

## Online Lectures Series

By

## MUHAMMAD SAFDAR

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University of Sargodha, Sargodha

## Lecture-3\&4: Preliminaries

## OBJECTIVES

1 Sets and Types of Sets
2 Types of Sets
3 Operation On Sets
4 Venn Diagram
5 De Morgan's
6 Ordered Pair
7 Cartesian Product

## OBJECTIV1. Sets and Types of Sets

A set is a collection of distinct objects(elements) which have common property. For example, cat, elephant, tiger, and rabbit are animals. When, these animals are considered collectively, it's called set.

## Set Notation

The members(elements) of set is separated by comma and braces \{ \} are used outside the comma separated elements. \{cat, elephant, tiger, rabbit\}
For convenience, sets are denoted by a capital letter. For example, $A=\{c a t$, elephant, tiger, rabbit $\}$

Here, $A$ is a set containing 4 elements.

## Important Points Regarding Sets

## 1. A Set is collection of distinct members.

Wrong: Elements are not distinct $A=\{a, b, c, b, c, d\}$
Right: Elements are distinct $A=\{a, b, c, d\}$
2. The number of elements in a set can be both finite and infinite.

$$
\begin{aligned}
& N=\{2,4,6\} \text { (finite) } \\
& N=\{\ldots,-2,-1,0,1,2, \ldots\} \text { (infinite) }
\end{aligned}
$$

3. The elements of a set can be in any order. Changing the order of elements doesn't change anything.

$$
A=\{1,2,3\}
$$

This set can also be written as:

$$
\begin{aligned}
& A=\{2,1,3\} \\
& A=\{3,1,2\} \text { and so on }
\end{aligned}
$$

## OBJECTIV2. Types of Sets

Empty set
A set which do not have any element is known as empty set.
It is also called Null Set,
Vacuous Set or Void Set. Empty set is denoted by $\phi$.
A $=\{ \}=\varphi$

## Singleton set

If a set has only one element, it's known as singleton set.
A = \{ moon $\}$

## Finite set

Set with finite number of elements is called finite set.
$S=\{1,2,3\}$

## Infinite set

A set with have infinite numbers of elements is called infinite set.
$A=\{x: x$ is an integer $\}$
$B=\{5,10,15,20,25, \ldots\}$

Equivalent sets
Two sets are said to be equivalent sets if they have same number of elements. For Example,
$A=\{a, b, c, d\}$
$B=\{e, f, g, h\}$
$A=B$
$\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}=\{\mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}\}$
Here, $A$ and $B$ are equivalent sets because both sets have 4 elements.

## Equal Sets

Two sets are said to be equal sets if they both have exactly same elements.
$\mathrm{A}=\{1,2,3,4\}$
$B=\{2,4,3,1\}$
$A=B$
$\{1,2,3,4\}=\{2,4,3,1\}$
Here, $A$ and $B$ are equal sets because both set have same elements (order of elements doesn't matter).

## Subset

A set $P$ is a subset of set $Q$ if every element of set $P$ is also the member of set $Q$. Simply, if set $P$ is contained in set $Q, P$ is called subset of superset $Q$. It is denoted by $\mathbf{P} \subset \mathbf{Q}$.
$Q=\{1,2,4,3,9\}$
$P=\{1,2,3\}$
Here, all three elements 1,2 , and 3 of set P is also member of set $Q$. Hence, $P$ is subset of $Q$.

## Overlapping Sets

Two sets are said to be overlapping sets if they have at least one element common.
$A=\{1,2,3,4\}$
$B=\{3,4,5\}$
Here $A$ and $B$ are overlapping sets because elements 3 and 4 are common in both sets.

## Disjoint Sets

Two sets are said to be disjoint sets if they don't have common elements.
$A=\{1,2,3,4\}$
$B=\{5,6\}$
Here $A$ and $B$ are disjoint sets because these two sets don't have common element.

## Cardinal Number of a Set:

The number of distinct elements in a given set $A$ is called the cardinal number of $A$. It is denoted by $n(A)$. For example:

- $A\{x: x \in N, x<5\}$
$A=\{1,2,3,4\}$
Therefore, $n(A)=4$
- $B=$ set of letters in the word ALGEBRA
$B=\{A, L, G, E, B, R\}$
Therefore, $n(B)=6$


## Superset

If set $A$ is a subset of set $B$ and all the elements of set $B$ are the elements of $\operatorname{set} A$, then $A$ is a superset of set $B$. It is denoted by AゝB.
Example: If Set $A=\{1,2,3,4\}$ is a subset of $B=\{1,2,3,4\}$. Then $A$ is superset of $B$.

## Power Set

Definition: The power set of a set $A$ is the set which consists of all the subsets of the set $A$. It is denoted by $P(A)$.
For a set $A$ which consists of $n$ elements, the total number of subsets that can be formed is $2^{n}$. From this, we can say that $P(A)$ will have $2^{n}$ elements.
Example:
If set $A=\{-9,13,6\}$, then power set of $A$ will be:
$P(A)=\{\phi,\{-9\},\{13\},\{6\},\{-9,13\},\{13,6\},\{6,-9\},\{-9,13,6\}\}$
Proper Subset
If $A \subseteq B$ and $A \neq B$, then $A$ is called the proper subset of $B$ and it can be written as $A \subset B$.
Example:
If $A=\{2,5,7\}$ is a subset of $B=\{2,5,7\}$ then it is not a proper subset of $B=\{2,5,7\}$
But, $A=\{2,5\}$ is a subset of $B=\{2,5,7\}$ and is a proper subset also.

## Universal Set

A set which contains all the sets relevant to a certain condition is called the universal set. It is the set of all possible values. Example: If $A=\{1,2,3\}$ and $B=\{2,3,4,5\}$, then universal set here will be:
$U=\{1,2,3,4,5\}$

## OBJECTIVE3. Operations on Sets

In set theory, the operations of the sets are carried when two or more sets combine to form a single set under some of the given conditions. The basic operations on sets are:
-Union of sets
-Intersection of sets
-A complement of a set

- Cartesian product of sets.
-Set difference


## Union of Sets

The union of two sets $A$ and $B$ is defined as the set of all the elements which lie in set $A$ and set $B$ or both the elements in $A$ and $B$ altogether. The union of the set is denoted by the symbol 'U'.

## OBJECTIVE4. Venn diagram

A Venn diagram is an illustration that uses circles to show the relationships among things or finite groups of things. Circles that overlap have a commonality while circles that do not overlap do not share those traits. Venn diagrams help to visually represent the similarities and differences between two concepts.

## Examples:

Let $A=\{2,4,6,8\}$ and $B=\{6,8,10,12\}$. Represent $A \cup B$ through a welllabeled Venn diagram.
$A \cup B=\{2,4,6,8\} \cup\{6,8,10,12\}$
$A \cup B=\{2,4,6,8,10,12\}$


The orange colored patch represents the common elements $\{6,8\}$ and the quadrilateral represents $\mathrm{A} \cup \mathrm{B}$.

| Properties of Unions of Sets |  |
| :---: | :---: |
| Commutative Property | $A \cup B=B \cup A$ |
| Associative Property | $(A \cup B) \cup C=A \cup(B \cup C)$ |
| Identity Property | $A \cup \varnothing=\varnothing \cup A$ |
| Distributive Property | $A \cup(B \cap C)=(A \cup B) \cap(A \cup C)$ |
|  | $A \cap(B \cup C)=(A \cap B) \cup(A \cap C)$ |

## Example:

Given $U=\{1,2,3,4,5,6,7,8,10\}$
$X=\{1,2,6,7\}$ and $Y=\{1,3,4,5,8\}$
Find $\mathrm{X} \cup \mathrm{Y}$ and draw a Venn diagram to illustrate $\mathrm{X} \cup \mathrm{Y}$.

## Solution:

$X \cup Y=\{1,2,6,7\} \cup\{1,3,4,5,8\} \leftarrow 1$ is written only once.
$X \cup Y=\{1,2,3,4,5,6,7,8\} \cup$
If $X \subset Y$ then $X \cup Y=Y$.

## Example:

Given $U=\{1,2,3,4,5,6,7,8,10\}$
$X=\{1,6,9\}$ and $Y=\{1,3,5,6,8,9\}$
Find $\mathrm{X} \cup \mathrm{Y}$ and draw a Venn diagram to illustrate $\mathrm{X} \cup \mathrm{Y}$. Solution:
$X \cup Y=\{1,6,9\} \cup\{1,3,5,6,8,9\}$
$X \cup Y=\{1,3,5,6,8,9\}$


## Intersection of Sets

Intersection of two given sets is the largest set which contains all the elements that are common to both the sets.
To find the intersection of two given sets $A$ and $B$ is a set which consists of all the elements which are common to both A and B .
The symbol for denoting intersection of sets is ' $\cap$ '.
Let $A=\{2,4,6,8\}$ and $B=\{6,8,10,12\}$ then $\mathbf{A} \cap \boldsymbol{B}$ is represented through a Venn diagram as per following:

The orange colored patch represents the common elements $\{6,8\}$ as well as the $\mathbf{A} \cap \mathbf{B}$.


The intersection of 2 or more sets is the overlapped part(s) of the individual circles with the elements written in the overlapped parts. Example:


Examples:

1. $A=\{a, b, c\}, B=\{k, \ell, m\}$. These two sets are disjoint as they have no common elements. Their intersection is the empty set.
$A \cap B=\{a, b, c\} \cap\{k, e, m\} .=\varnothing$.
$A \cap B=\{a, b, c\} \cap\{k, \ell, m\}=\varnothing$
2. $C=\{1,2,3,4\}, D=\{2,4,6,7\}$. The intersection of these sets is
$C \cap D=\{1,2,3,4\} \cap\{2,4,6,7\}=\{2,4\}$.

## Some properties of the operation of intersection

(i) $\mathrm{A} \cap \mathrm{B}=\mathrm{B} \cap \mathrm{A}$ (Commutative law)
(ii) $(A \cap B) \cap C=A \cap(B \cap C)$ (Associative law)
(iii) $\varphi \cap \mathrm{A}=\varphi($ Law of $\varphi)$
(iv) $\mathrm{U} \cap \mathrm{A}=\mathrm{A}($ Law of U$)$
(v) $\mathrm{A} \cap \mathrm{A}=\mathrm{A}$ (Idempotent law)
(vi) $A \cap(B \cup C)=(A \cap B) \cup(A \cap C)$ (Distributive law) Here $\cap$ distributes over $u$

Also, $A \cup(B \cap C)=(A \cup B) \cap(A U C)$ (Distributive law) Here $u$ distributes over $\cap$

1. If $A=\{2,4,6,8,10\}$ and $\mathbf{B}=\{1,3,8,4,6\}$. Find intersection of two set $A$ and $B$.
Solution:
$A \cap B=\{2,4,6,8,10\} \cap\{1,3,8,4,6\}$
$A \cap B=\{4,6,8\}$
Therefore, 4, 6 and 8 are the common elements in both the sets.
2. If $X=\{a, b, c\}$ and $\mathbf{Y}=\{\phi\}$. Find intersection of two given sets $X$ and $Y$.
Solution:
$X \cap Y=\{a, b, c\} \cap\{\phi\}$
$X \cap Y=\{ \}$
3. If set $A=\{4,6,8,10,12\}$, set $B=\{3,6,9,12,15$, $18\}$ and set $C=\{1,2,3,4,5,6,7,8,9,10\}$.
(i) Find the intersection of sets $A$ and $B$.
(ii) Find the intersection of two set B and C .
(iii) Find the intersection of the given sets A and C .

## Solution:

(i) Intersection of sets $A$ and $B$ is $A \cap B$

Set of all the elements which are common to both set $A$ and set $B$ is $\{6,12\}$.
(ii) Intersection of two set $B$ and $C$ is $B \cap C$

Set of all the elements which are common to both set $B$ and set $C$ is $\{3,6,9\}$.
(iii) Intersection of the given sets $A$ and $C$ is $A \cap C$ Set of all the elements which are common to both set $A$ and set $C$ is $\{4,6,8,10\}$.

## Notes:

$A \cap B$ is a subset of $A$ and $B$.
Intersection of a set is commutative, i.e., $A \cap B=B \cap A$ Operations are performed when the set is expressed in the roster form.

## Notes:

$A \cap \varphi=\varphi \cap A=\varphi$ i.e. intersection of any set with the empty set is always the empty set.

## Difference of Sets

If set $A$ and set $B$ are two sets, then set $A$ difference set $B$ is a set which has elements of $A$ but no elements of $B$.
The difference of two sets $A$ and $B$ is denoted by $A \backslash B$ or $A-B$.
Example:
$A=\{1,2,3\}$ and $B=\{2,3,4\}$
$A-B=\{1,2,3\}-\{2,3,4\}=\{1\}$

## Examples:

1. $A=\{a, b, c\}, B=\{k, \ell, m\}$. The difference between two disjoint sets is equal to the initial set. So, we have
$A \backslash B=A \backslash(A \cap B)=A \backslash \varnothing=A=\{a, b, c\}$.
2. $C=\{1,2,3,4\}, D=\{2,4,6,7\}$. The difference of two sets $C$ and $D$ is given by $C \backslash D=\{1,2,3,4\}-\{2,4,6,7\}=\{1,3\}$

## Some properties of the operation of Difference

1. If set $A$ and $B$ are equal then, $A-B=A-A=\phi$ (empty set) 2. When an empty set is subtracted from a set (suppose set $A$ ) then, the result is that set itself, i.e, $\mathbf{A}-\boldsymbol{\phi}=\mathbf{A}$. 3.When a set is subtracted from an empty set then, the result is an empty set, i.e, $\boldsymbol{\phi}-\mathbf{A}=\boldsymbol{\phi}$.
2. When a superset is subtracted from a subset, then result is an empty set, i.e, $\mathbf{A}-\mathbf{B}=\boldsymbol{\phi}$ if $\mathbf{A} \subset \mathbf{B}$
3. If A and B are disjoint sets then, $\mathrm{A}-\mathrm{B}=\mathbf{A}$ and $\mathrm{B}-\mathbf{A}=\mathbf{B}$


## Solved example to find the difference of sets using Venn

 diagram:1. If $A=\{2,3,4,5,6,7\}$ and $B=\{3,5,7,9,11,13\}$ then find (i) A - B and (ii) B - A.

## Solution:

According to the given statement; $A=\{2,3,4,5,6,7\}$
and $B=\{3,5,7,9,11,13\}$
(i) $\mathrm{A}-\mathrm{B}=\{2,3,4,5,6,7\}-\{3,5,7,9,11,13\}$

$$
A-B=\{2,4,6\}
$$

## Solved Examples on Difference of Sets

1. If $A=\{1,2,3,4,5,6,7,8,9\}, B=\{2,4,6,8,10,12,14,16$, 18\}. Find Difference of Sets $A-B$ and represent the Same using Venn Diagram?

## Solution:

Given Sets $A=\{1,2,3,4,5,6,7,8,9\}$

$$
B=\{2,4,6,8,10,12,14,16,18\}
$$

$A-B=\{1,2,3,4,5,6,7,8,9\}-\{2,4,6,8,10,12,14,16,18\}$
$A-B=\{1,3,5,7,9\}$
A-B is the Set of Elements that are present in Set A and doesn't belong to Set B.

Therefore, $\mathrm{A}-\mathrm{B}=\{1,3,5,7,9\}$

2. If $A=\{1,2,3,4\} B=\{3,4,5,6\}$. Find Difference of Sets $A$ and $B$ using Venn Diagram?

## Solution:

$A=\{1,2,3,4\}$ and $B=\{3,4,5,6\}$
$A-B=\{1,2,3,4\}-\{3,4,5,6\}$
$A-B=\{1,2\}$


A-B denotes the Elements in Set A but doesn't belong to Set B. The Difference of Sets A-B is shaded for your reference.
3. Let us consider Set A = \{blue, green, red $\}$ \& Set
$B=\{r e d$, orange, yellow $\}$. Find the Difference of Sets $A$ and $B$ ?

## Solution:

Given Set A = \{blue, green, red $\}$ \& Set B = \{red, orange, yellow $\}$ are
$A-B=\{b l u e$, green, red $\}-\{$ red, orange, yellow $\}$ $A-B=\{b l u e$, green $\}$


A-B denotes the colors that belong to Set A and don't belong to Set B. A-B is shaded so that you can understand easily.

## Symmetric Difference

The symmetric difference of two sets $A$ and $B$ is the set of all elements which belong to exactly one of the two original sets. This operation is written as $\mathrm{A} \triangle \mathrm{B}$ or $\mathrm{A} \oplus \mathrm{B}$.
In terms of unions and intersections, the symmetric difference of two sets A and B can be expressed as
$A \triangle B=(A \cup B) \backslash(A \cap B)$.

## Examples:

1. $A=\{a, b, c\}, B=\{k, \ell, m\}$. The symmetric difference of two disjoint sets is equal to their union:
$A \triangle B=(A \cup B) \backslash(A \cap B)=(A \cup B) \backslash \emptyset=A \cup B=\{a, b, c, k, \ell, m\}$.
$2 . C=\{1,2,3,4\}, D=\{2,4,6,7\}$. The symmetric difference of the sets $C$ and $D$ is given by
$C \triangle D=(C \cup D) \backslash(C \cap D)=\{1,2,3,4,6,7\}-\{2,4\}=\{1,3,6,7\}$.

## Some properties of the operation of Symmetric Difference

1. $A \Delta \emptyset=A$
2. $A \Delta A=A$
3. $A \cap(B \Delta C)=(A \cap B) \Delta(A \cap C)$
4.if $A \Delta B=A \Delta C$ then $B=C$
4. $A \Delta B=B \Delta A$
5. $(A \Delta B) \Delta C=A \Delta(B \Delta C)$

Complement of a Set
The complement of a set A is the set of elements in the given universal set U that are not elements of A. The complement of A is denoted by A ${ }^{\mathrm{c}}$ or $\mathrm{A}^{\prime}$, or sometimes $\overline{\mathrm{A}}$.

Figure 5.
So by definition, we have

$$
\mathrm{A}^{\mathrm{c}}=\mathrm{U} \backslash \mathrm{~A} .
$$

## Some properties of complement sets

(i) $\mathrm{A} \cup \mathrm{A}^{\prime}=\mathrm{A}^{\prime} \cup \mathrm{A}=\cup$ (Complement law)
(ii) $\left(\mathrm{A} \cap \mathrm{B}^{\prime}\right)=\varphi$ (Complement law)
(iii) $(A \cup B)^{\prime}=\left(A^{\prime} \cap B^{\prime}\right)($ De Morgan's law)
(iv) $(A \cap B)^{\prime}=\left(A^{\prime} \cup B^{\prime}\right)($ De Morgan's law)
(v) $\left(A^{\prime}\right)^{\prime}=A($ Law of complementation)
(vi) $\varphi^{\prime}=u($ Law of empty set)
(vii) $u^{\prime}=\varphi$ (and universal set)

## Examples:

1 Let the universal set be $\mathrm{U}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}\}$. If $\mathrm{A}=\{\mathrm{a}, \mathrm{b}, \mathrm{d}, \mathrm{f}\}$, then the complement of A is given by

$$
\mathrm{A}^{\mathrm{c}}=\mathrm{U} \backslash \mathrm{~A}=\{\mathrm{c}, \mathrm{e}\} .
$$

2 Suppose the universal set is $U=\left\{\begin{array}{ll}x & Z \mid x^{2}<20\end{array}\right\}$ and the set $A$ is given by $A=\left\{\begin{array}{ll}x & Z \mid-3 \leq x<3\end{array}\right\}$. Find the complement of $A$.

$$
U=\{-4,-3,-2,-1,0,1,2,3,4\}, \quad A=\{-3,-2,-1,0,1,2\}, \quad A^{c}=\{-4,3,4\} .
$$

Example \#1: Complement of a set with Venn Diagram If $U=\{1,2,3,4,5,6\}$ and $A=\{2,3,4\}$, find $A^{\text {c. }}$
Here, $U=\{1,2,3,4,5,6\}$

$$
A=\{2,3,4\}
$$

$\therefore A^{c}=U-A=\{1,2,3,4,5,6\}-\{2,3,4\}=\{1,5,6\}$


Figure: Complement of set $A\left(A^{c}\right)$

Example \#2: Complement of Union of Set If $\mathrm{U}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}\}$ and $\mathrm{A} \cup \mathrm{B}=\{\mathrm{a}, \mathrm{e}, \mathrm{g}, \mathrm{h}\}$, find $(\mathrm{A} \cup \mathrm{B})^{\mathrm{c}}$. Here, $U=\{a, b, c, d, e, f, g, h\}$
$A \cup B=\{a, e, g, h\}$
$(A \cup B)^{c}=U-(A \cup B)$
$\therefore(A \cup B)^{c}=\{b, c, d, f\}$


Figure: $(A \cup B)^{C}$

## OBJECTIVE5. De Morgan's Laws

## Definition:

The complement of the union of two sets is equal to the intersection of their complements and the complement of the intersection of two sets is equal to the union of their complements. These are called De Morgan's laws. For any two finite sets $A$ and $B$;
(i) $(A \cup B)^{\prime}=\left(A^{\prime} \cap B^{\prime}\right)$ (which is a De Morgan's law of union).
(ii) $(A \cap B)^{\prime}=\left(A^{\prime} \cup B^{\prime}\right)$ (which is a De Morgan's law of intersection).

## Distributive Laws

The distributive laws tell how unions and intersections pass through parentheses containing the other symbol: $A \cap(B \cup C)=(A \cap B) \cup(A \cap C$
$A \cup(B \cap C)=(A \cup B) \cap(A \cup C)$

Examples on De Morgan's law:

1. If $U=\{j, k, I, m, n\}, X=\{j, k, m\}$ and $Y=\{k, m, n\}$.

Proof of De Morgan's law: $(\mathrm{X} \cap \mathrm{Y})^{\prime}=\mathrm{X}^{\prime} \mathrm{U} \mathrm{Y}^{\prime}$.
Solution:
We know, $\mathrm{U}=\{\mathrm{j}, \mathrm{k}, \mathrm{l}, \mathrm{m}, \mathrm{n}\}$

$$
\begin{aligned}
& X=\{j, k, m\} \\
& Y=\{k, m, n\} \\
& \begin{aligned}
(X \cap Y) & =\{j, k, m\} \cap\{k, m, n\} \\
& =\{k, m\}
\end{aligned}
\end{aligned}
$$

Therefore, $(X \cap Y)^{\prime}=U-(X \cap Y)=\{j, k, I, m, n\}-\{k, m\}=\{j, I, n\} \quad \ldots$ (i)
Again, $X=\{j, k, m\}$ so, $X^{\prime}=U-X=\{j, k, I, m, n\}-\{j, k, m\}=\{I, n\}$
And $Y=\{k, m, n\}$ so, $Y^{\prime}=U-Y=\{j, k, I, m, n\}-\{k, m, n\}=\{j, l\}$

$$
\begin{equation*}
X^{\prime} \cup Y^{\prime}=\{I, n\} \cup\{j, I\} \tag{ii}
\end{equation*}
$$

Therefore, $\mathrm{X}^{\prime} \cup \mathrm{Y}^{\prime}=\{\mathrm{j}, \mathrm{I}, \mathrm{n}\}$
Combining (i) and (ii) we get;

$$
(X \cap Y)^{\prime}=X^{\prime} \cup Y^{\prime} . \quad(X \cup Y)^{\prime}=X^{\prime} \cap Y^{\prime} . \text { EXERCISE }
$$

2. Let $U=\{1,2,3,4,5,6,7,8\}, P=\{4,5,6\}$ and $\mathrm{Q}=\{5,6,8\}$.
Show that $(P \cup Q)^{\prime}=P^{\prime} \cap Q^{\prime}$.

## Solution:

We know, $\mathrm{U}=\{1,2,3,4,5,6,7,8\}$
$P=\{4,5,6\}$
$Q=\{5,6,8\}$
$(P \cup Q)=\{4,5,6\} \cup\{5,6,8\}$

$$
\begin{equation*}
=\{4,5,6,8\} \tag{i}
\end{equation*}
$$

Therefore, $(P \cup Q)^{\prime}=\mathrm{U}-(\mathrm{P} \cup \mathrm{Q})=\{1,2,3,7\}$
Now $P=\{4,5,6\}$ so, $P^{\prime}=(U-P)=\{1,2,3,7,8\}$
and $Q=\{5,6,8\}$ so, $Q^{\prime}=(U-Q)=\{1,2,3,4,7\}$
$P^{\prime} \cap Q^{\prime}=\{1,2,3,7,8\} \cap\{1,2,3,4,7\}$
Therefore, $\mathrm{P}^{\prime} \cap \mathrm{Q}^{\prime}=\{1,2,3,7\}$
Combining (i)and (ii) we get;
$(\mathrm{P} \cup \mathrm{Q})^{\prime}=\mathrm{P}^{\prime} \cap \mathrm{Q}^{\prime} . \quad$ Proved EXERCISE $(\mathrm{P} \cap \mathrm{Q})^{\prime}=\mathrm{P}^{\prime} \mathrm{UQ}{ }^{\prime}$

## OBJECTIVE6. Ordered Pair:

An ordered pair is a pair of objects where one element is designated first and the other element is designated second, denoted $(a, b)$. OR

An ordered pair is defined as a set of two objects together with an order associated with them. Ordered pairs are usually written in parentheses (as opposed to curly braces, which are used for writing sets).
In the ordered pair (a,b), the element a is called the first entry or first component, and $b$ is called the second entry or second component of the pair.
Two ordered pairs $(\mathrm{a}, \mathrm{b})$ and $(\mathrm{c}, \mathrm{d})$ are equal if and only if $a=c$ and $b=d$. In general, $(a, b) \neq(b, a)$.
The equality $(a, b)=(b, a)$ is possible only if $a=b$.

The ordered pair $(2,5)$ is not equal to ordered pair $(3,2)$ i.e., $(2,5) \neq(5,2)$. Thus, in a pair, the order of elements is important. An ordered pair consists of two elements that are written in the fixed order. So, we define an ordered pair as:

- The pair of elements that occur in particular order and are enclosed in brackets are called a set of ordered pairs.
- If ' $a$ ' and ' $b$ ' are two elements, then the two different pairs are (a, b); (b, a) and (a, b); (b, a).
- In an ordered pair ( $a, b$ ), $a$ is called the first component and $b$ is called the second component.

Suppose, if $A$ and $B$ are two sets such that $a \in A$ and $b \in B$, then by the ordered pair of elements we mean $(a, b)$ where ' $a$ ' is called the Ist component and ' $b$ ' is called the II ${ }^{\text {nd }}$ component of the ordered pair.

If the position of the components is changed, then the ordered pair is changed, i.e., it becomes (b, a) but $(a, b) \neq(b, a)$.

## Note:

Ordered pair is not a set consisting of two elements.

## OBJECTIVE7. Cartesian Product:

The Cartesian product of two sets $A$ and $B$, denoted $A \times B$, is the set of all possible ordered pairs where the elements of $A$ are first and the elements of $B$ are second.
In set-builder notation, $A \times B=\{(a, b): a \in A$ and $b \in B\}$.
In set-builder notation, $A \times B=\{(a, b): a \in A$ and $b \in B\}$.
Example: Let $A=\{H, T\}$ and $B=\{1,2,3,4,5,6\}$.
$A \times B=\{H, T\} \times\{1,2,3,4,5,6\}$
$A \times B=\{(\mathrm{H}, 1),(\mathrm{H}, 2),(\mathrm{H}, 3),(\mathrm{H}, 4),(\mathrm{H}, 5),(\mathrm{H}, 6),(\mathrm{T}, 1),(\mathrm{T}, 2),(\mathrm{T}, 3),(\mathrm{T}$, 4), (T, 5), (T, 6) \}
$B \times A=\{(1, H),(2, H),(3, H),(4, H),(5, H),(6, H),(1, T),(2, T),(3, T),(4$, $\mathrm{T}),(5, \mathrm{~T}),(6, \mathrm{~T})\}$
Note that in this case $A \times B \neq B \times A$, i.e., the Cartesian product is not commutative.
Also, note that $n(A) \cdot n(B)=2(6)=12=n(A \times B)$.
Example: $A \times \emptyset=\emptyset$ since no ordered pairs can be formed when one of the sets is empty.
Also, note that $n(A) \cdot n(\emptyset)=2(0)=0=n(A \times \varnothing)$.

## OR

## Cartesian Product of Two Sets

Suppose that A and B are non-empty sets.
The Cartesian product of two sets $A$ and $B$, denoted $A \times B$, is the set of all possible ordered pairs ( $a, b$ ), where $a \in A$ and $b \in B$ :
$A \times B=\{(a, b) \mid a \in A$ and $b \in B\}$.
The Cartesian product is also known as the cross product. The Cartesian product of the sets $A=\{1,2,3\}$ and $B=\{x, y\}$.
It consists of 6 ordered pairs:
$A \times B=\{(1, x),(2, x),(3, x),(1, y),(2, y),(3, y)\}$.
Similarly, we can find the Cartesian product $B \times A$ :
$B \times A=\{(x, 1),(y, 1),(x, 2),(y, 2),(x, 3),(y, 3)\}$.
As you can see from this example, the Cartesian products $A \times B$ and $B \times A$ do not contain exactly the same ordered pairs. So, in general, $A \times B \neq B \times A$.

## Some Properties of Cartesian Product

1.The Cartesian product is non-commutative: $\mathrm{A} \times \mathrm{B} \neq \mathrm{B} \times \mathrm{A}$
2. $A \times B=B \times A$, if only $A=B$.
3. $A \times B=\emptyset$, if either $A=\varnothing$ or $B=\varnothing$
4.The Cartesian product is non-associative: $(A \times B) \times C \neq A \times(B \times C)$
5.Distributive property over seintersection: $A \times(B \cap C)=(A \times B) \cap(A \times C)$ 6. Distributive property over set union: $A \times(B \cup C)=(A \times B) \cup(A \times C)$
7.Distributive property over set difference: $A \times(B \backslash C)=(A \times B) \backslash(A \times C)$ 8.If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set $C$.
9. $|A \times B|=|B \times A|=|A| \times|B|$.

Similarly,
10. $|A 1 \times \ldots \times A n|=|A 1| \times \ldots \times|A n|$.

## Note:

If either $A$ or $B$ are null sets, then $A \times B$ will also be an empty set, i.e., if $A=\varnothing$ or
$B=\varnothing$, then $A \times B=\emptyset$

## Example \#

Suppose two sets $A=\{a, b\}$ and $B=\{1,2,3\}$. Find $A \times B$ and $B \times A$.
Here, $A=\{a, b\}, B=\{1,2,3\}$ Now,
$A \times B=\{a, b\} \times\{1,2,3\}$
$A \times B=\{(a, 1),(a, 2),(a, 3),(b, 1),(b, 2),(b, 3)\}$
$B \times A=\{(1, a),(1, b),(2, a),(2, b),(3, a),(3, b)\}$

## Example \#

If $A \times B=\{(x, 1),(x, 2),(x, 3),(y, 1),(y, 2),(y, 3)\}$. Find $A$ and $B$. Since, set A contains first element of each ordered pair only, A $=\{x, y\}$
Since, set B contain second element of ordered pair only, $B=\{1,2,3\}$

## Example \#

Suppose two sets $A=\{-1,-2\}, B=\{1,2\}$ and $C=\{0\}$. Find $A \times B \times C$.
Here, $A=\{-1,-2\}$

$$
\begin{aligned}
B & =\{1,2\} \\
C & =\{0\} \text { Now, } \\
A \times B & =\{-1,-2\} \times\{1,2\} \\
A \times B & =\{(-1,1,0),(-1,2,0),(-2,1,0),(-2,2,0)\}
\end{aligned}
$$

## For Example;

1. If $A=\{7,8\}$ and $B=\{2,4,6\}$, find $A \times B$.

## Solution:

$A \times B=\{7,8\} \times\{2,4,6\}$
$A \times B=\{(7,2) ;(7,4) ;(7,6) ;(8,2) ;(8,4) ;(8,6)\}$
The 6 ordered pairs thus formed can represent the position of points in a plane, if $A$ and $B$ are subsets of a set of real numbers.
2. If $A \times B=\{(p, x) ;(p, y) ;(q, x) ;(q, y)\}$, find $A$ and B.

## Solution:

A is a set of all first entries in ordered pairs in $A \times B$. $B$ is a set of all second entries in ordered pairs in $A \times B$. Thus $A=\{p, q\}$ and $B=\{x, y\}$
3. If $A$ and $B$ are two sets, and $A \times B$ consists of 6 elements: If three elements of $A \times B$ are $(2,5)(3,7)(4$, 7) find $A \times B$.

## Solution:

Since, $(2,5)(3,7)$ and $(4,7)$ are elements of $A \times B$. So, we can say that $2,3,4$ are the elements of $A$ and 5 , 7 are the elements of $B$.
So, $A=\{2,3,4\}$ and $B=\{5,7\}$
Now, $A \times B=\{(2,5) ;(2,7) ;(3,5) ;(3,7) ;(4,5) ;(4$,
7) \}

Thus, $A \times B$ contain six ordered pairs.
4. If $A=\{1,3,5\}$ and $B=\{2,3\}$, then

Find: (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $(B \times B)$
Solution:
(i) $A \times B=\{1,3,5\} \times\{2,3\}$

$$
=[\{1,2\},\{1,3\},\{3,2\},\{3,3\},\{5,2\},\{5,3\}]
$$

(ii) $B \times A=\{2,3\} \times\{1,3,5\}$

$$
=[\{2,1\},\{2,3\},\{2,5\},\{3,1\},\{3,3\},\{3,5\}]
$$

(iii) $A \times A=\{1,3,5\} \times\{1,3,5\}$

$$
=[\{1,1\},\{1,3\},\{1,5\},\{3,1\},\{3,3\},\{3,
$$

$5\},\{5,1\},\{5,3\},\{5,5\}]$
(iv) $B \times B=\{2,3\} \times\{2,3\}$

$$
=[\{2,2\},\{2,3\},\{3,2\},\{3,3\}]
$$

## Exercise 1.

Let $A=\{1,5,31,56,101\}, B=\{22,56,5,103,87\}, C=\{41,13,7,101,48\}$, and $D=\{1,3,5,7 \ldots\}$
Give the sets resulting from:

1. $A \cap B$
2.CuA
3.C
2. $(A \cup B) \cup(C \cup D)$

## Exercise 2: True or False

$1.7 \in\{6,7,8,9\}$
2. $5 \notin\{6,7,8,9\}$
3. $\{2\} \nsubseteq\{1,2\}$
4. $\emptyset \nsubseteq\{\alpha, \beta, x\}$
5. $\varnothing=\{\varnothing\}$

## Exercise 3: Subsets

List all the subsets of:

1. $\{1,2,3\}$
2. $\{\varphi, \lambda, \Delta, \mu\}$
3. $\{\varnothing\}$

## Exercise 4

Assume that the universal set is the set of all integers.
Let
$A=\{-7,-5,-3,-1,1,3,5,7\}$
$B=\{x \in Z \mid x 2<9\}$
$C=\{2,3,4,5,6\}$
$D=\{x \in Z \mid x \leq 9\}$
Give the sets resulting from:

1. $\mathrm{A} \cap \mathrm{B}$
2.CuA
3.CRD
2. $(A \cup B) \cup(C \cup D)$

## Exercise 5: Set operations

Let $A=\{r, e, a, s, o, n, i, g\}$,
$B=\{m, a, t, h, e, t, i, c, l\}$ and
$\mathrm{C}=\{$ the set of vowels $\}$. Calculate:
1.AUBUC
2.A $\cap B$

$$
\text { 3. } c^{c} \text {. }
$$

## Exercise 6: Equal Sets

Consider the following sets:
$A=\{x \in Z \mid x=2 m, m \in Z\}$ and
$B=\{x \in Z \mid x=2(n-1), n \in Z\}$
Are $A$ and $B$ equal? Justify your answer.

## Exercise 7

Let $A=\{1,3,5\}$, and
$B=\{a, b\}$.
Then

1. Find $A \times B$ and $B \times A$.
2. Are $A \times B$ and $B \times A$ equal? Justify your answer.

## Exercise 8

## Solved basic word problems on sets:

1. Let $A$ and $B$ be two finite sets such that $n(A)=20$, $n(B)=28$ and $n(A \cup B)=36$, find $n(A \cap B)$.
Solution:
Using the formula $n(A \cup B)=n(A)+n(B)-n(A \cap B)$.
then $n(A \cap B)=n(A)+n(B)-n(A \cup B)$

$$
\begin{aligned}
& =20+28-36 \\
& =48-36 \\
& =12
\end{aligned}
$$

2. If $n(A-B)=18, n(A \cup B)=70$ and $n(A \cap B)=25$, then find $n(B)$.

## Solution:

Using the formula $n(A \cup B)=n(A-B)+n(A \cap B)+n(B-A)$

$$
\begin{aligned}
70 & =18+25+n(B-A) \\
70 & =43+n(B-A) \\
n(B-A) & =70-43 \\
n(B-A) & =27
\end{aligned}
$$

Now $n(B)=n(A \cap B)+n(B-A)$

$$
\begin{aligned}
& =25+27 \\
& =52
\end{aligned}
$$

## Different types on word problems on sets

3. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?

## Solution:

Let $A=$ Set of people who like cold drinks.
$B=$ Set of people who like hot drinks.
Given

$$
\begin{aligned}
& (A \cup B)=60 \quad n(A)=27 \quad n(B)=42 \text { then; } \\
& n(A \cap B)=n(A)+n(B)-n(A \cup B) \\
& =27+42-60 \\
& =69-60=9 \\
& =9
\end{aligned}
$$

Therefore, 9 people like both tea and coffee.
4. There are 35 students in art class and 57 students in dance class. Find the number of students who are either in art class or in dance class.

- When two classes meet at different hours and 12 students are enrolled in both activities.
- When two classes meet at the same hour.


## Solution:

$$
n(A)=35, \quad n(B)=57, \quad n(A \cap B)=12
$$

(Let A be the set of students in art class. $B$ be the set of students in dance class.)
(i) When 2 classes meet at different hours $n(A \cup B)=n(A)+n(B)-n(A \cap B)$

$$
\begin{aligned}
& =35+57-12 \\
& =92-12 \\
& =80
\end{aligned}
$$

(ii) When two classes meet at the same hour, $A \cap B=\varnothing n(A \cup B)=n(A)+n(B)-n(A \cap B)$

$$
\begin{aligned}
& =n(A)+n(B) \\
& =35+57 \\
& =92
\end{aligned}
$$

## Further concept to solve word problems on sets:

5. In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French? Solution:

Let $A$ be the set of people who speak English.
$B$ be the set of people who speak French.
A - B be the set of people who speak English and not French.

B - A be the set of people who speak French and not English.
$A \cap B$ be the set of people who speak both French and English.
Given,
$n(A)=72 \quad n(B)=43 \quad n(A \cup B)=100$
Now, $n(A \cap B)=n(A)+n(B)-n(A \cup B)$
$=72+43-100$
$=115-100$
$=15$
Therefore, Number of persons who speak both French and English $=15$

$$
n(A)=n(A-B)+n(A \cap B)
$$

$\Rightarrow \mathrm{n}(\mathrm{A}-\mathrm{B})=\mathrm{n}(\mathrm{A})-\mathrm{n}(\mathrm{A} \cap \mathrm{B})$

$$
\begin{aligned}
& =72-15 \\
& =57
\end{aligned}
$$

and $n(B-A)=n(B)-n(A \cap B)$

$$
\begin{aligned}
& =43-15 \\
& =28
\end{aligned}
$$

Therefore, Number of people speaking English only = 57
Number of people speaking French only $=28$

Word problems on sets using the different properties (Union \& Intersection):
6. In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?

## Solution:

Let $A=$ set of persons who got medals in dance. $B=$ set of persons who got medals in dramatics. $\mathrm{C}=$ set of persons who got medals in music.
Given,
$n(A)=36$
$n(B)=12 \quad n(C)=18$
$n(A \cup B \cup C)=45$
$n(A \cap B \cap C)=4$

We know that number of elements belonging to exactly two of the three sets A, B, C
$=n(A \cap B)+n(B \cap C)+n(A \cap C)-3 n(A \cap B \cap C)$
$=n(A \cap B)+n(B \cap C)+n(A \cap C)-3 \times 4$
$n(A \cup B \cup C)=n(A)+n(B)+n(C)-n(A \cap B)-n(B \cap C)-n(A \cap C)+n(A \cap B$ © C)
Therefore, $n(A \cap B)+n(B \cap C)+n(A \cap C)=n(A)+n(B)+n(C)+n(A \cap B \cap$ C) $-n(A \cup B \cup C)$

From (i) required number

$$
\begin{aligned}
& =n(A)+n(B)+n(C)+n(A \cap B \cap C)-n(A \cup B \cup C)-12 \\
& =36+12+18+4-45-12 \\
& =70-57 \\
& =13
\end{aligned}
$$

Apply set operations to solve the word problems on sets:
7. Each student in a class of 40 plays at least one indoor game chess, carrom and scrabble. 18 play chess, 20 play scrabble and 27 play carrom. 7 play chess and scrabble, 12 play scrabble and carrom and 4 play chess, carrom and scrabble. Find the number of students who play (i) chess and carrom. (ii) chess, carrom but not scrabble.

## Solution:

Let A be the set of students who play chess $B$ be the set of students who play scrabble $C$ be the set of students who play carrom Therefore, We are given $n(A \cup B \cup C)=40$, $\mathrm{n}(\mathrm{A})=18, \quad \mathrm{n}(\mathrm{B})=20 \quad \mathrm{n}(\mathrm{C})=27$, $n(A \cap B)=7, \quad n(C \cap B)=12 \quad n(A \cap B \cap C)=4$

We have
$n(A \cup B \cup C)=n(A)+n(B)+n(C)-n(A \cap B)-n(B \cap C)$
$-n(C \cap A)+n(A \cap B \cap C)$
Therefore, $40=18+20+27-7-12-n(C \cap A)+4$
$40=69-19-n(C \cap A)$
$40=50-n(C \cap A) n(C \cap A)=50-40$
$n(C \cap A)=10$
Therefore, Number of students who play chess and carrom are 10.
Also, number of students who play chess, carrom and not scrabble.

$$
\begin{aligned}
& =n(C \cap A)-n(A \cap B \cap C) \\
& =10-4 \\
& =6
\end{aligned}
$$

Therefore, we learned how to solve different types of word problems on sets without using Venn diagram.

## Exercise 9

Write the universal set for the following.
(a) $P=\{4,6,8\}$
$Q=\{1,3,9\}$
$R=\{0,2,5\}$
$S=\{7\}$
(b) $X=\{a, b, c\} \quad Y=\{c, b, f\} \quad Z=\{e, g\}$
(c) Prime numbers less than 10, even numbers less than 10, multiples of 3 less than 10 .
Exercise 10
If $\xi=\{1,2,3,4,5,6,7,8,9,10\}$
$A=\{2,4,6,8\}$
$B=\{3,5,7\}$
$C=\{1,5,7,8,9\}$
Find (a) $A^{\prime} \quad$ (b) $B^{\prime} \quad$ (c) $C^{\prime}$

## Exercise 11

Let the set of integer be the universal set and let $A=$ set of whole numbers, then what is $A^{\prime}$ ?

## Exercise 12

If $U=\{2,3,4,5,6,7,8,9\} \quad X=\{3,5,7,9\}$
$Y=\{2,4,6,8\}$
Show that $X=Y^{\prime}$ and $Y=X^{\prime}$
Exercise 13
Let $P=\{3,5,7,9,11\} \quad Q=\{9,11,13\}$
$R=\{3,5,9\} \quad S=\{13,11\}$
Write Yes or No for the following.
(a) $R \subset P$
(b) $\mathrm{Q} \subset P$
(c) $R \subset S$
(d) $S \subset Q$
(e) $S \subset P$
(f) $\mathrm{P} \not \subset \mathrm{Q}$
(g) $Q \not \subset R$
(h) $S \not \subset Q$
$\mathbf{U}=\{$ natural numbers $\} ; A=\{2,4,6,8,10\} ; B=\{1,3,6,7,8\}$ State whether each of the following is true or false:
(a) $A \subset \mathbf{U}$
(b) $B \subseteq A$
(c) $\varnothing \subset \mathbf{U}$

## Exercise 15

$\mathbf{U}=\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}\} ; P=\{\mathrm{c}, \mathrm{f}\} ; Q=\{\mathrm{a}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{h}\} ;$
$R=\{\mathrm{c}, \mathrm{d}, \mathrm{h}\}$
(a) Draw a Venn diagram, showing these sets with all the elements entered into the appropriate regions. If necessary, redraw the diagram to eliminate any empty regions.
(b) Which of sets $P, Q$ and $R$ are proper subsets of others? Write your answer(s) using the $\subset$ symbol.
(c) $P$ and $R$ are disjoint sets. True or False?

## Exercise 16

Sketch Venn diagrams that show the universal set, $\mathbf{U}$, the sets $A$ and $B$, and a single element $x$ in each of the following cases:
(a) $x \in A ; A \subset B$
(b) $x \in A ; A$ and $B$ are disjoint
(c) $x \in A ; x \notin B ; B \subset A$
(d) $x \in A ; x \in B$; $A$ is not a subset of $B ; B$ is not a subset of $A$

## Exercise 17

$\mathbf{U}=\{1,2,3,4,5,6,7,8,9,10\}$
$A=\{2,4,6,8,10\}$
$B=\{1,3,6,7,8\}$
$C=\{3,7\}$
(a) Illustrate the sets $\mathbf{U}, A, B$ and $C$ in a Venn diagram, marking all the elements in the appropriate places. (Note: if any region in your diagram does not contain any elements, re-draw the set loops to correct this.)
(b) Using your Venn diagram, list the elements in each of the following sets:
$A \cap B, A \cup C, A^{\prime}, B^{\prime}, B \cap A^{\prime}, B \cap C^{\prime}, A-B, A \Delta B$
(c) Complete the statement using a single symbol: $C-B=\ldots$.

## Exercise 18

What can you say about two sets $P$ and $Q$ if:
(a) $P \cap Q^{\prime}=\varnothing$
(b) $P \cup Q=P$

## Exercise 19

Make six copies of the Venn diagram shown alongside, and then shade the areas represented by:
(a) $A^{\prime} \cup B$
(b) $A \cap B^{\prime}$
(c) $(A \cap B)^{\prime}$
(d) $A^{\prime} \cup B^{\prime}$
(e) $(A \cup B)^{\prime}$
(f) $A^{\prime} \cap B^{\prime}$


## Exercise 20

Identify the sets represented by each of the shaded areas below, using the set notation symbols $\cap, \cup$ and ' only:


## Exercise 21

$X=\{a, c\}$ and
$Y=\{a, b, e, f\}$.
Write down the elements of:
(a) $X \times Y$
(b) $Y \times X$
(c) $X^{2}=(X \times X)$
(d) What could you say about two sets $A$ and $B$ if $A \times B=B \times A$ ?

