]$ and the range is $[0, 2]$. ■

Graphing Piecewise Defined Functions

A piecewise defined function is defined by different formulas on different parts of its domain. As you might expect, the graph of such a function consists of separate pieces.

Example 5 Graph of a Piecewise Defined Function

Sketch the graph of the function

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

Solution

If $x \leq 1$, then $f(x) = x^2$, so the part of the graph to the left of $x = 1$ coincides with the graph of $y = x^2$, which we sketched in Figure 3. If $x > 1$, then $f(x) = 2x + 1$, so the part of the graph to the right of $x = 1$ coincides with the line $y = 2x + 1$, which we graphed in Figure 2. This enables us to sketch the graph in Figure 8.

The solid dot at $(1, 1)$ indicates that this point is included in the graph; the open dot at $(1, 3)$ indicates that this point is excluded from the graph.

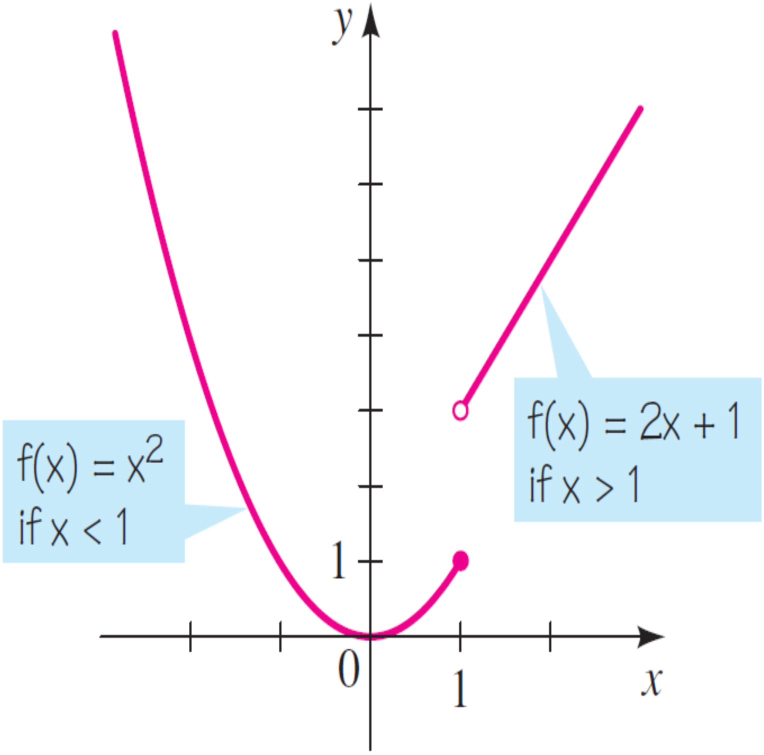


Figure 8

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

Example 6 Graph of the Absolute Value Function

Sketch the graph of the absolute value function $f(x) = |x|$.

Solution

Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 2, we note that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 9).

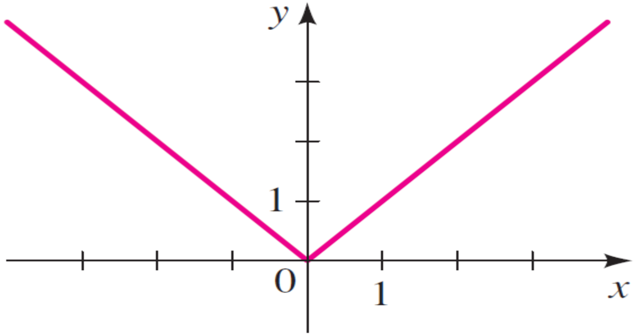


Figure 9
Graph of $f(x) = |x|$

The **greatest integer function** is defined by

$$[\![x]\!] = \text{greatest integer less than or equal to } x$$

For example, $[\![2]\!] = 2$, $[\![2.3]\!] = 2$, $[\![1.999]\!] = 1$, $[\![0.002]\!] = 0$, $[\![-3.5]\!] = -4$, $[\![-0.5]\!] = -1$.

Example 7 Graph of the Greatest Integer Function

Sketch the graph of $f(x) = [\![x]\!]$.

Solution

The table shows the values of f for some values of x. Note that $f(x)$ is constant between consecutive integers so the graph between integers is a horizontal line segment as shown in Figure 10.

x	$[\![x]\!]$
$\vdots$	$\vdots$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$\vdots$	$\vdots$

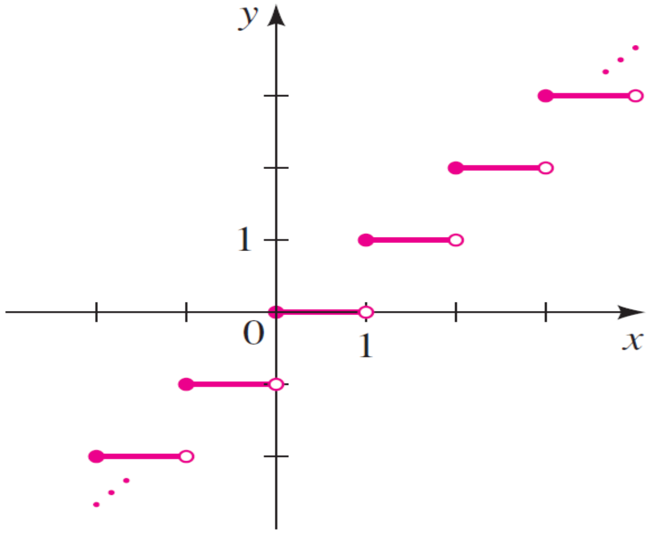


Figure 10

The greatest integer function, $y = [\![x]\!]$ ■

The greatest integer function is an example of a **step function**. The next example gives a real-world example of a step function.

Equations That Define Functions

Any equation in the variables x and y defines a relationship between these variables. For example, the equation

$$y - x^2 = 0$$

defines a relationship between y and x. Does this equation define y as a *function* of x? To find out, we solve for y and get

$$y = x^2$$

We see that the equation defines a rule, or function, that gives one value of y for each value of x. We can express this rule in function notation as

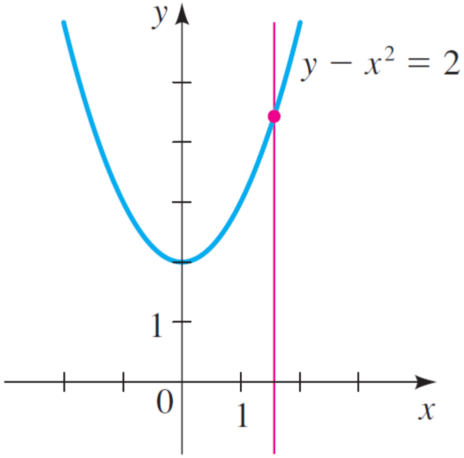
$$f(x) = x^2$$

But not every equation defines y as a function of x, as the following example shows.

Example 10 Equations That Define Functions

Does the equation define y as a function of x?

(a) $y - x^2 = 2$ (b) $x^2 + y^2 = 4$



(a)

Solution

(a) Solving for y in terms of x gives

$$y - x^2 = 2$$

$$y = x^2 + 2 \qquad \text{Add } x^2$$

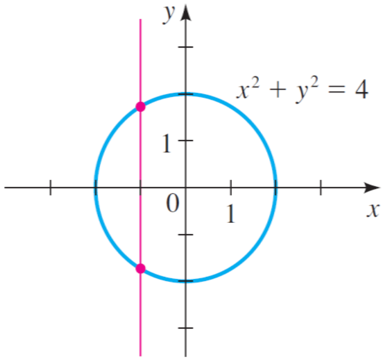
The last equation is a rule that gives one value of y for each value of x, so it defines y as a function of x. We can write the function as $f(x) = x^2 + 2$.

(b) We try to solve for y in terms of x:

$$x^2 + y^2 = 4$$

$$y^2 = 4 - x^2 \qquad \text{Subtract } x^2$$

$$y = \pm\sqrt{4 - x^2} \qquad \text{Take square roots}$$



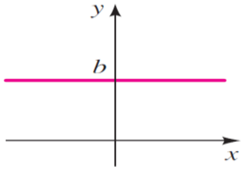
(b)

The last equation gives two values of y for a given value of x. Thus, the equation does not define y as a function of x. ■

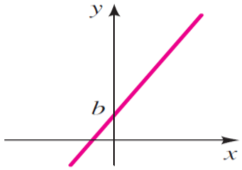
The following table shows the graphs of some functions that you will see frequently in this book.

SOME FUNCTIONS AND THEIR GRAPHS

Linear functions
$f(x) = mx + b$

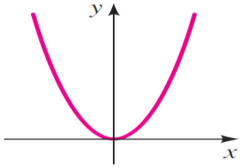


$f(x) = b$

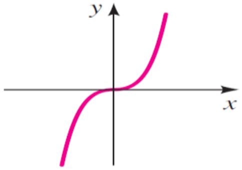


$f(x) = mx + b$

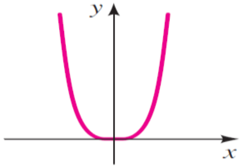
Power functions
$f(x) = x^n$



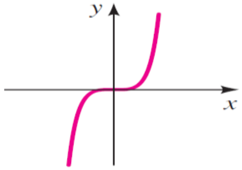
$f(x) = x^2$



$f(x) = x^3$

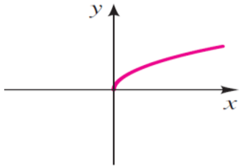


$f(x) = x^4$

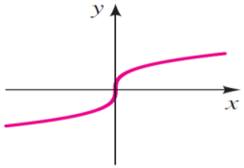


$f(x) = x^5$

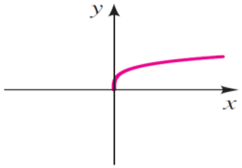
Root functions
$f(x) = \sqrt[n]{x}$



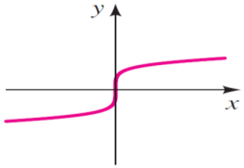
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

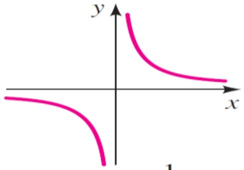


$f(x) = \sqrt[4]{x}$

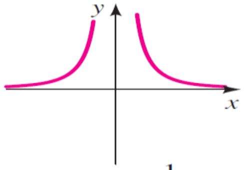


$f(x) = \sqrt[5]{x}$

Reciprocal functions
$f(x) = 1/x^n$

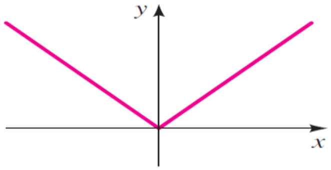


$f(x) = \frac{1}{x}$



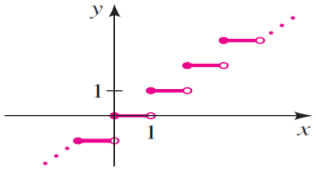
$f(x) = \frac{1}{x^2}$

Absolute value function
$f(x) = |x|$



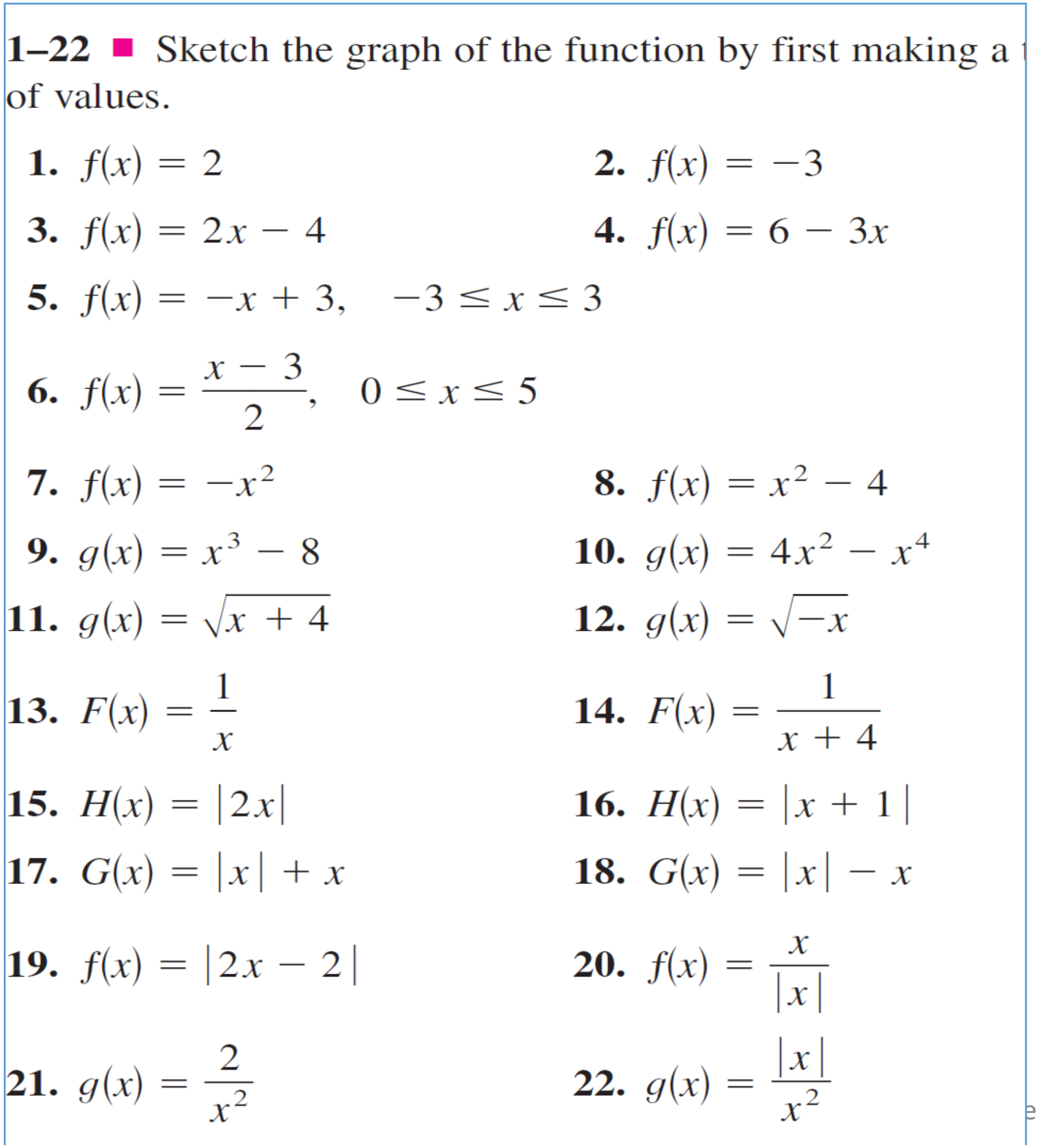
$f(x) = |x|$

Greatest integer function
$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$

3.2 Exercises



1–22 ■ Sketch the graph of the function by first making a
of values.

1. $f(x) = 2$

2. $f(x) = -3$

3. $f(x) = 2x - 4$

4. $f(x) = 6 - 3x$

5. $f(x) = -x + 3, \quad -3 \le x \le 3$

6. $f(x) = \dfrac{x - 3}{2}, \quad 0 \le x \le 5$

7. $f(x) = -x^2$

8. $f(x) = x^2 - 4$

9. $g(x) = x^3 - 8$

10. $g(x) = 4x^2 - x^4$

11. $g(x) = \sqrt{x + 4}$

12. $g(x) = \sqrt{-x}$

13. $F(x) = \dfrac{1}{x}$

14. $F(x) = \dfrac{1}{x + 4}$

15. $H(x) = |2x|$

16. $H(x) = |x + 1|$

17. $G(x) = |x| + x$

18. $G(x) = |x| - x$

19. $f(x) = |2x - 2|$

20. $f(x) = \dfrac{x}{|x|}$

21. $g(x) = \dfrac{2}{x^2}$

22. $g(x) = \dfrac{|x|}{x^2}$

27–36 ■ A function *f* is given.

(a) Use a graphing calculator to draw the graph of *f*.

(b) Find the domain and range of *f* from the graph.

27. $f(x) = x - 1$

28. $f(x) = 2(x + 1)$

29. $f(x) = 4$

30. $f(x) = -x^2$

31. $f(x) = 4 - x^2$

32. $f(x) = x^2 + 4$

33. $f(x) = \sqrt{16 - x^2}$

34. $f(x) = -\sqrt{25 - x^2}$

35. $f(x) = \sqrt{x - 1}$

36. $f(x) = \sqrt{x + 2}$

37–50 ■ Sketch the graph of the piecewise defined function.

37. $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

38. $f(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

39. $f(x) = \begin{cases} 3 & \text{if } x < 2 \\ x - 1 & \text{if } x \geq 2 \end{cases}$

40. $f(x) = \begin{cases} 1 - x & \text{if } x < -2 \\ 5 & \text{if } x \geq -2 \end{cases}$

41. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$

43. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ 1 & \text{if } -1 \leq x \leq 1 \\ -1 & \text{if } x > 1 \end{cases}$

44. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$

45. $f(x) = \begin{cases} 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

46. $$f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 2 \\ x & \text{if } x > 2 \end{cases}$$

47. $$f(x) = \begin{cases} 0 & \text{if } |x| \le 2 \\ 3 & \text{if } |x| > 2 \end{cases}$$

48. $$f(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

49. $$f(x) = \begin{cases} 4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x \le 2 \\ -x + 6 & \text{if } x > 2 \end{cases}$$

50. $$f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 9 - x^2 & \text{if } 0 < x \le 3 \\ x - 3 & \text{if } x > 3 \end{cases}$$

59–70 ■ Determine whether the equation defines y as a function of x. (See Example 10.)

59. $x^2 + 2y = 4$

60. $3x + 7y = 21$

61. $x = y^2$

62. $x^2 + (y - 1)^2 = 4$

63. $x + y^2 = 9$

64. $x^2 + y = 9$

65. $x^2y + y = 1$

66. $\sqrt{x} + y = 12$

67. $2|x| + y = 0$

68. $2x + |y| = 0$

69. $x = y^3$

70. $x = y^4$

3.3 Increasing and Decreasing Functions; Average Rate of Change

Increasing and Decreasing Functions

It is very useful to know where the graph of a function rises and where it falls. The graph shown in Figure 1 rises, falls, then rises again as we move from left to right: It rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*. The function *f* is said to be *increasing* when its graph rises and *decreasing* when its graph falls.

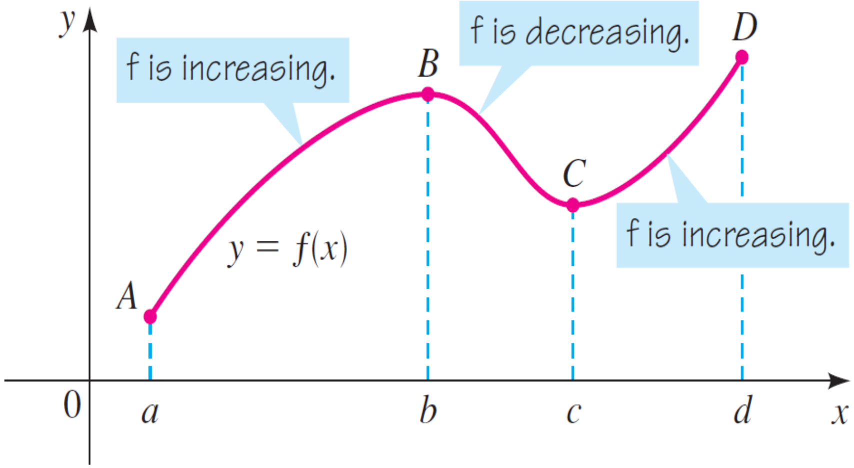


Figure 1

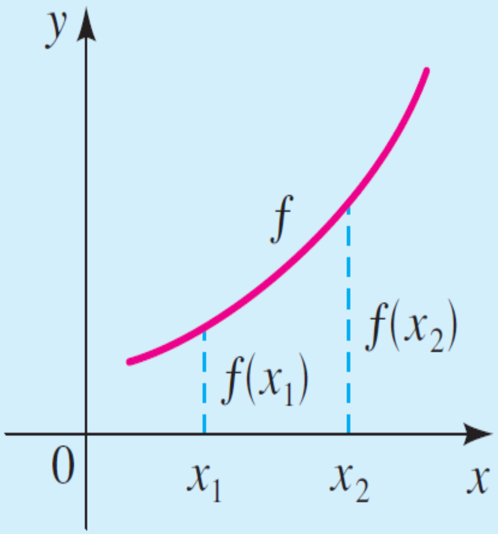
f is increasing on [*a*, *b*] and [*c*, *d*].
f is decreasing on [*b*, *c*].

We have the following definition.

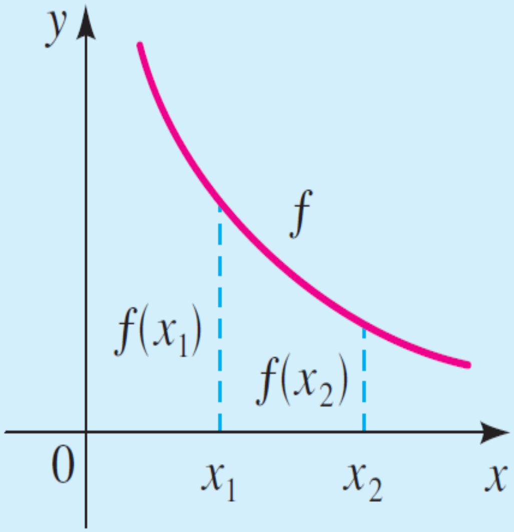
Definition of Increasing and Decreasing Functions

f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



f is increasing



f is decreasing

Example 1 Intervals on which a Function Increases and Decreases

The graph in Figure 2 gives the weight W of a person at age x. Determine the intervals on which the function W is increasing and on which it is decreasing.

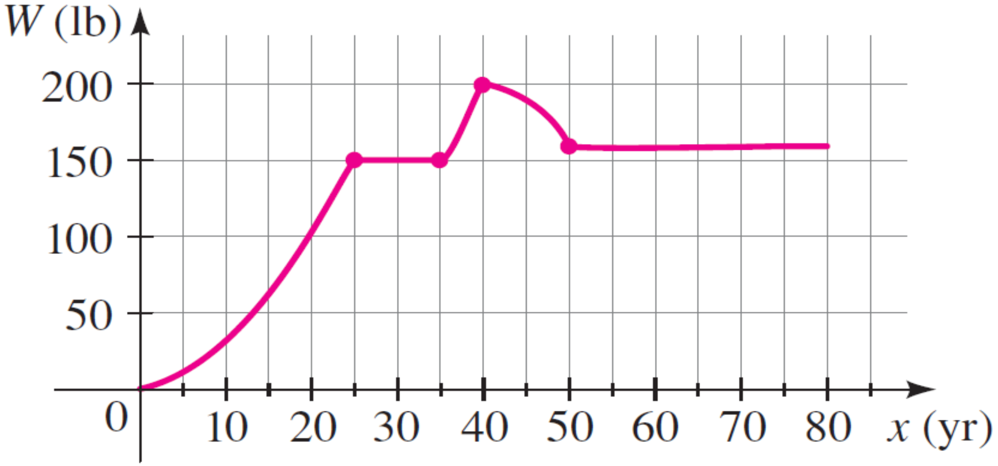


Figure 2

Weight as a function of age

Solution

The function is increasing on $[0, 25]$ and $[35, 40]$. It is decreasing on $[40, 50]$. The function is constant (neither increasing nor decreasing) on $[25, 30]$ and $[50, 80]$. This means that the person gained weight until age 25, then gained weight again between ages 35 and 40. He lost weight between ages 40 and 50. ■

Example 2 Using a Graph to Find Intervals where a Function Increases and Decreases

(a) Sketch the graph of the function $f(x) = x^{2/3}$.

(b) Find the domain and range of the function.

(c) Find the intervals on which f increases and decreases.

Solution

(a) We use a graphing calculator to sketch the graph in Figure 3.

(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $[0, \infty)$.

(c) From the graph we see that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

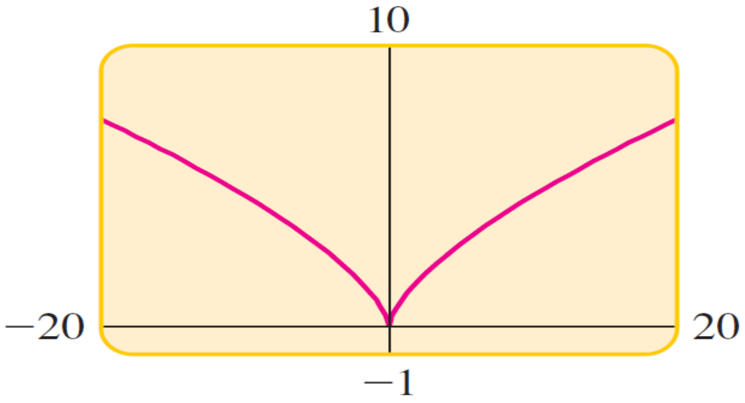


Figure 3
Graph of $f(x) = x^{2/3}$

3.3 Exercises

5–12 ■ A function f is given.

(a) Use a graphing device to draw the graph of f.

(b) State approximately the intervals on which f is increasing and on which f is decreasing.

5. $f(x) = x^{2/5}$

6. $f(x) = 4 - x^{2/3}$

7. $f(x) = x^2 - 5x$

8. $f(x) = x^3 - 4x$

9. $f(x) = 2x^3 - 3x^2 - 12x$

10. $f(x) = x^4 - 16x^2$

11. $f(x) = x^3 + 2x^2 - x - 2$

12. $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$

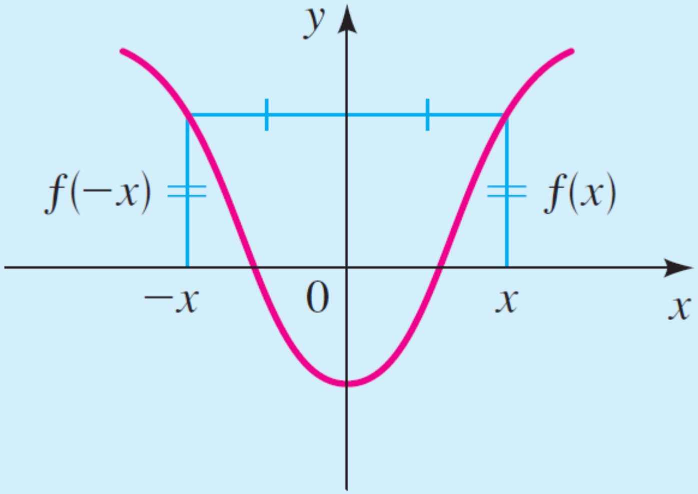
3.4 Transformations of Functions

Even and Odd Functions

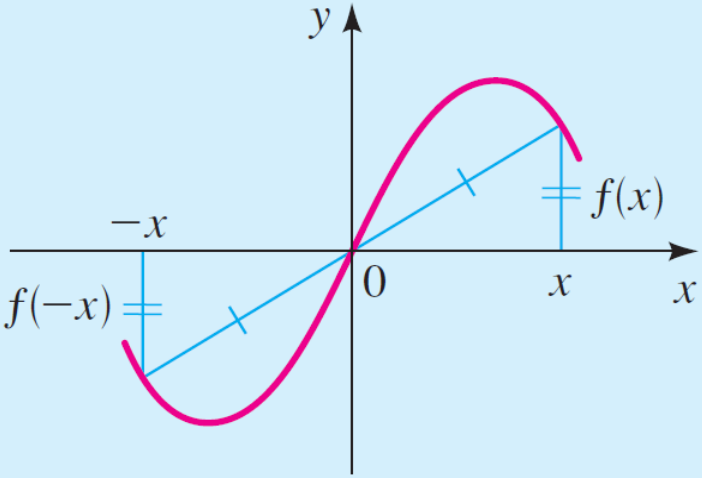
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

Example 9 Even and Odd Functions

Determine whether the functions are even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

Solution

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore, f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h\,(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

3.4 Exercises

61–68 ■ Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

61. $f(x) = x^{-2}$

62. $f(x) = x^{-3}$

63. $f(x) = x^2 + x$

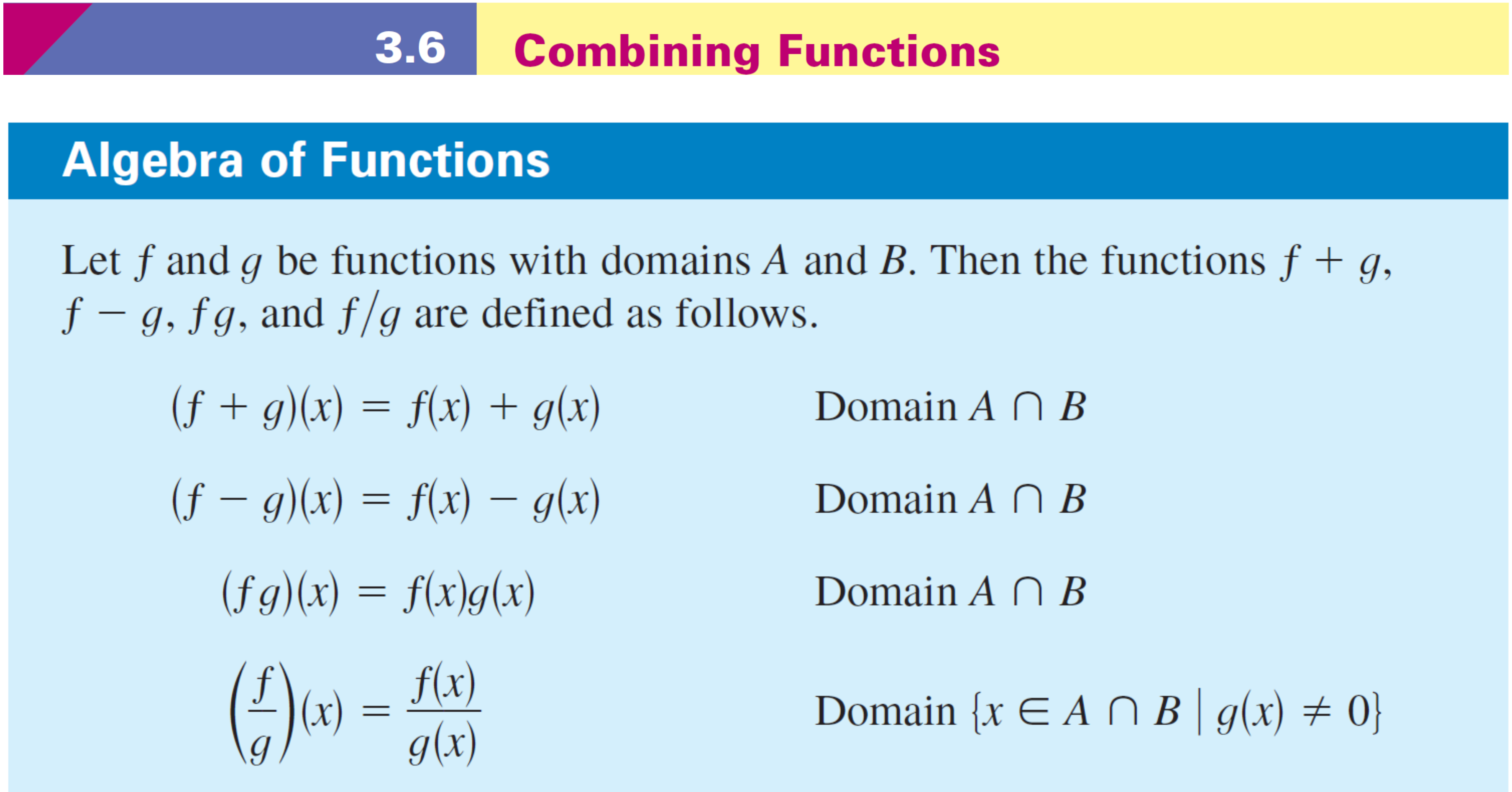
64. $f(x) = x^4 - 4x^2$

65. $f(x) = x^3 - x$

66. $f(x) = 3x^3 + 2x^2 + 1$

67. $f(x) = 1 - \sqrt[3]{x}$

68. $f(x) = x + \dfrac{1}{x}$



3.6 Combining Functions

Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f + g$, $f - g$, fg, and f/g are defined as follows.

$(f + g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f - g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

The sum of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

The name of the new function is "$f + g$." So this + sign stands for the operation of addition of *functions*. The + sign on the right side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Example 1 Combinations of Functions and Their Domains

Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$.

(a) Find the functions $f + g$, $f - g$, fg, and f/g and their domains.

(b) Find $(f + g)(4)$, $(f - g)(4)$, $(fg)(4)$, and $(f/g)(4)$.

Solution

(a) The domain of f is $\{x \mid x \neq 2\}$ and the domain of g is $\{x \mid x \geq 0\}$. The intersection of the domains of f and g is

$$\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0, 2) \cup (2, \infty)$$

Thus, we have

$$(f + g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(f - g)(x) = f(x) - g(x) = \frac{1}{x-2} - \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(fg)(x) = f(x)g(x) = \frac{\sqrt{x}}{x-2} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}} \qquad \text{Domain } \{x \mid x > 0 \text{ and } x \neq 2\}$$

Note that in the domain of f/g we exclude 0 because $g(0) = 0$.

(b) Each of these values exist because $x = 4$ is in the domain of each function.

$$(f + g)(4) = f(4) + g(4) = \frac{1}{4-2} + \sqrt{4} = \frac{5}{2}$$

$$(f - g)(4) = f(4) - g(4) = \frac{1}{4-2} - \sqrt{4} = -\frac{3}{2}$$

$$(fg)(4) = f(4)g(4) = \left(\frac{1}{4-2}\right)\sqrt{4} = 1$$

$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4-2)\sqrt{4}} = \frac{1}{4}$$

■

To divide fractions, invert the denominator and multiply:

$$\frac{1/(x-2)}{\sqrt{x}} = \frac{1/(x-2)}{\sqrt{x}/1} = \frac{1}{x-2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{(x-2)\sqrt{x}}$$

The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates, as illustrated in the next example.

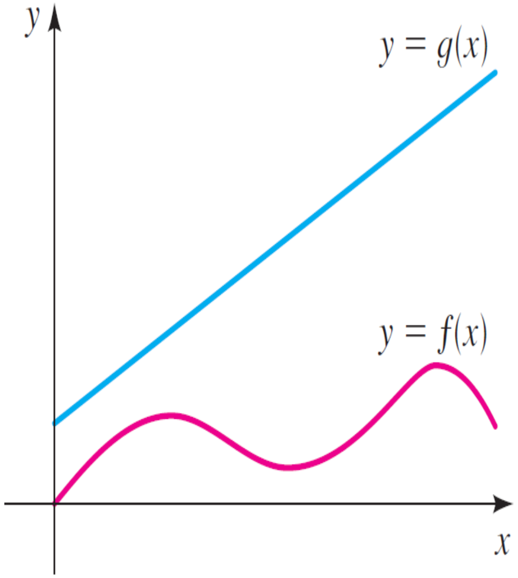


Figure 1

Example 2 Using Graphical Addition

The graphs of f and g are shown in Figure 1. Use graphical addition to graph the function $f + g$.

Solution

We obtain the graph of $f + g$ by “graphically adding” the value of $f(x)$ to $g(x)$ as

shown in Figure 2. This is implemented by copying the line segment PQ on top of PR to obtain the point S on the graph of $f + g$.

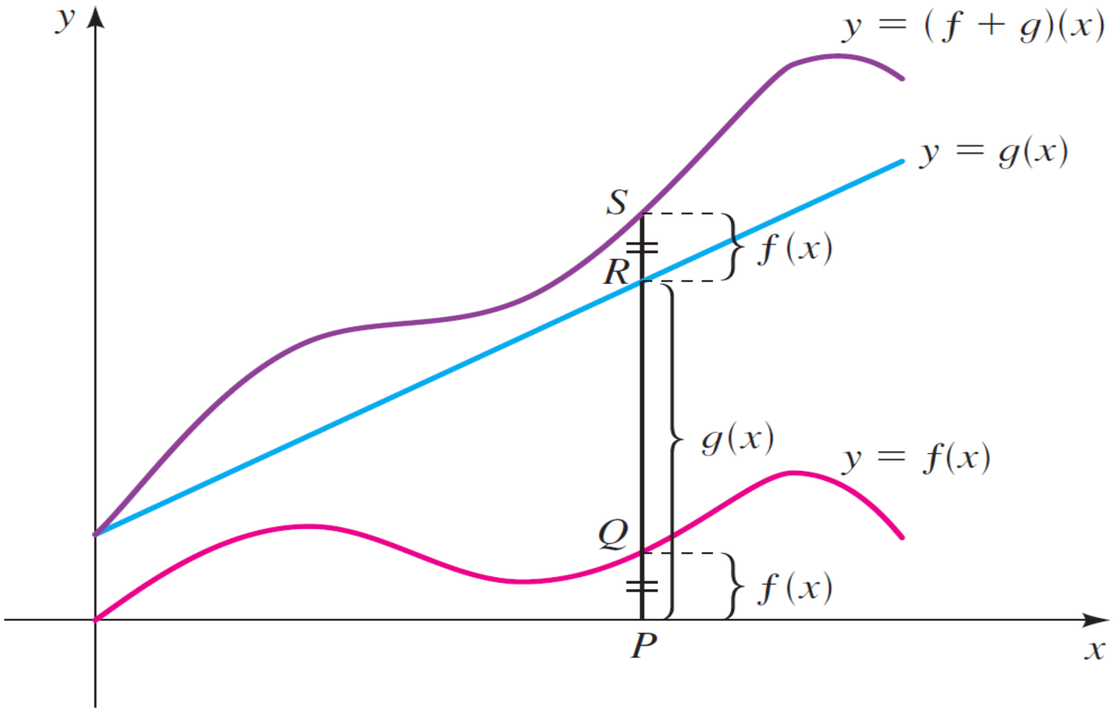


Figure 2
Graphical addition

Composition of Functions

Now let's consider a very important way of combining two functions to get a new function. Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. We may define a function h as

$$h(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The function h is made up of the functions f and g in an interesting way: Given a number x, we first apply to it the function g, then apply f to the result. In this case, f is the rule "take the square root," g is the rule "square, then add 1," and h is the rule "square, then add 1, then take the square root." In other words, we get the rule h by applying the rule g and then the rule f.

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f composed with g").

Composition of Functions

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. We can picture $f \circ g$ using a machine diagram (Figure 3) or an arrow diagram (Figure 4).

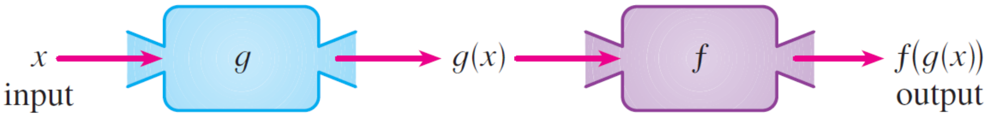


Figure 3

The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

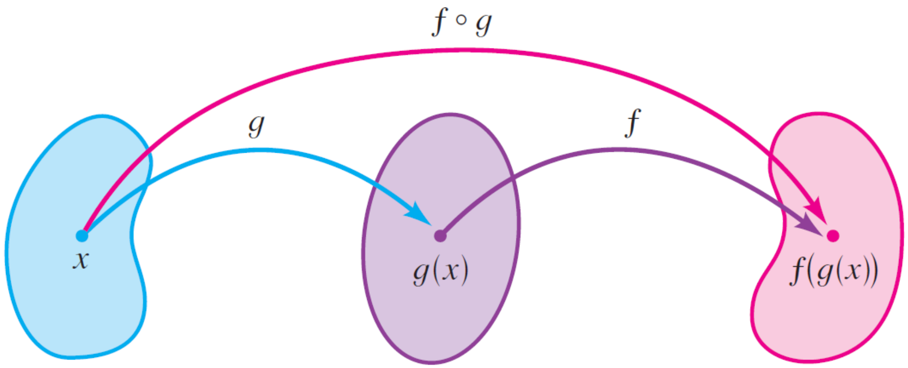


Figure 4

Arrow diagram for $f \circ g$

Example 3 Finding the Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.
(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

In Example 3, f is the rule "square" and g is the rule "subtract 3." The function $f \circ g$ *first* subtracts 3 and *then* squares; the function $g \circ f$ *first* squares and *then* subtracts 3.

Solution

(a) We have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(x - 3) && \text{Definition of } g\\ &= (x - 3)^2 && \text{Definition of } f\end{aligned}$$

and

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(x^2) && \text{Definition of } f\\ &= x^2 - 3 && \text{Definition of } g\end{aligned}$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have

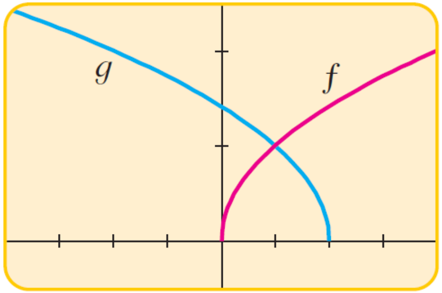
$$(f \circ g)(5) = f(g(5)) = f(2) = 2^2 = 4$$

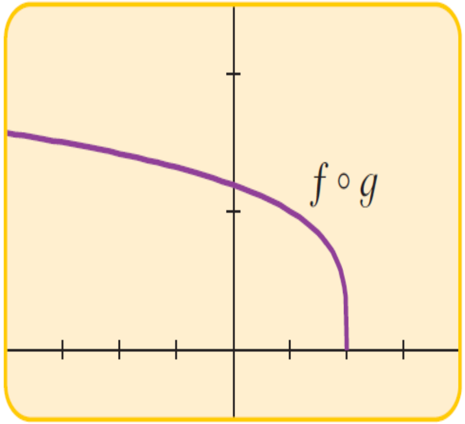
$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

■

You can see from Example 3 that, in general, $f \circ g \neq g \circ f$. Remember that the notation $f \circ g$ means that the function g is applied first and then f is applied second.

The graphs of f and g of Example 4, as well as $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, are shown below. These graphs indicate that the operation of composition can produce functions quite different from the original functions.





Example 4 Finding the Composition of Functions

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the following functions and their domains.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Solution

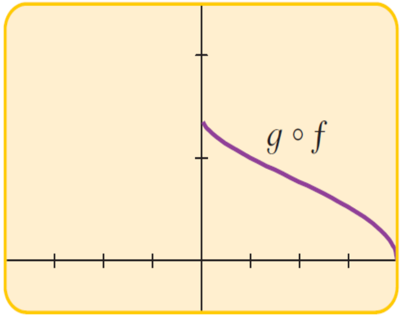
(a)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f\left(\sqrt{2 - x}\right) && \text{Definition of } g \\ &= \sqrt{\sqrt{2 - x}} && \text{Definition of } f \\ &= \sqrt[4]{2 - x} \end{aligned}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b)

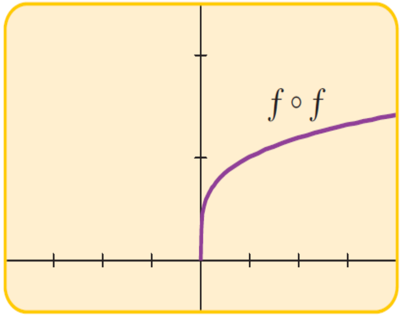
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g\left(\sqrt{x}\right) && \text{Definition of } f \\ &= \sqrt{2 - \sqrt{x}} && \text{Definition of } g \end{aligned}$$



For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined, we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c)

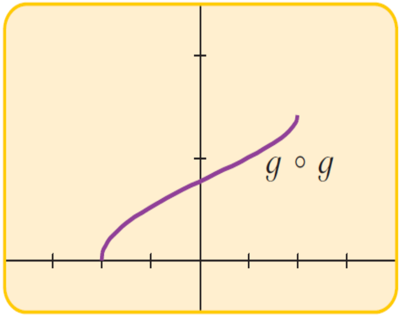
$$\begin{aligned} (f \circ f)(x) &= f(f(x)) && \text{Definition of } f \circ f \\ &= f(\sqrt{x}) && \text{Definition of } f \\ &= \sqrt{\sqrt{x}} && \text{Definition of } f \\ &= \sqrt[4]{x} \end{aligned}$$



The domain of $f \circ f$ is $[0, \infty)$.

(d)

$$\begin{aligned} (g \circ g)(x) &= g(g(x)) && \text{Definition of } g \circ g \\ &= g(\sqrt{2 - x}) && \text{Definition of } g \\ &= \sqrt{2 - \sqrt{2 - x}} && \text{Definition of } g \end{aligned}$$



This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, or $x \geq -2$. Thus, $-2 \leq x \leq 2$, so the domain of $g \circ g$ is $[-2, 2]$. ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then

f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Example 5 A Composition of Three Functions

Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

Solution

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) && \text{Definition of } f \circ g \circ h \\
&= f(g(x + 3)) && \text{Definition of } h \\
&= f((x + 3)^{10}) && \text{Definition of } g \\
&= \frac{(x + 3)^{10}}{(x + 3)^{10} + 1} && \text{Definition of } f
\end{aligned}$$

■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is useful to be able to "decompose" a complicated function into simpler ones, as shown in the following example.

Example 6 Recognizing a Composition of Functions

Given $F(x) = \sqrt[4]{x + 9}$, find functions f and g such that $F = f \circ g$.

Solution

Since the formula for F says to first add 9 and then take the fourth root, we let

$$g(x) = x + 9 \qquad \text{and} \qquad f(x) = \sqrt[4]{x}$$

Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(x + 9) && \text{Definition of } g\\ &= \sqrt[4]{x + 9} && \text{Definition of } f\\ &= F(x)\end{aligned}$$

■

3.6 Exercises

1–6 ■ Find $f + g$, $f - g$, fg, and f/g and their domains.

1. $f(x) = x^2, \quad g(x) = x + 2$

2. $f(x) = x^3 + 2x^2, \quad g(x) = 3x^2 - 1$

3. $f(x) = \sqrt{1 + x^2}, \quad g(x) = \sqrt{1 - x}$

4. $f(x) = \sqrt{9 - x^2}, \quad g(x) = \sqrt{x^2 - 1}$

5. $f(x) = \dfrac{2}{x}, \quad g(x) = -\dfrac{2}{x + 4}$

6. $f(x) = \dfrac{1}{x + 1}, \quad g(x) = \dfrac{x}{x + 1}$

7–10 ■ Find the domain of the function.

7. $f(x) = \sqrt{x} + \sqrt{1 - x}$

8. $g(x) = \sqrt{x + 1} - \dfrac{1}{x}$

9. $h(x) = (x - 3)^{-1/4}$

10. $k(x) = \dfrac{\sqrt{x + 3}}{x - 1}$

17–22 ■ Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate the expression.

17. **(a)** $f(g(0))$ **(b)** $g(f(0))$

18. **(a)** $f(f(4))$ **(b)** $g(g(3))$

19. **(a)** $(f \circ g)(-2)$ **(b)** $(g \circ f)(-2)$

20. **(a)** $(f \circ f)(-1)$ **(b)** $(g \circ g)(2)$

21. **(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

22. **(a)** $(f \circ f)(x)$ **(b)** $(g \circ g)(x)$

45–50 ■ Express the function in the form $f \circ g$.

45. $F(x) = (x - 9)^5$

46. $F(x) = \sqrt{x} + 1$

47. $G(x) = \dfrac{x^2}{x^2 + 4}$

48. $G(x) = \dfrac{1}{x + 3}$

49. $H(x) = |1 - x^3|$

50. $H(x) = \sqrt{1 + \sqrt{x}}$

29–40 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

29. $f(x) = 2x + 3, \quad g(x) = 4x - 1$

30. $f(x) = 6x - 5, \quad g(x) = \dfrac{x}{2}$

31. $f(x) = x^2, \quad g(x) = x + 1$

32. $f(x) = x^3 + 2, \quad g(x) = \sqrt[3]{x}$

33. $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 4$

34. $f(x) = x^2, \quad g(x) = \sqrt{x - 3}$

35. $f(x) = |x|, \quad g(x) = 2x + 3$

36. $f(x) = x - 4, \quad g(x) = |x + 4|$

37. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = 2x - 1$

38. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x^2 - 4x$

39. $f(x) = \sqrt[3]{x}, \quad g(x) = \sqrt[4]{x}$

40. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{x}{x - 2}$

41–44 ■ Find $f \circ g \circ h$.

41. $f(x) = x - 1, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1$

42. $f(x) = \dfrac{1}{x}, \quad g(x) = x^3, \quad h(x) = x^2 + 2$

43. $f(x) = x^4 + 1, \quad g(x) = x - 5, \quad h(x) = \sqrt{x}$

44. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

3.7 One-to-One Functions and Their Inverses

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.

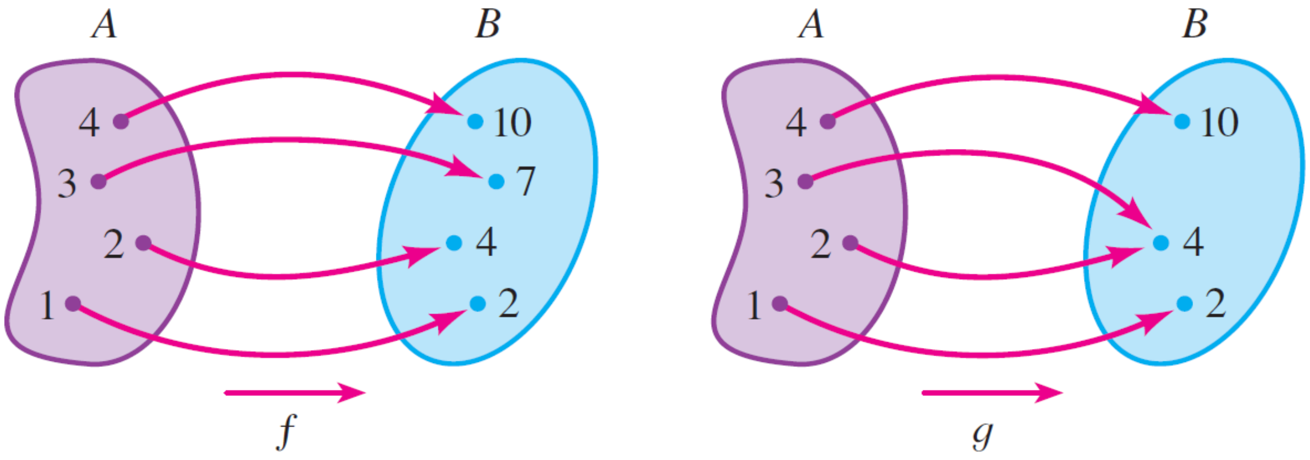


Figure 1 f is one-to-one g is not one-to-one

Definition of a One-to-One Function

A function within domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

An equivalent way of writing the condition for a one-to-one function is this:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

If a horizontal line intersects the graph of f at more than one point, then we see from Figure 2 that there are numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore, we have the following geometric method for determining whether a function is one-to-one.

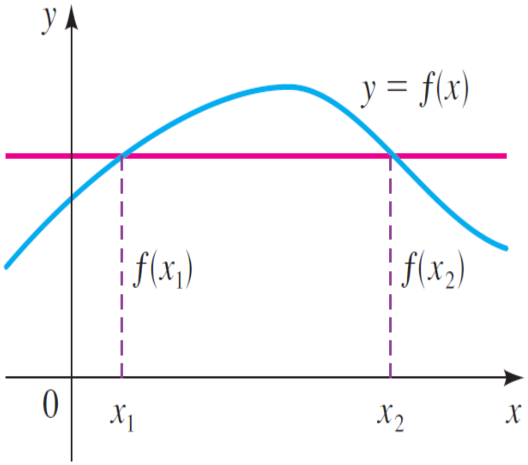


Figure 2

This function is not one-to-one because $f(x_1) = f(x_2)$.

Horizontal Line Test

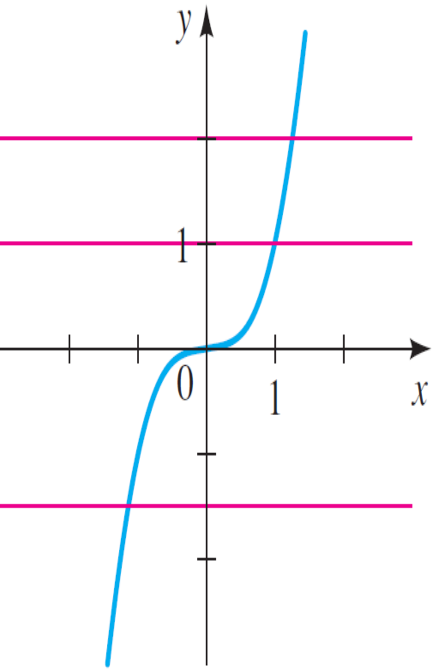
A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1 Deciding whether a Function Is One-to-One

Is the function $f(x) = x^3$ one-to-one?

Solution 1

If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube). Therefore, $f(x) = x^3$ is one-to-one.



Solution 2

From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

Notice that the function f of Example 1 is increasing and is also one-to-one. In fact, it can be proved that every increasing function and every decreasing function is one-to-one.

Example 2 Deciding whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

Solution 1

This function is not one-to-one because, for instance,

$$g(1) = 1 \qquad \text{and} \qquad g(-1) = 1$$

and so 1 and -1 have the same image.

Solution 2

From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. ■

Although the function g in Example 2 is not one-to-one, it is possible to restrict its domain so that the resulting function is one-to-one. In fact, if we define

$$h(x) = x^2, \qquad x \geq 0$$

then h is one-to-one, as you can see from Figure 5 and the Horizontal Line Test.

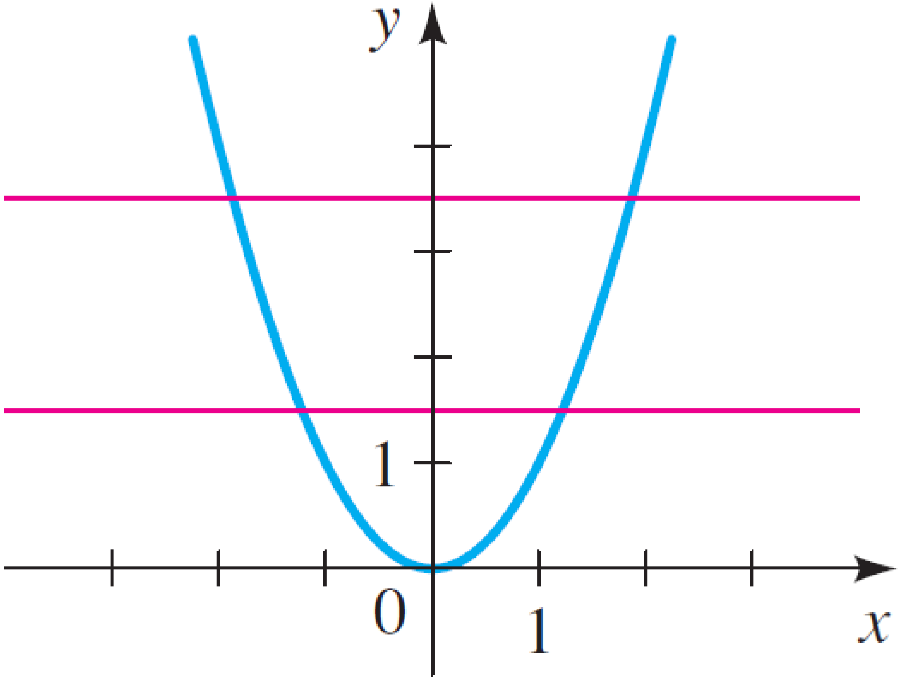


Figure 4

$g(x) = x^2$ is not one-to-one.

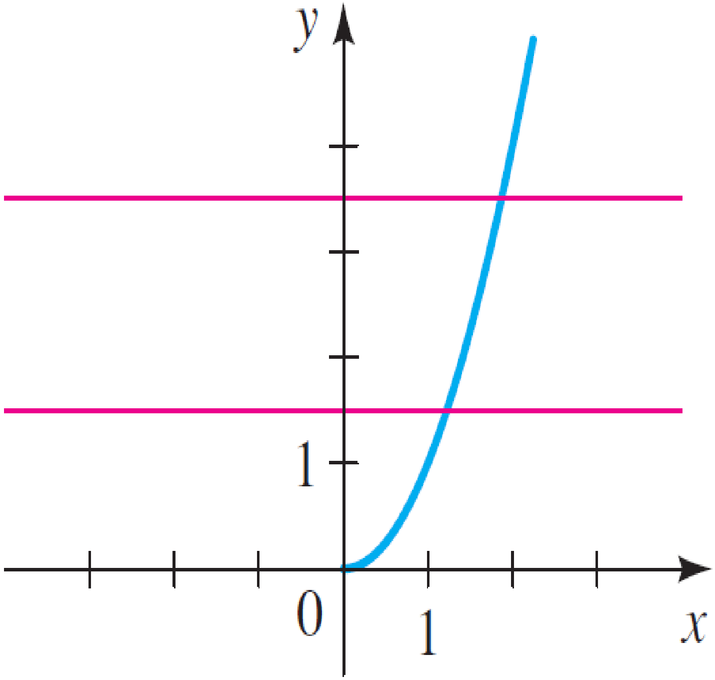


Figure 5

$h(x) = x^2$ $(x \geq 0)$ is one-to-one.

Example 3 Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

Solution

Suppose there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. Then

$$3x_1 + 4 = 3x_2 + 4 \qquad \text{Suppose } f(x_1) = f(x_2)$$

$$3x_1 = 3x_2 \qquad \text{Subtract 4}$$

$$x_1 = x_2 \qquad \text{Divide by 3}$$

Therefore, *f* is one-to-one.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

This definition says that if f takes x into y, then f^{-1} takes y back into x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Don't mistake the -1 in f^{-1} for an exponent.

$$f^{-1} \quad \textit{does not mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ is written as $(f(x))^{-1}$.

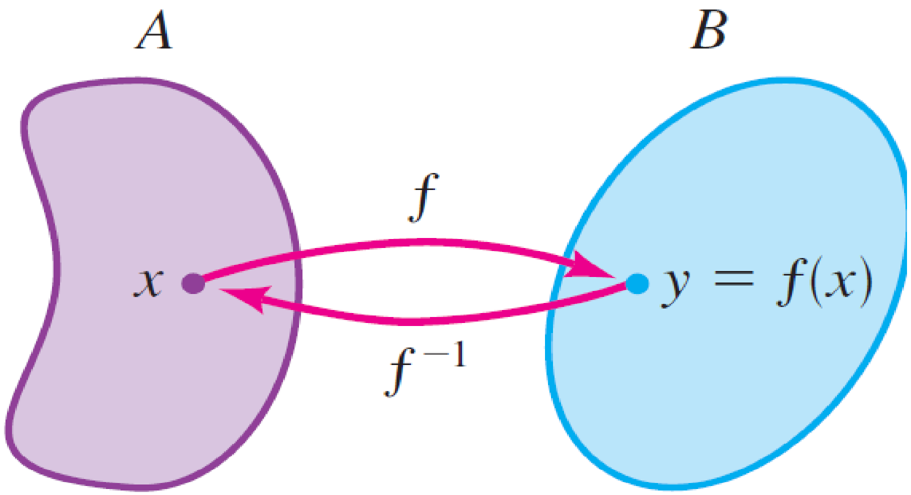


Figure 6

Example 4 Finding f^{-1} for Specific Values

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

Solution

From the definition of f^{-1} we have

$$\begin{aligned} f^{-1}(5) &= 1 \qquad \text{because} \qquad f(1) = 5 \\ f^{-1}(7) &= 3 \qquad \text{because} \qquad f(3) = 7 \\ f^{-1}(-10) &= 8 \qquad \text{because} \qquad f(8) = -10 \end{aligned}$$

Figure 7 shows how f^{-1} reverses the effect of f in this case.

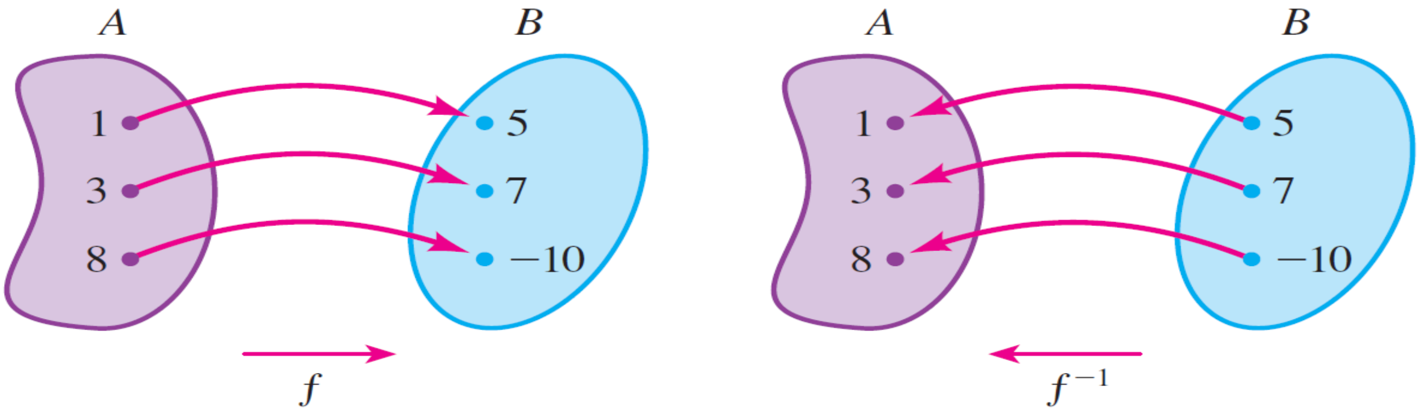


Figure 7

■

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

Property of Inverse Functions

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties.

$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1}, so we say that f and f^{-1} are *inverses of each other*.

Example 5 Verifying That Two Functions Are Inverses

Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.

Solution

Note that the domain and range of both f and g is $\mathbb{R}$. We have

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

So, by the Property of Inverse Functions, f and g are inverses of each other. These equations simply say that the cube function and the cube root function, when composed, cancel each other. ■

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So, if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

How to Find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Note that Steps 2 and 3 can be reversed. In other words, we can interchange x and y first and then solve for y in terms of x.

Example 6 Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

Solution

First we write $y = f(x)$.

$$y = 3x - 2$$

Then we solve this equation for x:

$$3x = y + 2 \qquad \text{Add 2}$$

$$x = \frac{y + 2}{3} \qquad \text{Divide by 3}$$

Finally, we interchange x and y:

$$y = \frac{x + 2}{3}$$

Therefore, the inverse function is $f^{-1}(x) = \dfrac{x + 2}{3}$.

Example 7 Finding the Inverse of a Function

Find the inverse of the function $f(x) = \dfrac{x^5 - 3}{2}$.

Solution

We first write $y = (x^5 - 3)/2$ and solve for x.

$$\begin{aligned}
y &= \frac{x^5 - 3}{2} && \text{Equation defining function} \\
2y &= x^5 - 3 && \text{Multiply by 2} \\
x^5 &= 2y + 3 && \text{Add 3} \\
x &= (2y + 3)^{1/5} && \text{Take fifth roots}
\end{aligned}$$

Then we interchange x and y to get $y = (2x + 3)^{1/5}$. Therefore, the inverse function is $f^{-1}(x) = (2x + 3)^{1/5}$.

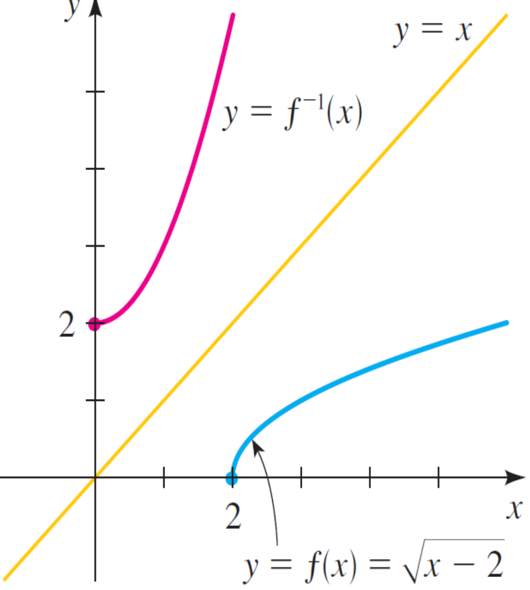


Figure 10

Example 8 Finding the Inverse of a Function

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.
(b) Use the graph of f to sketch the graph of f^{-1}.
(c) Find an equation for f^{-1}.

Solution

(a) Using the transformations from Section 3.4, we sketch the graph of $y = \sqrt{x - 2}$ by plotting the graph of the function $y = \sqrt{x}$ (Example 1(c) in Section 3.2) and moving it to the right 2 units.

(b) The graph of f^{-1} is obtained from the graph of f in part (a) by reflecting it in the line $y = x$, as shown in Figure 10.

(c) Solve $y = \sqrt{x - 2}$ for x, noting that $y \geq 0$.

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2 \qquad \text{Square each side}$$

$$x = y^2 + 2, \quad y \geq 0 \qquad \text{Add 2}$$

Interchange x and y:

$$y = x^2 + 2, \qquad x \geq 0$$

Thus $$f^{-1}(x) = x^2 + 2, \qquad x \geq 0$$

This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$ and, from the graph shown in Figure 10, this seems reasonable. ■

In Example 8 note how f^{-1} reverses the effect of f. The function f is the rule "subtract 2, then take the square root," whereas f^{-1} is the rule "square, then add 2."

3.7 Exercises

7–16 ■ Determine whether the function is one-to-one.

7. $f(x) = 3x + 1$

8. $f(x) = -2x + 5$

9. $g(x) = \sqrt{x}$

10. $g(x) = |x|$

11. $h(x) = x^2 - 2x$

12. $h(x) = x^3 + 8$

13. $f(x) = x^4 + 5$

14. $f(x) = x^4 + 5, \quad 0 \le x \le 2$

15. $f(x) = \dfrac{1}{x^2}$

16. $f(x) = \dfrac{1}{x}$

19. If $f(x) = 5 - 2x$, find $f^{-1}(3)$.

20. If $g(x) = x^2 + 4x$ with $x \ge -2$, find $g^{-1}(5)$.

17–20 ■ Assume f is a one-to-one function.

17. **(a)** If $f(2) = 7$, find $f^{-1}(7)$.
(b) If $f^{-1}(3) = -1$, find $f(-1)$.

18. **(a)** If $f(5) = 18$, find $f^{-1}(18)$.
(b) If $f^{-1}(4) = 2$, find $f(2)$.

21–30 ■ Use the Property of Inverse Functions to show that f and g are inverses of each other.

21. $f(x) = x + 3, \quad g(x) = x - 3$

22. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$

23. $f(x) = 2x - 5; \quad g(x) = \dfrac{x + 5}{2}$

24. $f(x) = \dfrac{3 - x}{4}; \quad g(x) = 3 - 4x$

25. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

26. $f(x) = x^5, \quad g(x) = \sqrt[5]{x}$

27. $f(x) = x^2 - 4, \quad x \geq 0;$
$g(x) = \sqrt{x + 4}, \quad x \geq -4$

28. $f(x) = x^3 + 1; \quad g(x) = (x - 1)^{1/3}$

29. $f(x) = \dfrac{1}{x - 1}, \quad x \neq 1;$
$g(x) = \dfrac{1}{x} + 1, \quad x \neq 0$

30. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2;$
$g(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$

31–50 ■ Find the inverse function of f.

31. $f(x) = 2x + 1$

32. $f(x) = 6 - x$

33. $f(x) = 4x + 7$

34. $f(x) = 3 - 5x$

35. $f(x) = \dfrac{x}{2}$

36. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

37. $f(x) = \dfrac{1}{x + 2}$

38. $f(x) = \dfrac{x - 2}{x + 2}$

39. $f(x) = \dfrac{1 + 3x}{5 - 2x}$

40. $f(x) = 5 - 4x^3$

41. $f(x) = \sqrt{2 + 5x}$

42. $f(x) = x^2 + x, \quad x \geq -\frac{1}{2}$

43. $f(x) = 4 - x^2, \quad x \geq 0$

44. $f(x) = \sqrt{2x - 1}$

45. $f(x) = 4 + \sqrt[3]{x}$

46. $f(x) = (2 - x^3)^5$

47. $f(x) = 1 + \sqrt{1 + x}$

48. $f(x) = \sqrt{9 - x^2}, \quad 0 \leq x \leq 3$

4.1 **Polynomial Functions and Their Graphs**

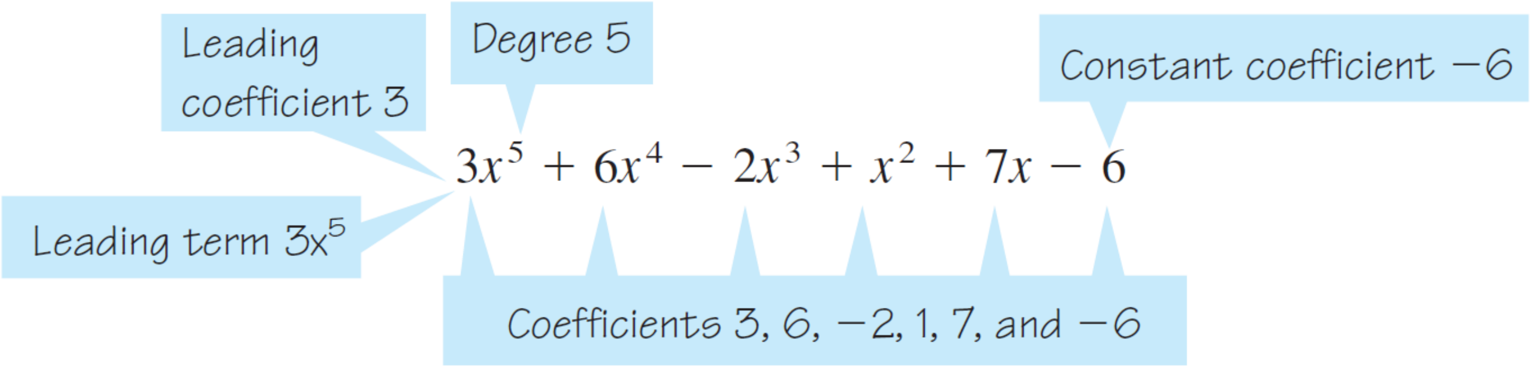
Polynomial Functions

A **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$. The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial. The number a_0 is the **constant coefficient** or **constant term**. The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

We often refer to polynomial functions simply as *polynomials*. The following polynomial has degree 5, leading coefficient 3, and constant term −6.



Here are some more examples of polynomials.

$$P(x) = 3 \qquad \text{Degree 0}$$

$$Q(x) = 4x - 7 \qquad \text{Degree 1}$$

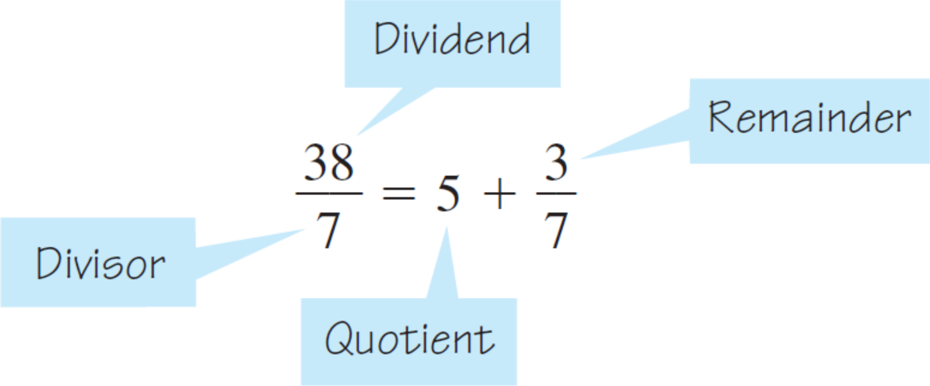
$$R(x) = x^2 + x \qquad \text{Degree 2}$$

$$S(x) = 2x^3 - 6x^2 - 10 \qquad \text{Degree 3}$$

If a polynomial consists of just a single term, then it is called a **monomial**. For example, $P(x) = x^3$ and $Q(x) = -6x^5$ are monomials.

Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write



$$\frac{38}{7} = 5 + \frac{3}{7}$$

To divide polynomials, we use long division.

Example 1 Long Division of Polynomials

Divide $6x^2 - 26x + 12$ by $x - 4$.

Solution

The *dividend* is $6x^2 - 26x + 12$ and the *divisor* is $x - 4$. We begin by arranging them as follows:

$$x - 4 \,\overline{\big)\, 6x^2 - 26x + 12}$$

Next we divide the leading term in the dividend by the leading term in the divisor to get the first term of the quotient: $6x^2/x = 6x$. Then we multiply the divisor by $6x$ and subtract the result from the dividend.

$$\begin{array}{rrrr}
 & 6x & & \\ \cline{2-4}
x - 4 \,\big) & 6x^2 & -\,26x & +\,12 \\
 & 6x^2 & -\,24x & \\ \cline{2-3}
 & & -2x & +\,12
\end{array}$$

Divide leading terms: $\dfrac{6x^2}{x} = 6x$

Multiply: $6x(x - 4) = 6x^2 - 24x$

Subtract and "bring down" 12

We repeat the process using the last line $-2x + 12$ as the dividend.

$$\begin{array}{rrrr}
 & 6x & -\,2 & \\ \cline{2-4}
x - 4 \,\big) & 6x^2 & -\,26x & +\,12 \\
 & 6x^2 & -\,24x & \\ \cline{2-3}
 & & -2x & +\,12 \\
 & & -2x & +\,8 \\ \cline{3-4}
 & & & 4
\end{array}$$

Divide leading terms: $\dfrac{-2x}{x} = -2$

Multiply: $-2(x - 4) = -2x + 8$

Subtract

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in either of two ways.

$$\frac{6x^2 - 26x + 12}{x - 4} = 6x - 2 + \frac{4}{x - 4}$$

or

$$\underbrace{6x^2 - 26x + 12}_{\text{Dividend}} = \underbrace{(x - 4)}_{\text{Divisor}}\underbrace{(6x - 2)}_{\text{Quotient}} + \underbrace{4}_{\text{Remainder}}$$

Division Algorithm

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$; where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$\underbrace{P(x)}_{\text{Dividend}} = \underbrace{D(x)}_{\text{Divisor}} \cdot \underbrace{Q(x)}_{\text{Quotient}} + \underbrace{R(x)}_{\text{Remainder}}$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

Example 2 Long Division of Polynomials

Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

Solution

We use long division after first inserting the term $0x^3$ into the dividend to ensure that the columns line up correctly.

$$\begin{array}{rrl}
 & 4x^2 + 2x \phantom{{}- 3x + 1} & \\
2x^2 - x + 2 & \big)\overline{8x^4 + 0x^3 + 6x^2 - 3x + 1} & \\
 & \underline{8x^4 - 4x^3 + 8x^2} \phantom{{}- 3x + 1} & \text{Multiply divisor by } 4x^2 \\
 & 4x^3 - 2x^2 - 3x \phantom{{}+ 1} & \text{Subtract} \\
 & \underline{4x^3 - 2x^2 + 4x} \phantom{{}+ 1} & \text{Multiply divisor by } 2x \\
 & -7x + 1 & \text{Subtract}
\end{array}$$

The process is complete at this point because $-7x + 1$ is of lesser degree than the divisor $2x^2 - x + 2$. From the long division table we see that $Q(x) = 4x^2 + 2x$ and $R(x) = -7x + 1$, so

$$8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1) \quad \blacksquare$$

Synthetic Division

Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$. In synthetic division we write only the essential part of the long division table. Compare these long division and synthetic division tables, in which we divide $2x^3 - 7x^2 + 5$ by $x - 3$:

Long Division

$$
\begin{array}{rl}
 & 2x^2 - x - 3 \qquad \text{Quotient}\\
x - 3\,\big) & \overline{2x^3 - 7x^2 + 0x + 5}\\
 & \underline{2x^3 - 6x^2}\\
 & \qquad\quad -x^2 + 0x\\
 & \qquad\quad \underline{-x^2 + 3x}\\
 & \qquad\qquad\qquad -3x + 5\\
 & \qquad\qquad\qquad \underline{-3x + 9}\\
 & \qquad\qquad\qquad\qquad -4 \qquad \text{Remainder}
\end{array}
$$

Synthetic Division

$$
\begin{array}{c|cccc}
3 & 2 & -7 & 0 & 5\\ \cline{2-5}
 & & 6 & -3 & -9\\ \hline
 & 2 & -1 & -3 & -4\\
 & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad}_{\text{Quotient}}} & \text{Remainder}
\end{array}
$$

Note that in synthetic division we abbreviate $2x^3 - 7x^2 + 5$ by writing only the coefficients: 2 $\;-7\;$ 0 $\;$ 5, and instead of $x - 3$, we simply write 3. (Writing 3 instead of -3 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the gold boxes.)

The next example shows how synthetic division is performed.

Example 3 Synthetic Division

Use synthetic division to divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Solution

We begin by writing the appropriate coefficients to represent the divisor and the dividend.

Divisor $x - 3$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline \end{array}$$

Dividend
$2x^3 - 7x^2 + 0x + 5$

We bring down the 2, multiply $3 \cdot 2 = 6$, and write the result in the middle row. Then we add:

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline & & 6 & & \\ \hline & \boxed{2} & -1 & & \end{array}$$

Multiply: $3 \cdot 2 = 6$

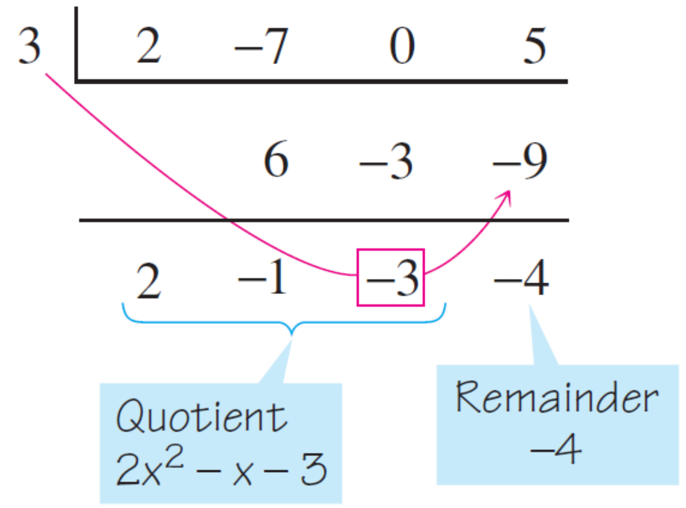
Add: $-7 + 6 = -1$

We repeat this process of multiplying and then adding until the table is complete.

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline & & 6 & -3 & \\ \hline & 2 & \boxed{-1} & -3 & \end{array}$$

Multiply: $3(-1) = -3$

Add: $0 + (-3) = -3$



$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ \hline & & 6 & -3 & -9 \\ \hline & 2 & -1 & \boxed{-3} & -4 \end{array}$$

Quotient
$2x^2 - x - 3$

Remainder
-4

Multiply: $3(-3) = 9$

Add: $5 + (-9) = -4$

From the last line of the synthetic division table, we see that the quotient is $2x^2 - x - 3$ and the remainder is -4. Thus

$$2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) - 4$$

■

The Remainder and Factor Theorems

The next theorem shows how synthetic division can be used to evaluate polynomials easily.

Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

■ **Proof** If the divisor in the Division Algorithm is of the form $x - c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant r, then

$$P(x) = (x - c) \cdot Q(x) + r$$

Setting $x = c$ in this equation, we get $P(c) = (c - c) \cdot Q(x) + r = 0 + r = r$, that is, $P(c)$ is the remainder r. ■

Example 4 Using the Remainder Theorem to Find the Value of a Polynomial

Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.
(b) Use the Remainder Theorem to find $P(-2)$.

Solution

(a) Since $x + 2 = x - (-2)$, the synthetic division table for this problem takes the following form.

$$
\begin{array}{r|rrrrrr}
-2 & 3 & 5 & -4 & 0 & 7 & 3 \\
\hline
 & & -6 & 2 & 4 & -8 & 2 \\
\hline
 & 3 & -1 & -2 & 4 & -1 & 5
\end{array}
$$

Remainder is 5, so $P(-2) = 5$.

The quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$ and the remainder is 5.

(b) By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x - (-2) = x + 2$. From part (a) the remainder is 5, so $P(-2) = 5$. ■

The next theorem says that *zeros* of polynomials correspond to *factors*; we used this fact in Section 4.1 to graph polynomials.

Factor Theorem

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

- **Proof** If $P(x)$ factors as $P(x) = (x - c) \cdot Q(x)$, then

$$P(c) = (c - c) \cdot Q(c) = 0 \cdot Q(c) = 0$$

Conversely, if $P(c) = 0$, then by the Remainder Theorem

$$P(x) = (x - c) \cdot Q(x) + 0 = (x - c) \cdot Q(x)$$

so $x - c$ is a factor of $P(x)$.

Example 5 Factoring a Polynomial Using the Factor Theorem

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

Solution

Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$. By the Factor Theorem, this means that $x - 1$ is a factor of $P(x)$. Using synthetic or long division (shown in the margin), we see that

$$
\begin{aligned}
P(x) &= x^3 - 7x + 6 \\
&= (x-1)(x^2 + x - 6) && \text{See margin} \\
&= (x-1)(x-2)(x+3) && \text{Factor quadratic } x^2 + x - 6
\end{aligned}
$$

■

$$
\begin{array}{r|rrrr}
1 & 1 & 0 & -7 & 6 \\
 & & 1 & 1 & -6 \\
\hline
 & 1 & 1 & -6 & 0
\end{array}
$$

$$
\begin{array}{r}
x^2 + x - 6 \\
x - 1 \,\big)\, \overline{x^3 + 0x^2 - 7x + 6} \\
\underline{x^3 - x^2} \\
x^2 - 7x \\
\underline{x^2 - x} \\
-6x + 6 \\
\underline{-6x + 6} \\
0
\end{array}
$$

Example 6 Finding a Polynomial with Specified Zeros

Find a polynomial of degree 4 that has zeros −3, 0, 1, and 5.

Solution

By the Factor Theorem, $x - (-3)$, $x - 0$, $x - 1$, and $x - 5$ must all be factors of the desired polynomial, so let

$$P(x) = (x + 3)(x - 0)(x - 1)(x - 5) = x^4 - 3x^3 - 13x^2 + 15x$$

Since $P(x)$ is of degree 4 it is a solution of the problem. Any other solution of the problem must be a constant multiple of $P(x)$, since only multiplication by a constant does not change the degree. ■

The polynomial P of Example 6 is graphed in Figure 1. Note that the zeros of P correspond to the x-intercepts of the graph.

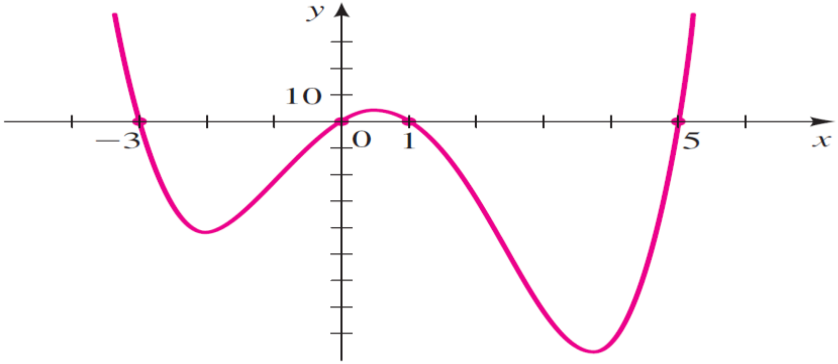


Figure 1
$P(x) = (x + 3)x(x - 1)(x - 5)$
has zeros −3, 0, 1, and 5.

4.2 Exercises

1–4 ■ Two polynomials P and D are given.

(a) Divide $P(x)$ by $D(x)$.

(b) Express P in the form $P(x) = D(x) \cdot Q(x) + R(x)$.

1. $P(x) = 3x^2 + 5x - 4, \quad D(x) = x + 3$

2. $P(x) = x^3 + 4x^2 - 6x + 1, \quad D(x) = x - 1$

3. $P(x) = x^4 - x^3 + 4x + 2, \quad D(x) = x^2 + 3$

4. $P(x) = 2x^5 + 4x^4 - 4x^3 - x - 3, \quad D(x) = x^2 - 2$

5–14 ■ Find the quotient and remainder using long division.

5. $\dfrac{x^2 - 6x - 8}{x - 4}$

6. $\dfrac{x^3 - x^2 - 2x + 6}{x - 2}$

7. $\dfrac{4x^3 + 2x^2 - 2x - 3}{2x + 1}$

8. $\dfrac{x^3 + 3x^2 + 4x + 3}{3x + 6}$

9. $\dfrac{x^3 + 6x + 3}{x^2 - 2x + 2}$

10. $\dfrac{3x^4 - 5x^3 - 20x - 5}{x^2 + x + 3}$

11. $\dfrac{6x^3 + 2x^2 + 22x}{2x^2 + 5}$

12. $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

13. $\dfrac{x^6 + x^4 + x^2 + 1}{x^2 + 1}$

14. $\dfrac{2x^5 - 7x^4 - 13}{4x^2 - 6x + 8}$

15–28 ■ Find the quotient and remainder using synthetic division.

15. $\dfrac{x^2 - 5x + 4}{x - 3}$

16. $\dfrac{x^2 - 5x + 4}{x - 1}$

17. $\dfrac{3x^2 + 5x}{x - 6}$

18. $\dfrac{4x^2 - 3}{x + 5}$

19. $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$

20. $\dfrac{3x^3 - 12x^2 - 9x + 1}{x - 5}$

21. $\dfrac{x^3 - 8x + 2}{x + 3}$

22. $\dfrac{x^4 - x^3 + x^2 - x + 2}{x - 2}$

23. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

24. $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

25. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

26. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

27. $\dfrac{x^3 - 27}{x - 3}$

28. $\dfrac{x^4 - 16}{x + 2}$

29–41 ■ Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

29. $P(x) = 4x^2 + 12x + 5, \quad c = -1$

30. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

31. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

32. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

33. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

34. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

35. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

36. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

37. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

38. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112,$
$c = -3$

39. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

40. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

41. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

43–46 ■ Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

43. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

44. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

45. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

46. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

5.1 Exponential Functions

Exponential Functions

The **exponential function with base a** is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

We assume $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function. Here are some examples of exponential functions:

$$f(x) = 2^x \qquad g(x) = 3^x \qquad h(x) = 10^x$$

Base 2 Base 3 Base 10

Example 1 Evaluating Exponential Functions

Let $f(x) = 3^x$ and evaluate the following:

(a) $f(2)$ (b) $f(-\frac{2}{3})$ (c) $f(\pi)$ (d) $f(\sqrt{2})$

Solution

We use a calculator to obtain the values of f.

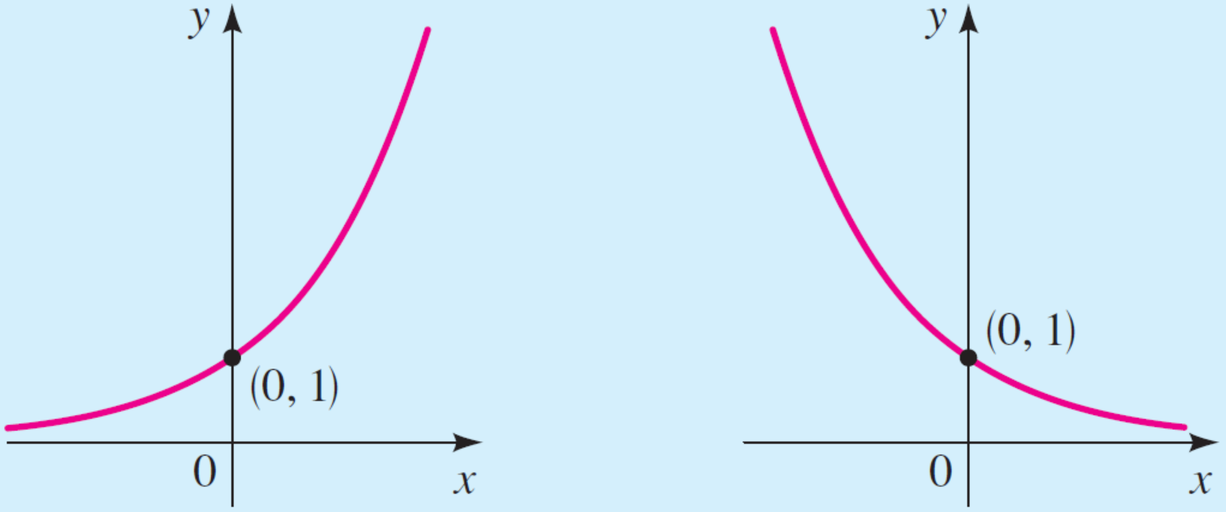
	Calculator keystrokes	Output
(a) $f(2) = 3^2 = 9$	3 ^ 2 ENTER	9
(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$	3 ^ (− 2 ÷ 3) ENTER	0.4807498
(c) $f(\pi) = 3^{\pi} \approx 31.544$	3 ^ π ENTER	31.5442807
(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$	3 ^ $\sqrt{\ }$ 2 ENTER	4.7288043 ■

Graphs of Exponential Functions

The exponential function

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

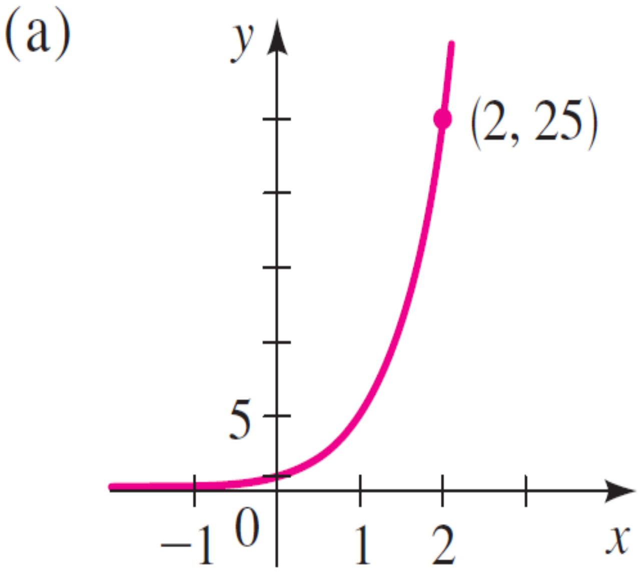
has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of f has one of these shapes.

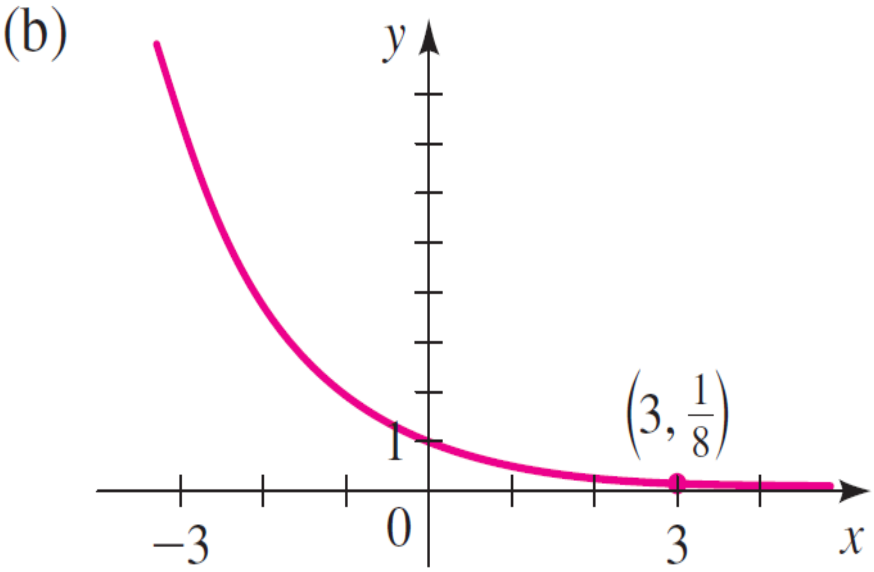


$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$

Example 3 Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.





Solution

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$.

Example 4 Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function.

(a) $g(x) = 1 + 2^x$ (b) $h(x) = -2^x$ (c) $k(x) = 2^{x-1}$

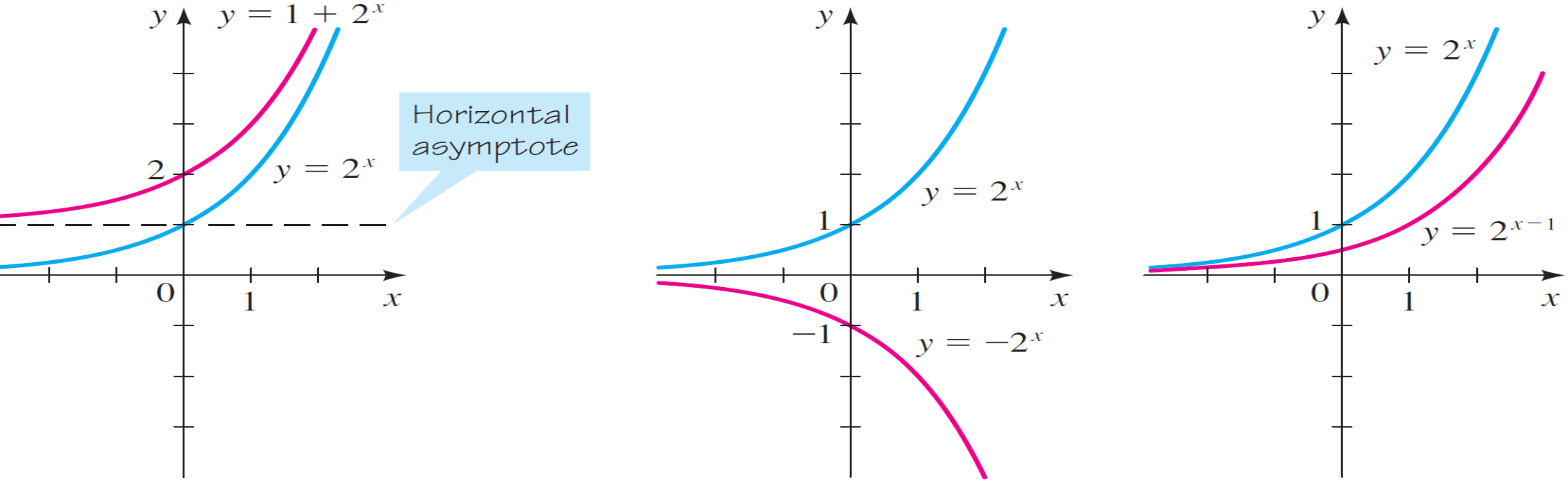
Solution

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit. Notice from Figure 3(a) that the line $y = 1$ is now a horizontal asymptote.

is

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b).

(c) This time we start with the graph of $f(x) = 2^x$ and shift it to the right by 1 unit, to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c).



(a) (b) (c)

■

Example 5 Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

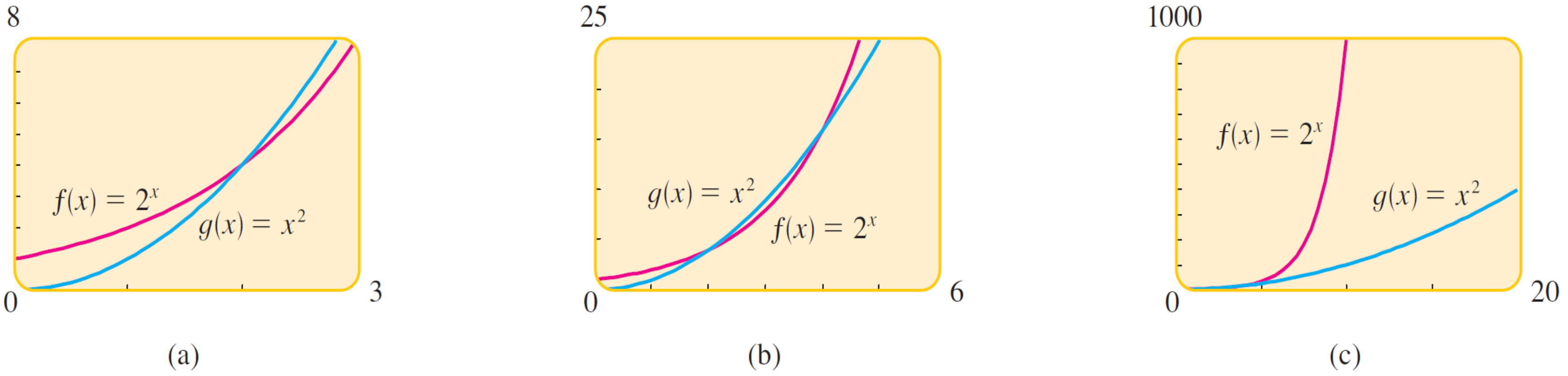
(a) $[0, 3]$ by $[0, 8]$ (b) $[0, 6]$ by $[0, 25]$ (c) $[0, 20]$ by $[0, 1000]$

Solution

(a) Figure 4(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 4(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 4(c) gives a more global view and shows that, when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.



(a) (b) (c)

Figure 4

■

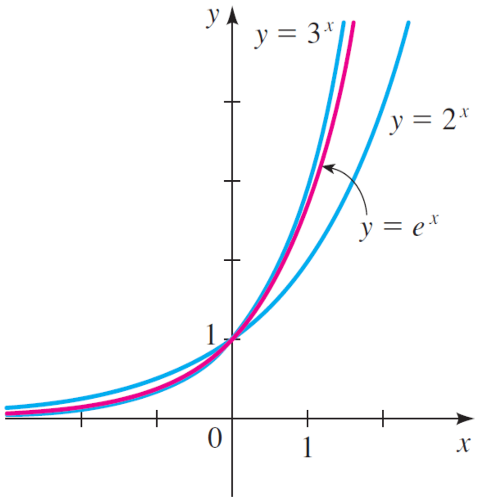


Figure 5

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown in Figure 5.

Scientific calculators have a special key for the function $f(x) = e^x$. We use this key in the next example.

Example 6 Evaluating the Exponential Function

Evaluate each expression correct to five decimal places.

(a) e^3 (b) $2e^{-0.53}$ (c) $e^{4.8}$

Solution

We use the $\boxed{\mathrm{e}^{\mathrm{x}}}$ key on a calculator to evaluate the exponential function.

(a) $e^3 \approx 20.08554$

(b) $2e^{-0.53} \approx 1.17721$

(c) $e^{4.8} \approx 121.51042$ ■

(c) $e \approx 121.51042$

Example 7 Transformations of the Exponential Function

Sketch the graph of each function.

(a) $f(x) = e^{-x}$ (b) $g(x) = 3e^{0.5x}$

Solution

(a) We start with the graph of $y = e^{x}$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$ as in Figure 6.

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in Figure 7.

x	$f(x) = 3e^{0.5x}$
23	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

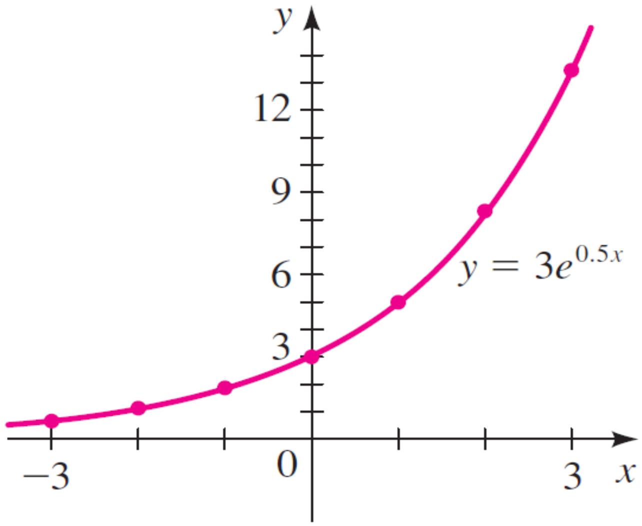


Figure 7

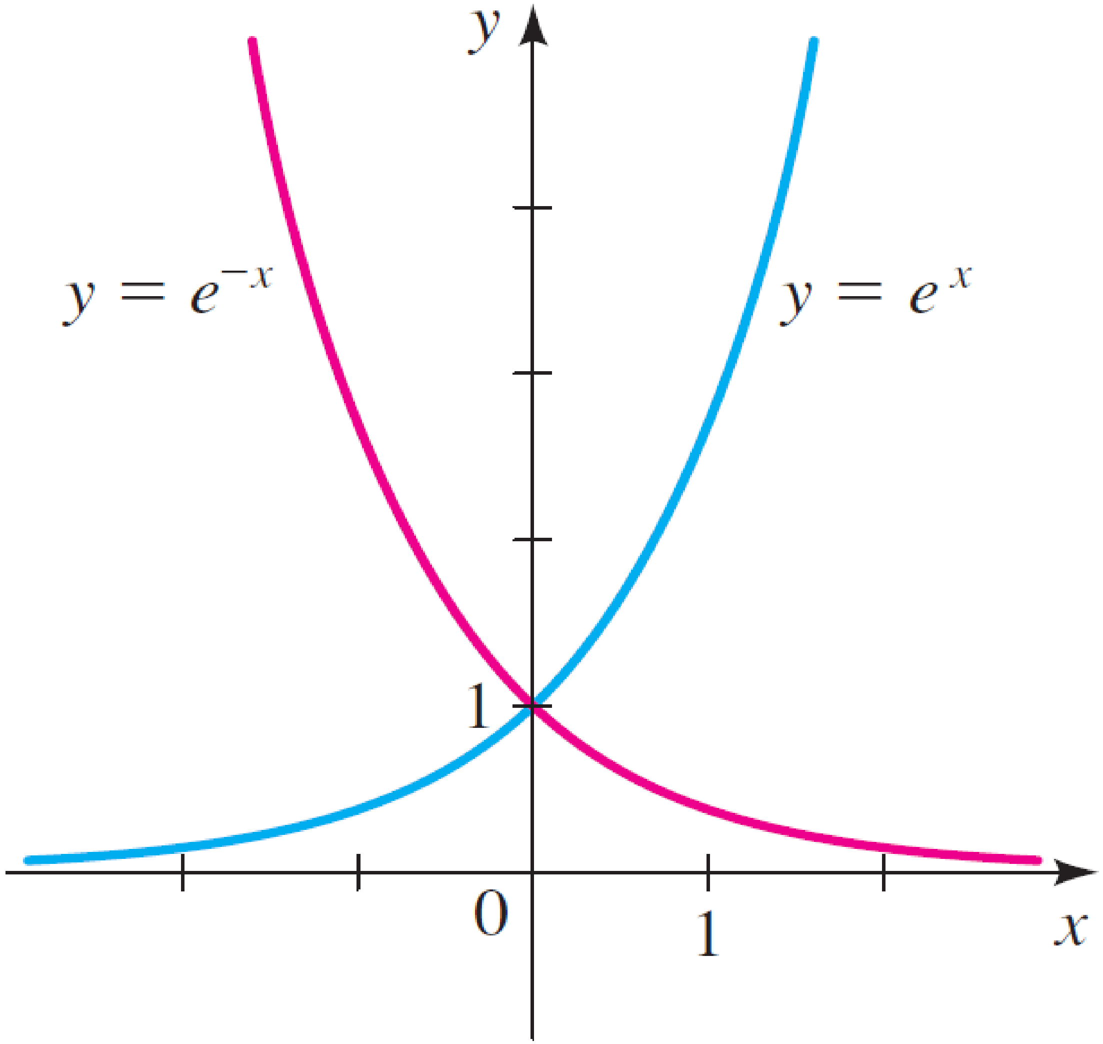


Figure 6

5.1 Exercises

1–6 ■ Sketch the graph of the function by making a table of values. Use a calculator if necessary.

1. $f(x) = 2^x$

2. $g(x) = 8^x$

3. $f(x) = \left(\frac{1}{3}\right)^x$

4. $h(x) = (1.1)^x$

5. $g(x) = 3e^x$

6. $h(x) = 2e^{-0.5x}$

7–8 ■ Graph both functions on one set of axes.

7. $y = 4^x$ and $y = 7^x$

8. $y = \left(\frac{2}{3}\right)^x$ and $y = \left(\frac{4}{3}\right)^x$

19–32 ■ Graph the function, not by plotting points, but by starting from the graphs in Figures 2 and 5. State the domain, range, and asymptote.

19. $f(x) = -3^x$

20. $f(x) = 10^{-x}$

21. $g(x) = 2^x - 3$

22. $g(x) = 2^{x-3}$

23. $h(x) = 4 + \left(\frac{1}{2}\right)^x$

24. $h(x) = 6 - 3^x$

25. $f(x) = 10^{x+3}$

26. $f(x) = -\left(\frac{1}{5}\right)^x$

27. $f(x) = -e^x$

28. $y = 1 - e^x$

29. $y = e^{-x} - 1$

30. $f(x) = -e^{-x}$

31. $f(x) = e^{x-2}$

32. $y = e^{x-3} + 4$

5.2 Logarithmic Functions

Definition of the Logarithmic Function

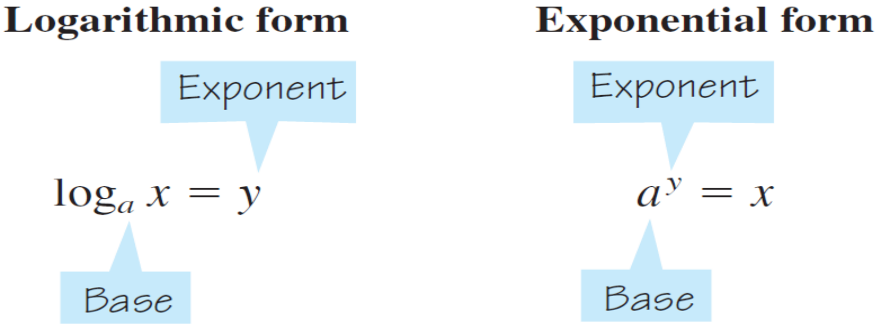
Let a be a positive number with $a \neq 1$. The **logarithmic function with base *a***, denoted by $\mathbf{log}_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

In words, this says that

$\log_a x$ is the exponent to which the base a must be raised to give x.

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it's helpful to notice that, in both forms, the base is the same:



$$\log_a x = y \qquad\qquad a^y = x$$

Example 1 Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations—if one is true, then so is the other. So, we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log^{10} 100{,}000\ 5\ 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

■

Properties of Logarithms

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Example 2 Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$ ■

In Section 3.7 we saw that a function f and its inverse function f^{-1} satisfy the equations

$$f^{-1}(f(x)) = x \qquad \text{for } x \text{ in the domain of } f$$

$$f(f^{-1}(x)) = x \qquad \text{for } x \text{ in the domain of } f^{-1}$$

When applied to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, these equations become

$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad x > 0$$

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

Example 3 Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$ Property 1

$\log_5 5 = 1$ Property 2

$\log_5 5^8 = 8$ Property 3

$5^{\log_5 12} = 12$ Property 4

Example 4 Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

Solution

To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve as in Figure 3.

x	$\log_2 x$
2^4	4
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

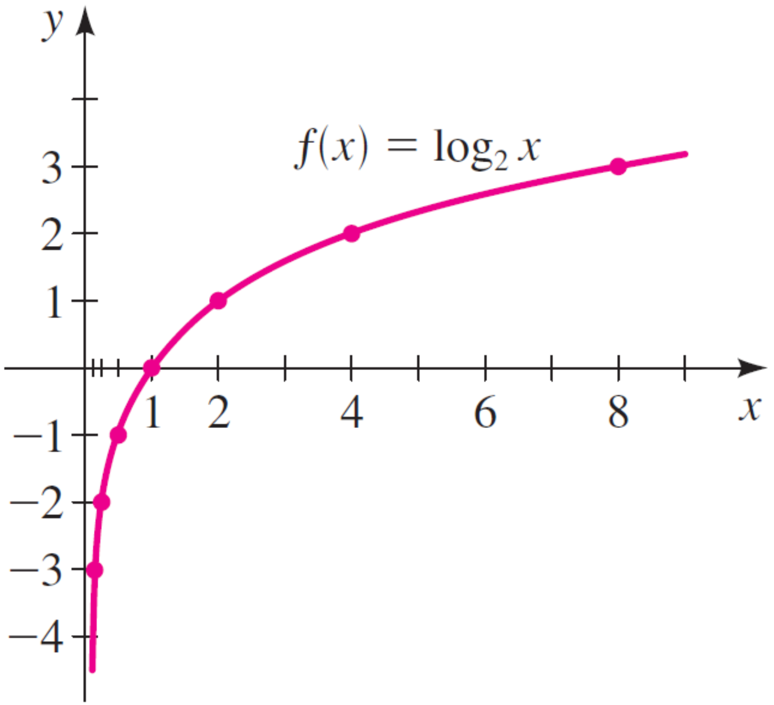


Figure 3

■

Example 5 Reflecting Graphs of Logarithmic Functions

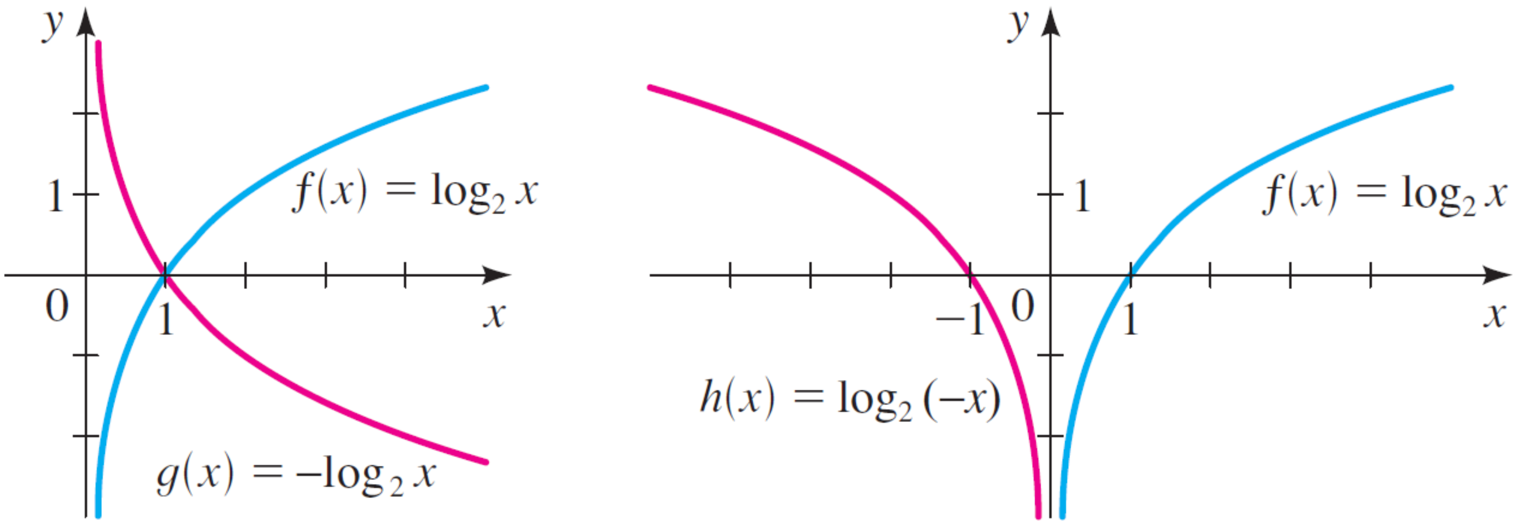
Sketch the graph of each function.

(a) $g(x) = -\log_2 x$ (b) $h(x) = \log_2 (-x)$

Solution

(a) We start with the graph of $f(x) = \log_2 x$ and reflect in the x-axis to get the graph of $g(x) = -\log_2 x$ in Figure 5(a).

(b) We start with the graph of $f(x) = \log_2 x$ and reflect in the y-axis to get the graph of $h(x) = \log_2 (-x)$ in Figure 5(b).



gure 5 (a) (b)

■

Common Logarithms

We now study logarithms with base 10.

Common Logarithm

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

From the definition of logarithms we can easily find that

$$\log 10 = 1 \qquad \text{and} \qquad \log 100 = 2$$

Example 7 Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$ and use the values to sketch the graph.

Solution

We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1
15	1.176

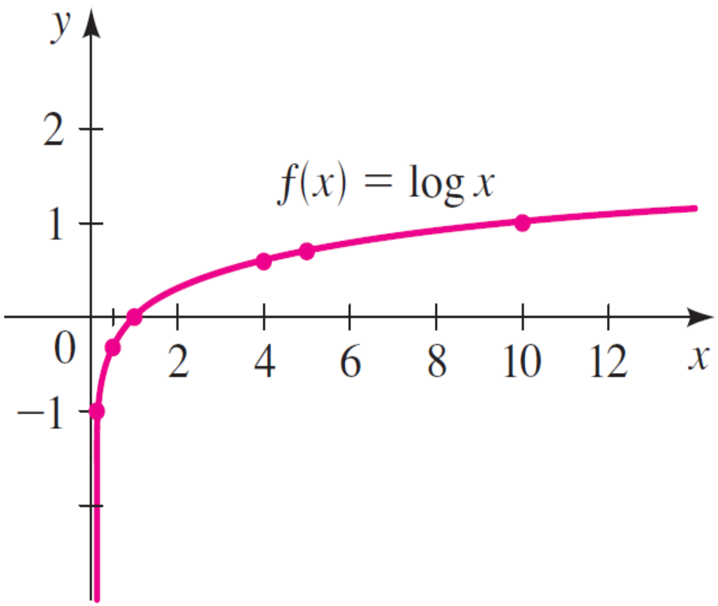


Figure 8

■

Natural Logarithm

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

The natural logarithmic function $y = \ln x$ is the inverse function of the exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$$

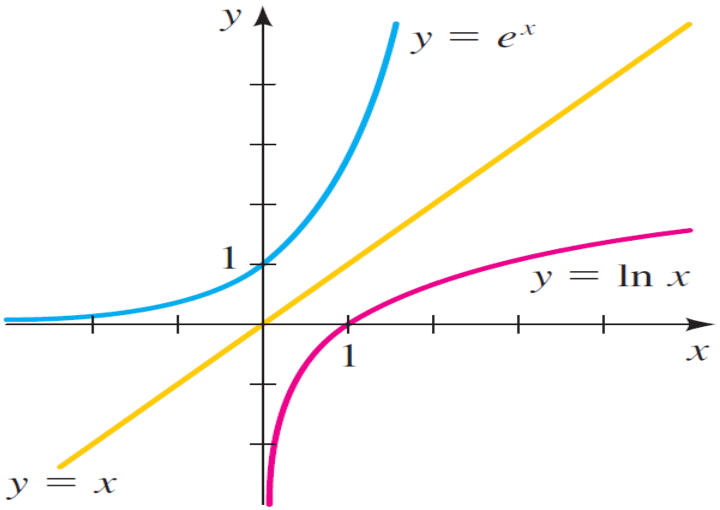


Figure 9

Graph of the natural logarithmic function

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

Properties of Natural Logarithms

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Example 10 Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

Solution

As with any logarithmic function, $\ln x$ is defined when $x > 0$.
Thus, the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}$$

$$= \{x \mid -2 < x < 2\} = (-2, 2)$$

5.2 Exercises

1–6 ■ Express the equation in exponential form.

1. **(a)** $\log_5 25 = 2$ **(b)** $\log_5 1 = 0$

2. **(a)** $\log_{10} 0.1 = -1$ **(b)** $\log_8 512 = 3$

3. **(a)** $\log_8 2 = \frac{1}{3}$ **(b)** $\log_2\left(\frac{1}{8}\right) = -3$

4. **(a)** $\log_3 81 = 4$ **(b)** $\log_8 4 = \frac{2}{3}$

5. **(a)** $\ln 5 = x$ **(b)** $\ln y = 5$

6. **(a)** $\ln(x + 1) = 2$ **(b)** $\ln(x - 1) = 4$

7–12 ■ Express the equation in logarithmic form.

7. **(a)** $5^3 = 125$ **(b)** $10^{-4} = 0.000$

8. **(a)** $10^3 = 1000$ **(b)** $81^{1/2} = 9$

9. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

10. **(a)** $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

11. **(a)** $e^x = 2$ **(b)** $e^3 = y$

12. **(a)** $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

57–62 ■ Find the domain of the function.

57. $f(x) = \log_{10}(x + 3)$

58. $f(x) = \log_5(8 - 2x)$

59. $g(x) = \log_3(x^2 - 1)$

60. $g(x) = \ln(x - x^2)$

61. $h(x) = \ln x + \ln(2 - x)$

62. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

13–22 ■ Evaluate the expression.

13. **(a)** $\log_3 3$ **(b)** $\log_3 1$ **(c)** $\log_3 3^2$

14. **(a)** $\log_5 5^4$ **(b)** $\log_4 64$ **(c)** $\log_9 9$

15. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

16. **(a)** $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

17. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10} \sqrt{10}$ **(c)** $\log_5 0.2$

18. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9 \sqrt{3}$

19. **(a)** $2^{\log_2 37}$ **(b)** $3^{\log_3 8}$ **(c)** $e^{\ln \sqrt{5}}$

20. **(a)** $e^{\ln \pi}$ **(b)** $10^{\log 5}$ **(c)** $10^{\log 87}$

21. **(a)** $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

22. **(a)** $\log_4 \sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

23–30 ■ Use the definition of the logarithmic function to find x.

23. **(a)** $\log_2 x = 5$ **(b)** $\log_2 16 = x$

24. **(a)** $\log_5 x = 4$ **(b)** $\log_{10} 0.1 = x$

25. **(a)** $\log_3 243 = x$ **(b)** $\log_3 x = 3$

26. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

27. **(a)** $\log_{10} x = 2$ **(b)** $\log_5 x = 2$

28. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

29. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

30. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

5.3 Laws of Logarithms

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

Example 1 Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$ (c) $-\frac{1}{3}\log 8$

Solution

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ Law 3

$= \log\left(\frac{1}{2}\right)$ Property of negative exponents

≈ -0.301 Calculator

Example 2 Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ (b) $\log_5(x^3y^6)$ (c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

Solution

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ Law 1

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3\log_5 x + 6\log_5 y$ Law 3

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3

Example 2 Expanding Logarithmic Expressions

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Solution

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ Law 1

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3\log_5 x + 6\log_5 y$ Law 3

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3

Example 3 Combining Logarithmic Expressions

Combine $3 \log x + \frac{1}{2} \log(x + 1)$ into a single logarithm.

Solution

$$\begin{aligned} 3 \log x + \tfrac{1}{2} \log(x + 1) &= \log x^3 + \log(x + 1)^{1/2} && \text{Law 3} \\ &= \log(x^3(x + 1)^{1/2}) && \text{Law 1} \end{aligned}$$

Example 4 Combining Logarithmic Expressions

Combine $3 \ln s + \frac{1}{2} \ln t - 4 \ln(t^2 + 1)$ into a single logarithm.

Solution

$$\begin{aligned} 3 \ln s + \tfrac{1}{2} \ln t - 4 \ln(t^2 + 1) &= \ln s^3 + \ln t^{1/2} - \ln(t^2 + 1)^4 && \text{Law 3} \\ &= \ln(s^3 t^{1/2}) - \ln(t^2 + 1)^4 && \text{Law 1} \\ &= \ln\left(\frac{s^3 \sqrt{t}}{(t^2 + 1)^4}\right) && \text{Law 2} \end{aligned}$$

WARNING Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference.* For instance,

$$\log_a(x + y) \xcancel{=} \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \xcancel{=} \log\left(\frac{6}{2}\right) \quad \text{and} \quad (\log_2 x)^3 \xcancel{=} 3 \log_2 x$$

Logarithmic functions are used to model a variety of situations involving human behavior. One such behavior is how quickly we forget things we have learned. For example, if you learn algebra at a certain performance level (say 90% on a test) and then don't use algebra for a while, how much will you retain after a week, a month, or a year? Hermann Ebbinghaus (1850–1909) studied this phenomenon and formulated the law described in the next example.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if we put $x = a$, then $\log_a a = 1$ and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

Example 6 Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct to five decimal places.

(a) $\log_8 5$ (b) $\log_9 20$

Solution

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

Example 7 Using the Change of Base Formula to Graph a Logarithmic Function

Use a graphing calculator to graph $f(x) = \log_6 x$.

Solution

Calculators don’t have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

5.3 Exercises

1–12 ■ Evaluate the expression.

1. $\log_5 \sqrt{125}$

2. $\log_2 112 - \log_2 7$

3. $\log 2 + \log 5$

4. $\log \sqrt{0.1}$

5. $\log_4 192 - \log_4 3$

6. $\log_{12} 9 + \log_{12} 16$

7. $\log_2 6 - \log_2 15 + \log_2 20$

8. $\log_3 100 - \log_3 18 - \log_3 50$

9. $\log_4 16^{100}$

10. $\log_2 8^{33}$

11. $\log(\log 10^{10,000})$

12. $\ln(\ln e^{e^{200}})$

13–38 ■ Use the Laws of Logarithms to expand the expression.

13. $\log_2(2x)$

14. $\log_3(5y)$

15. $\log_2(x(x-1))$

16. $\log_5\left(\dfrac{x}{2}\right)$

17. $\log 6^{10}$

18. $\ln\left(\sqrt{z}\right)$

19. $\log_2(AB^2)$

20. $\log_6 \sqrt[4]{17}$

21. $\log_3\left(x\sqrt{y}\right)$

22. $\log_2(xy)^{10}$

23. $\log_5 \sqrt[3]{x^2 + 1}$

24. $\log_a\left(\dfrac{x^2}{yz^3}\right)$

25. $\ln \sqrt{ab}$

26. $\ln \sqrt[3]{3r^2s}$

27. $\log\left(\dfrac{x^3y^4}{z^6}\right)$

28. $\log\left(\dfrac{a^2}{b^4\sqrt{c}}\right)$

29. $\log_2\left(\dfrac{x(x^2 + 1)}{\sqrt{x^2 - 1}}\right)$

30. $\log_5 \sqrt{\dfrac{x - 1}{x + 1}}$

31. $\ln\left(x\sqrt{\dfrac{y}{z}}\right)$

32. $\ln \dfrac{3x^2}{(x + 1)^{10}}$

33. $\log \sqrt[4]{x^2 + y^2}$

34. $\log\left(\dfrac{x}{\sqrt[3]{1 - x}}\right)$

35. $\log \sqrt{\dfrac{x^2 + 4}{(x^2 + 1)(x^3 - 7)^2}}$

36. $\log \sqrt{x\sqrt{y\sqrt{z}}}$

37. $\ln\left(\dfrac{x^3\sqrt{x - 1}}{3x + 4}\right)$

38. $\log\left(\dfrac{10^x}{x(x^2 + 1)(x^4 + 2)}\right)$

Example 2 Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

Solution

We first divide by 8 in order to isolate the exponential term on one side of the equation.

$$
\begin{aligned}
8e^{2x} &= 20 && \text{Given equation} \\
e^{2x} &= \tfrac{20}{8} && \text{Divide by 8} \\
\ln e^{2x} &= \ln 2.5 && \text{Take ln of each side} \\
2x &= \ln 2.5 && \text{Property of ln} \\
x &= \frac{\ln 2.5}{2} && \text{Divide by 2} \\
&\approx 0.458 && \text{Calculator}
\end{aligned}
$$

≈ 0.458 *Calculator*

Example 3 Solving an Exponential Equation Algebraically and Graphically

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

Solution 1: Algebraic

Since the base of the exponential term is e, we use natural logarithms to solve this equation.

$$\begin{aligned} e^{3-2x} &= 4 && \textit{Given equation} \\ \ln(e^{3-2x}) &= \ln 4 && \textit{Take ln of each side} \\ 3 - 2x &= \ln 4 && \textit{Property of ln} \\ 2x &= 3 - \ln 4 \\ x &= \tfrac{1}{2}(3 - \ln 4) \approx 0.807 \end{aligned}$$

You should check that this answer satisfies the original equation.

Solution 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in Figure 1. The solutions occur where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x \approx 0.81$.

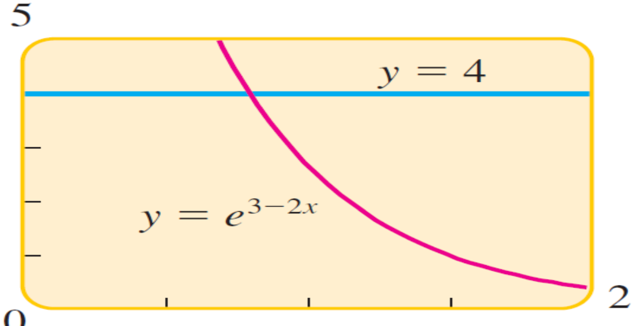

Figure 1