

# Online Lectures Series 

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## Lecture-1\&2: Preliminaries

## OBJECTIVES

1 Write sets using set notation.

2 Use number lines.
3 Know the common sets of numbers.

4 Find additive inverses.
5 Use absolute value.
6 Use inequality symbols.
7 Graph sets of real numbers.

## Real Numbers and their properties

Natural numbers $=N=\{1,2,3, \ldots\}$

$$
x+5=5
$$

Whole numbers $=W=\{0,1,2,, \ldots\}$

$$
x+6=5
$$

Integers $=Z=\{-2,-1,0,1,2, \ldots\}$

$$
2 x=5
$$

Rational Numbers $=Q=\left\{\frac{p}{q}, p, q \in Z\right\}$

$$
x^{2}=5
$$

Irrational numbers $=Q^{\prime}=\left\{\right.$ Numbers that can not be written in $\frac{p}{q}$ form $\}$

OBJECTIVE 1 Write sets using set notation. A settis a collection of objects called the elements or members of the set. In algebra, the elements of a set are usually numbers. Set braces, $\}$, are used to enclose the elements. For example, 2 is an element of the set $\{1,2,3\}$. Since we can count the number of elements in the set $\{1,2,3\}$, it is a finite set.

To write the fact that 2 is an element of the set $\{1,2,3\}$, we use the symbol $\in$ (read "is an element of").

$$
2 \in\{1,2,3\}
$$

The number 2 is also an element of the set of natural numbers $N$, so we may write

$$
2 \in N .
$$

To show that 0 is not an element of set $N$, we draw a slash through the symbol $\in$.

$$
0 \notin N
$$

Two sets are equal if they contain exactly the same elements. For example, $\{1,2\}=\{2,1\}$, because the sets contain the same elements. (Order doesn't matter.) On the other hand, $\{1,2\} \neq\{0,1,2\}$ ( $\neq$ means "is not equal to") since one set contains the element 0 while the other does not.

In algebra, letters called variables are often used to represent numbers or to define sets of numbers. For example,
$\{x \mid x$ is a natural number between 3 and 15$\}$
(read "the set of all elements $x$ such that $x$ is a natural number between 3 and 15 ") defines the set

$$
\{4,5,6,7, \ldots, 14\} .
$$

The notation $\{x \mid x$ is a natural number between 3 and 15$\}$ is an example of setbuilder notation.


## EXAMPLE 1 Listing the Elements in Sets

List the elements in each set.
(a) $\{x \mid x$ is a natural number less than 4$\}$

The natural numbers less than 4 are 1,2 , and 3 . This set is $\{1,2,3\}$.
(b) $\{y \mid y$ is one of the first five even natural numbers $\}=\{2,4,6,8,10\}$
(c) $\{z \mid z$ is a natural number greater than or equal to 7$\}$

The set of natural numbers greater than or equal to 7 is an infinite set, writter with three dots as $\{7,8,9,10, \ldots\}$.

## EXAMPLE 2 Using Set-Builder Notation to Describe Sets

Use set-builder notation to describe each set.
(a) $\{1,3,5,7,9\}$

There are often several ways to describe a set with set-builder notation. One way to describe this set is
$\{y \mid y$ is one of the first five odd natural numbers $\}$.
(b) $\{5,10,15, \ldots\}$

This set can be described as $\{x \mid x$ is a multiple of 5 greater than 0$\}$.

OBJECTIVE 2 Use number lines. A good way to get a picture of a set of numbers is to use a number line. To construct a number line, choose any point on a horizontal line and label it 0 . Next, choose a point to the right of 0 and label it 1 . The distance from 0 to 1 establishes a scale that can be used to locate more points, with positive numbers to the right of 0 and negative numbers to the left of 0 . The number 0 is neither positive nor negative. A number line is shown in Figure 1.


FIGURE 1

The set of numbers identified on the number line in Figure 1 including nositive
Each number on a number line is called the coordinate of the point that it labels, while the point is the graph of the number. Figure 2 shows a number line with several selected points graphed on it.


FIGURE 2

The fractions $-\frac{1}{2}$ and $\frac{3}{4}$, graphed on the number line in Figure 2, are examples of rational numbers. A rational number can be expressed as the quotient of two integers, with denominator not 0 . Rational numbers can also be written in decimal form, either as terminating decimals such as $\frac{3}{5}=.6, \frac{1}{8}=.125$, or $\frac{11}{4}=2.75$, or as repeating decimals such as $\frac{1}{3}=.33333 \ldots$ or $\frac{3}{11}=.272727 \ldots$ A repeating decimal is often written with a bar over the repeating digit(s). Using this notation, 2727 . . is written 27.

Decimal numbers that neither terminate nor repeat are not rational, and thus are called irrational numbers. Many square roots are irrational numbers; for example, $\sqrt{2}=1.4142136 \ldots$ and $-\sqrt{7}=-2.6457513 \ldots$ repeat indefinitely without pattern. (Some square roots are rational: $\sqrt{16}=4, \sqrt{100}=10$, and so on.) Another irrational number is $\pi$, the ratio of the distance around or circumference of a circle to its diameter.

Some of the rational and irrational numbers just discussed are graphed on the number line in Figure 3. The rational numbers together with the irrational numbers make up the set of real numbers. Every point on a number line corresponds to a real number, and every real number corresponds to a point on the number line.

## Real numbers



FIGURE 3

OBJECTIVE 3 Know the common sets of numbers. The following sets of numbers will be used throughout the rest of this text.

## Sets of Numbers

Natural numbers or counting numbers

Whole numbers
Integers

$$
\begin{aligned}
& \{1,2,3,4,5,6, \ldots\} \\
& \{0,1,2,3,4,5,6, \ldots\} \\
& \{\ldots,-3,-2,-1,0,1,2,3, \ldots\}
\end{aligned}
$$

$$
\left\{\left.\frac{p}{q} \right\rvert\, p \text { and } q \text { are integers, } q \neq 0\right\}
$$

$$
\text { Examples: } \frac{4}{1} \text { or } 4,1.3,-\frac{9}{2} \text { or }-4 \frac{1}{2}, \frac{16}{8} \text { or } 2, \sqrt{9} \text { or } 3, . \overline{6}
$$

Irrational numbers $\{x \mid x$ is a real number that is not rational $\}$ Examples: $\sqrt{3},-\sqrt{2}, \pi$
Real numbers
$\{x \mid x \text { is represented by a point on a number line }\}^{*}$

The relationships among these various sets of numbers are shown in Figure 4; in particular, the figure shows that the set of real numbers includes both the rational and irrational numbers. Every real number is either rational or irrational. Also, notice that the integers are elements of the set of rational numbers and that whole numbers and natural numbers are elements of the set of integers.

## Real numbers



FIGuRF 4 The Real Numbers

## EXAMPLE 3 Identifying Examples of Number Sets

Which numbers in

$$
\left\{-8,-\sqrt{6},-\frac{9}{64}, 0, .5, \frac{2}{3}, 1 . \overline{12}, \sqrt{3}, 2\right\}
$$

are elements of each set?
(a) Integers
$-8,0$, and 2 are integers.
(c) Irrational numbers
$-\sqrt{6}$ and $\sqrt{3}$ are irrational numbers.
(b) Rational numbers
$-8,-\frac{9}{64}, 0, .5, \frac{2}{3}, 1 . \overline{12}$, and 2 are rational numbers.
(d) Real numbers

All the numbers in the given set are real numbers.

## EXAMPLE 4 Determining Relationships between Sets of Numbers

Decide whether each statement is true or false.
(a) All irrational numbers are real numbers.

This is true. As shown in Figure 4, the set of real numbers includes all irrational numbers.
(b) Every rational number is an integer.

This statement is false. Although some rational numbers are integers, other rational numbers, such as $\frac{2}{3}$ and $-\frac{1}{4}$, are not.

OBJECTIVE 4 Find additive inverses. Look again at the number line in Figure 1. For each positive number, there is a negative number on the opposite side of 0 that lies the same distance from 0 . These pairs of numbers are called additive inverses, negatives, or opposites of each other. For example, 5 is the additive inverse of -5 , and -5 is the additive inverse of 5 .

## Additive Inverse

For any real number $a$, the number $-a$ is the additive inverse of $a$.

Change the sign of a number to get its additive inverse. The sum of a number and its additive inverse is always 0 .

The symbol "-" can be used to indicate any of the following:

1. a negative number, such as -9 or -15 ;
2. the additive inverse of a number, as in " -4 is the additive inverse of 4 ";
3. subtraction, as in $12-3$.

In the expression $-(-5)$, the symbol "-" is being used in two ways: the first indicates the additive inverse of -5 , and the second indicates a negative number, -5 . Since the additive inverse of -5 is 5 , then $-(-5)=5$. This example suggests the following property.

For any real number $a, \quad-(-\boldsymbol{a})=\mathbf{a}$.

Numbers written with positive or negative signs, such as $+4,+8,-9$, and -5 , are called signed numbers. A positive number can be called a signed number even though the positive sign is usually left off. The table in the margin shows the additive inverses of several signed numbers. The number 0 is its own additive inverse.

| Number | Additive Inverse |
| :---: | :---: |
| 6 | -6 |
| -4 | 4 |
| $\frac{2}{3}$ | $-\frac{2}{3}$ |
| -8.7 | 8.7 |
| 0 | 0 |

OBJECTIVE 5 Use absolute value. Geometrically, the absolute value of a number $a$, written $|a|$, is the distance on the number line from 0 to $a$. For example, the absolute value of 5 is the same as the absolute value of -5 because each number lies five units from 0. See Figure 5. That is,

$$
|5|=5 \quad \text { and } \quad|-5|=5
$$


figure 5

The formal definition of absolute value follows.

## Absolute Value

$$
|a|=\left\{\begin{aligned}
a & \text { if } a \text { is positive or } 0 \\
-a & \text { if } a \text { is negative }
\end{aligned}\right.
$$

The second part of this definition, $|a|=-a$ if $a$ is negative, requires careful thought. If $a$ is a negative number, then $-a$, the additive inverse or opposite of $a$, is a positive number, so $|a|$ is positive. For example, if $a=-3$, then

$$
|a|=|-3|=-(-3)=3 . \quad|a|=-a \text { if } a \text { is negative. }
$$

## EXAMPLE 5 Evaluating Absolute Value Expressions

Find the value of each expression.
(a) $|13|=13$
(b) $|-2|=-(-2)=2$
(c) $|0|=0$
(d) $-|8|$

Evaluate the absolute value first. Then find the additive inverse.

$$
-|8|=-(8)=-8
$$

(e) $-|-8|$

Work as in part (d): $|-8|=8$, so

$$
-|-8|=-(8)=-8
$$

(f) $|-2|+|5|$

Evaluate each absolute value first, then add.

$$
|-2|+|5|=2+5=7
$$

## EXAMPLE 6 Comparing Rates of Change in Industries

The projected annual rates of employment change (in percent) in some of the fastest growing and most rapidly declining industries from 1994 through 2005 are shown in the table on the next page.

| Industry (1994-2005) | Percent Rate <br> of Change |
| :--- | :---: |
| Health services | 5.7 |
| Computer and data processing services | 4.9 |
| Child day care services | 4.3 |
| Footware, except rubber and plastic | -6.7 |
| Household audio and video equipment | -4.2 |
| Luggage, handbags, and leather products | -3.3 |

Source: U.S. Bureau of Labor Statistics.

What industry in the list is expected to see the greatest change? the least change?
We want the greatest change, without regard to whether the change is an increase or a decrease. Look for the number in the list with the largest absolute value. That number is found in footware, since $|-6.7|=6.7$. Similarly, the least change is in the luggage, handbags, and leather products industry: $|-3.3|=3.3$.

OBJECTIVE 6 Use inequality symbols. The statement $4+2=6$ is an equation; it states that two quantities are equal. The statement $4 \neq 6$ (read " 4 is not equal to 6 ") is an inequality, a statement that two quantities are not equal. When two numbers are not equal, one must be less than the other. The symbol $<$ means "is less than." For example,

$$
8<9, \quad-6<15, \quad-6<-1, \quad \text { and } \quad 0<\frac{4}{3}
$$

The symbol $>$ means "is greater than." For example,

$$
12>5, \quad 9>-2, \quad-4>-6, \quad \text { and } \quad \frac{6}{5}>0
$$

Notice that in each case, the symbol "points" toward the smaller number.
The number line in Figure 6 shows the graphs of the numbers 4 and 9. We know that $4<9$. On the graph, 4 is to the left of 9 . The smaller of two numbers is always to the left of the other on a number line.


FIGURE 6

## Inequalities on a Number Line

On a number line,
$\boldsymbol{a}<\boldsymbol{b}$ if $a$ is to the left of $b ; \quad \boldsymbol{a}>\boldsymbol{b}$ if $a$ is to the right of $b$.
We can use a number line to determine order. As shown on the number line in Figure 7, -6 is located to the left of 1 . For this reason, $-6<1$. Also, $1>-6$. From the same number line, $-5<-2$, or $-2>-5$.

figure 7
The following table summarizes results about positive and negative numbers in both words and symbols.

| Words | Symbols |
| :--- | :--- |
| Every negative number is less than 0. | If $a$ is negative, then $a<0$. |
| Every positive number is greater than 0. | If $a$ is positive, then $a>0$. |
| 0 is neither positive nor negative. |  |

In addition to the symbols $\neq,<$, and $>$, the symbols $\leq$ and $\geq$ are often $u$
INEQUALITY SYMBOLS

| Symbol | Meaning | Example |
| :---: | :--- | :---: |
| $\neq$ | is not equal to | $3 \neq 7$ |
| $<$ | is less than | $-4<-1$ |
| $>$ | is greater than | $3>-2$ |
| $\leq$ | is less than or equal to | $6 \leq 6$ |
| $\geq$ | is greater than or equal to | $-8 \geq-10$ |

The following table shows several inequalities and why each is true.

| Inequality | Why It Is True |
| :---: | :---: |
| $6 \leq 8$ | $6<8$ |
| $-2 \leq-2$ | $-2=-2$ |
| $-9 \geq-12$ | $-9>-12$ |
| $-3 \geq-3$ | $-3=-3$ |
| $6 \cdot 4 \leq 5(5)$ | $24<25$ |

Notice the reason why $-2 \leq-2$ is true. With the symbol $\leq$, if either the $<$ part or the $=$ part is true, then the inequality is true. This is also the case with the $\geq$ symbol. In the last row of the table, recall that the dot in $6 \cdot 4$ indicates the product $6 \times 4$, or 24 , and $5(5)$ means $5 \times 5$, or 25 . Thus, the inequality $6 \cdot 4 \leq 5(5)$ becomes $24 \leq 25$. which is true.

OBJECTIVE 7 Graph sets of real numbers. Inequality symbols and variables are used to write sets of real numbers. For example, the set $\{x \mid x>-2\}$ consists of all the real numbers greater than -2 . On a number line, we show the elements of this set (the set of all real numbers to the right of -2 ) by drawing an arrow from -2 to the right. We use a parenthesis at -2 to indicate that -2 is not an element of the given set. The result, shown in Figure 8, is the graph of the set $\{x \mid x>-2\}$.

figure 8
The set of numbers greater than -2 is an example of an interval on the number line. To write intervals, we use interval notation. Using this notation, we write the interval of all numbers greater than -2 as $(-2, \infty)$. The infinity symbol $\infty$ does not indicate a number; it shows that the interval includes all real numbers greater than -2 . The left parenthesis indicates that -2 is not included. A parenthesis is always used next to the infinity symbol in interval notation. The set of all real numbers is written in interval notation as $(-\infty, \infty)$.

## EXAMPLE 7 Graphing an Inequality Written in Interval Notation

Write $\{x \mid x<4\}$ in interval notation and graph the interval.
The interval is written $(-\infty, 4)$. The graph is shown in Figure 9. Since the elements of the set are all real numbers less than 4 , the graph extends to the left.

figure 9

The set $\{x \mid x \leq-6\}$ includes all real numbers less than or equal to -6 . To show that -6 is part of the set, a square bracket is used at -6 , as shown in Figure 10. In interval notation, this set is written $(-\infty,-6]$.


FIGURE 10

## EXAMPLE 8 Graphing an Inequality Written in Interval Notation

Write $\{x \mid x \geq-4\}$ in interval notation and graph the interval.
This set is written in interval notation as $[-4, \infty)$. The graph is shown in Figure 11 . We use a square bracket at -4 since -4 is part of the set.


FIGURE 11

It is common to graph sets of numbers that are between two given numbers. For example, the set $\{x \mid-2<x<4\}$ includes all real numbers between -2 and 4 , but not the numbers -2 and 4 themselves. This set is written in interval notation as $(-2,4)$. The graph has a heavy line between -2 and 4 with parentheses at -2 and 4 . See Figure 12. The inequality $-2<x<4$ is read " -2 is less than $x$ and $x$ is less than 4 ," or " $x$ is between -2 and 4 ."


FIGURE 12

## EXAMPLE 9 Graphing a Three-Part Inequality

Write $\{x \mid 3<x \leq 10\}$ in interval notation and graph the interval.
Use a parenthesis at 3 and a square bracket at 10 to get $(3,10$ ] in interval notation. The graph is shown in Figure 13. Read the inequality $3<x \leq 10$ as " 3 is less than $x$ and $x$ is less than or equal to 10 ," or " $x$ is between 3 and 10 , excluding 3 and including 10 ."

figure 13

## Interval Notation

| Type of Interval | Set | Interval Notation | Graph |
| :---: | :---: | :---: | :---: |
| Open interval | $\begin{aligned} & \{x \mid a<x\} \\ & \{x \mid a<x<b\} \\ & \{x \mid x<b\} \\ & \{x \mid x \text { is a real number }\} \end{aligned}$ | $\begin{aligned} & (a, \infty) \\ & (a, b) \\ & (-\infty, b) \\ & (-\infty, \infty) \end{aligned}$ |  |
| Half-open interval | $\begin{aligned} & \{x \mid a \leq x\} \\ & \{x \mid a<x \leq b\} \\ & \{x \mid a \leq x<b\} \\ & \{x \mid x \leq b\} \end{aligned}$ | $[a, \infty)$ <br> $(a, b]$ <br> $[a, b)$ <br> $(-\infty, b]$ |  |
| Closed interval | $\{x \mid a \leq x \leq b\}$ | [a, b] |  |

## EXERCISES

Write each set by listing its elements. See Example 1.

1. $\{x \mid x$ is a natural number less than 6$\}$
2. $\{z \mid z$ is an integer greater than 4$\}$
3. $\{z \mid z$ is an integer less than or equal to 4$\}$
4. $\{a \mid a$ is an even integer greater than 8$\}$
5. $\{m \mid m$ is a natural number less than 9$\}$
6. $\{y \mid y$ is an integer greater than 8$\}$
7. $\{p \mid p$ is an integer less than 3$\}$
8. $\{k \mid k$ is an odd integer less than 1$\}$
9. $\{x \mid x$ is an irrational number that is also rational $\}$
10. $\{r \mid r$ is a number that is both positive and negative $\}$
11. $\{p \mid p$ is a number whose absolute value is 4$\}$
12. $\{w \mid w$ is a number whose absolute value is 7$\}$

Write each set using set-builder notation. See Example 2. (More than one description is possible.)
13. $\{2,4,6,8\}$
14. $\{11,12,13,14\}$
15. $\{4,8,12,16, \ldots\}$
16. $\{\ldots,-6,-3,0,3,6, \ldots\}$
17. A student claimed that $\{x \mid x$ is a natural number greater than 3$\}$ and $\{y \mid y$ is a natural number greater than 3$\}$ actually name the same set, even though different variables are used. Was this student correct?
18. A student claimed that $\{\emptyset\}$ and $\emptyset$ name the same set. Was this student correct?

Graph the elements of each set on a number line.
19. $\{-3,-1,0,4,6\}$
21. $\left\{-\frac{2}{3}, 0, \frac{4}{5}, \frac{12}{5}, \frac{9}{2}, 4.8\right\}$
20. $\{-4,-2,0,3,5\}$
22. $\left\{-\frac{6}{5},-\frac{1}{4}, 0, \frac{5}{6}, \frac{13}{4}, 5.2, \frac{11}{2}\right\}$
23. Explain the difference between the graph of a number and the coordinate of a point.
24. Explain why the real numbers .36 and.$\overline{36}$ have different points as graphs on a number line.

Which elements of each set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, (e) irrational numbers, (f) real numbers? See Example 3.
25. $\left\{-8,-\sqrt{5},-.6,0, \frac{3}{4}, \sqrt{3}, \pi, 5, \frac{13}{2}, 17, \frac{40}{2}\right\}$
26. $\left\{-9,-\sqrt{6},-.7,0, \frac{6}{7}, \sqrt{7}, 4 . \overline{6}, 8, \frac{21}{2}, 13, \frac{75}{5}\right\}$

Give (a) the additive inverse and (b) the absolute value of each number. See the discussion of additive inverses and Example 5.

| 37.6 | 38. 8 | 39. -12 | 40. -15 | 41. $\frac{6}{5}$ | 42. 13 |
| :--- | :--- | :--- | :--- | :--- | :--- |

Find the value of each expression. See Example 5.

| 43. $\|-8\|$ | 44. $\|-11\|$ | 45. $\left\|\frac{3}{2}\right\|$ | 46. $\left\|\frac{7}{4}\right\|$ |
| :--- | :--- | :--- | :--- |
| 47. $-\|5\|$ | 48. $-\|17\|$ | 49. $-\|-2\|$ | 50. $-\|-8\|$ |
| 51. $-\|4.5\|$ | 52. $-\|12.6\|$ | 53. $\|-2\|+\|3\|$ | 54. $\|-16\|+\|12\|$ |
| 55. $\|-9\|-\|-3\|$ | 56. $\|-10\|-\|-5\|$ |  |  |
| 57. $\|-1\|+\|-2\|-\|-3\|$ | 58. $\|-6\|+\|-4\|-\|-10\|$ |  |  |

Solve each problem. See Example 6.
59. The table shows the percent change in population from 1990 through 1999 for some of the largest cities in the United States.

| City | Percent Change |
| :--- | ---: |
| New York | 1.4 |
| Los Angeles | 4.2 |
| Chicago | .6 |
| Philadelphia | -10.6 |
| Houston | 8.7 |
| Detroit | -6.1 |

Source: U.S. Bureau of the Census.
(a) Which city had the greatest change in population? What was this change? Was it an increase or a decline?
(b) Which city had the smallest change in population? What was this change? Was it an increase or a decline?
60. The table gives the net trade balance, in millions of dollars, for selected U.S. trade partners for April 2002.

| Country | Trade Balance <br> (in millions of dollars) |
| :--- | ---: |
| Germany | -2815 |
| China | -7552 |
| Netherlands | 823 |
| France | -951 |
| Turkey | 96 |
| Australia | 373 |

Source: U.S. Bureau of the Census.

A negative balance means that imports exceeded exports, while a positive balance means that exports exceeded imports.
(a) Which country had the greatest discrepancy between exports and imports? Explain.
(b) Which country had the smallest discrepancy between exports and imports? Explain.

Rewrite each statement with $>$ so that it uses $<$ instead; rewrite each statement with $<$ so that it uses $>$.
73. $6>2$
74. $4>1$
75. $-9<4$
76. $-5<1$
77. $-5>-10$
78. $-8>-12$
79. $0<x$
80. $-2<x$

Use an inequality symbol to write each statement.
81. 7 is greater than $y$.
83. 5 is greater than or equal to 5 .
85. $3 t-4$ is less than or equal to 10 .
87. $5 x+3$ is not equal to 0 .
89. $t$ is between -3 and 5 .
91. $3 x$ is between -3 and 4 , including -3 and excluding 4 .
82. -4 is less than 12 .
84. -3 is less than or equal to -3 .
86. $5 x+4$ is greater than or equal to 19 .
88. $6 x+7$ is not equal to -3 .
90. $r$ is between -4 and 12 .
92. $5 y$ is between -2 and 6 , excluding -2 and including 6 .

First simplify each side of the inequality. Then tell whether the resulting statement is true or false.

| 93. $-6<7+3$ | 94. $-7<4+2$ | 95. $2 \cdot 5 \geq 4+6$ |
| :--- | :--- | :--- |
| 96. $8+7 \leq 3 \cdot 5$ | 97. $-\|-3\| \geq-3$ | 98. $-\|-5\| \leq-5$ |
| 99. $-8>-\|-6\|$ | 100. $-9>-\|-4\|$ |  |
| Write each set using interval notation and graph the interval. See Examples $7-9$. |  |  |
| 101. $\{x \mid x>-1\}$ | 102. $\{x \mid x<5\}$ | 103. $\{x \mid x \leq 6\}$ |
| 104. $\{x \mid x \geq-3\}$ | 105. $\{x \mid 0<x<3.5\}$ | 106. $\{x \mid-4<x<6.1\}$ |
| 107. $\{x \mid 2 \leq x \leq 7\}$ | 108. $\{x \mid-3 \leq x \leq-2\}$ | 109. $\{x \mid-4<x \leq 3\}$ |
| 110. $\{x \mid 3 \leq x<6\}$ | 111. $\{x \mid 0<x \leq 3\}$ | 112. $\{x \mid-1 \leq x<6\}$ |

Match each inequality in Column I with the correct graph or interval notation in Column II.

| I $x \leq 3$ | A. |
| :--- | :--- |
| 2. $x>3$ | B. |
| 3. $x<3$ | C. $(3, \infty)$ |
| 4. $x \geq 3$ | D. $(-\infty, 3]$ |
| 5. $-3 \leq x \leq 3$ | E. $(-3,3)$ |
| 6. $-3<x<3$ | F. $[-3,3]$ |

7 Finglain hovi to determine whether to wee narentheces ar hranlzete when oranhine the

## Complex numbers

The solution of

$$
x^{2}+1=0
$$

leads to

$$
x= \pm \sqrt{-1}
$$

$$
\text { iot } a=i=\sqrt{-1}
$$

$$
\begin{aligned}
& i^{0}=1 \\
& i^{1}=i \\
& i^{2}=-1 \\
& i^{3}=i^{2} \times i=(-1) \times i=-i \\
& i^{4}=i^{2} \times i^{2}=(-1) \times(-1)=1 \\
& i^{5}=i^{4} \times i=i
\end{aligned}
$$



## Definition 1.1 Complex Number

A complex number is any number of the form $z=a+i b$ where $a$ and $b$ are real numbers and $i$ is the imaginary unit.

## Complex Numbers

A Complex Number consist of a Real Part and an Imaginary Part

Real Part Imaginary Part

$$
\begin{aligned}
& i^{2}=-1 \\
& i=\sqrt{-1}
\end{aligned}
$$

## Terminology

$$
z=a+i b \quad \operatorname{Re}(\mathrm{z})=\mathrm{a} \quad \text { and } \quad \operatorname{Im}(\mathrm{z})=\mathrm{b}
$$

Or
$z=x+i y \quad \operatorname{Re}(\mathrm{z})=\mathrm{x} \quad$ and $\quad \operatorname{Im}(\mathrm{z})=\mathrm{y}$

Example $z=4-9 i$, then $\operatorname{Re}(z)=4$ and $\operatorname{Im}(z)=-9$.

## What is a complex number ?



## Definition 1.2 Equality

Complex numbers $z_{1}=a_{1}+i b_{1}$ and $z_{2}=a_{2}+i b_{2}$ are equal, $z_{1}=z_{2}$, if $a_{1}=a_{2}$ and $b_{1}=b_{2}$.

## Arithmetic Operations Complex numbers can be added, subtracted,

 multiplied, and divided. If $z_{1}=a_{1}+i b_{1}$ and $z_{2}=a_{2}+i b_{2}$, these operations are defined as follows.Addition:

$$
z_{1}+z_{2}=\left(a_{1}+i b_{1}\right)+\left(a_{2}+i b_{2}\right)=\left(a_{1}+a_{2}\right)+i\left(b_{1}+b_{2}\right)
$$

Subtraction:

$$
z_{1}-z_{2}=\left(a_{1}+i b_{1}\right)-\left(a_{2}+i b_{2}\right)=\left(a_{1}-a_{2}\right)+i\left(b_{1}-b_{2}\right)
$$

Multiplication: $\quad z_{1} \cdot z_{2}=\left(a_{1}+i b_{1}\right)\left(a_{2}+i b_{2}\right)$

$$
=a_{1} a_{2}-b_{1} b_{2}+i\left(b_{1} a_{2}+a_{1} b_{2}\right)
$$

## EXAMPLE 1 Addition and Multiplication

If $z_{1}=2+4 i$ and $z_{2}=-3+8 i$, find (a) $z_{1}+z_{2}$ and (b) $z_{1} z_{2}$.

Solution (a) By adding real and imaginary parts, the sum of the two complex numbers $z_{1}$ and $z_{2}$ is

$$
z_{1}+z_{2}=(2+4 i)+(-3+8 i)=(2-3)+(4+8) i=-1+12 i .
$$

(b) By the distributive law and $i^{2}=-1$, the product of $z_{1}$ and $z_{2}$ is

$$
\begin{aligned}
z_{1} z_{2}=(2+4 i)(-3+8 i) & =(2+4 i)(-3)+(2+4 i)(8 i) \\
& =-6-12 i+16 i+32 i^{2} \\
& =(-6-32)+(16-12) i=-38+4 i .
\end{aligned}
$$

## Example

Let $z_{1}=8+3 i$ and $z_{2}=9-2 i$. Then $\operatorname{Re} z_{1}=8, \operatorname{Im} z_{1}=3, \operatorname{Re} z_{2}=9, \operatorname{Im} z_{2}=-2$ and

$$
\begin{gathered}
z_{1}+z_{2}=(8+3 i)+(9-2 i)=17+i \\
z_{1} z_{2}=(8+3 i)(9-2 i)=72+6+i(-16+27)=78+11 i
\end{gathered}
$$

## Division

To divide $z_{1}$ by $z_{2}$, multiply the numerator and denominator of $z_{1} / z_{2}$ by the conjugate of $z_{2}$. That is,

$$
\begin{equation*}
\frac{z_{1}}{z_{2}}=\frac{z_{1}}{z_{2}} \cdot \frac{\bar{z}_{2}}{\bar{z}_{2}}=\frac{z_{1} \bar{z}_{2}}{z_{2} \bar{z}_{2}} \tag{7}
\end{equation*}
$$

and then use the fact that $z_{2} \bar{z}_{2}$ is the sum of the squares of the real and imaginary parts of $z_{2}$.

## EXAMPLE 2 Division

If $z_{1}=2-3 i$ and $z_{2}=4+6 i$, find $z_{1} / z_{2}$.
Solution We multiply numerator and denominator by the conjugate $\bar{z}_{2}=4-6 i$ of the denominator $z_{2}=4+6 i$ and then use (4):

$$
\begin{aligned}
& \frac{z_{1}}{z_{2}}=\frac{2-3 i}{4+6 i}=\frac{2-3 i}{4+6 i} \frac{4-6 i}{4-6 i}=\frac{8-12 i-12 i+18 i^{2}}{4^{2}+6^{2}}=\frac{-10-24 i}{52} . \\
& \frac{z_{1}}{z_{2}}=-\frac{10}{52}-\frac{24}{52} i=-\frac{5}{26}-\frac{6}{13} i .
\end{aligned}
$$

## Example

For $z_{1}=8+3 i$ and $z_{2}=9-2 i$ we get $z_{1}-z_{2}=(8+3 i)-(9-2 i)=-1+5 i$ and

$$
\frac{z_{1}}{z_{2}}=\frac{8+3 i}{9-2 i}=\frac{(8+3 i)(9+2 i)}{(9-2 i)(9+2 i)}=\frac{66+43 i}{81+4}=\frac{66}{85}+\frac{43}{85} i .
$$

# Zero and Unity The zero in the complex number system is the num- 

 ber $0+0 i$ and the unity is $1+0 i$. The zero and unity are denoted by 0 and 1, respectively. The zero is the additive identity in the complex number system since, for any complex number $z=a+i b$, we have $z+0=z$. To see this, we use the definition of addition:$$
z+0=(a+i b)+(0+0 i)=a+0+i(b+0)=a+i b=z .
$$

Similarly, the unity is the multiplicative identity of the system since, for any complex number $z$, we have $z \cdot 1=z \cdot(1+0 i)=z$.

Conjugate If $z$ is a complex number, the number obtained by changing the sign of its imaginary part is called the complex conjugate, or simply conjugate, of $z$ and is denoted by the symbol $\bar{z}$. In other words, if $z=a+i b$, then its conjugate is $\bar{z}=a-i b$. For example, if $z=6+3 i$, then $\bar{z}=6-3 i$

$$
\begin{aligned}
& \overline{z_{1}+z_{2}}=\bar{z}_{1}+\bar{z}_{2}, \quad \overline{z_{1}-z_{2}}=\bar{z}_{1}-\bar{z}_{2} . \\
& \overline{z_{1} z_{2}}=\bar{z}_{1} \bar{z}_{2}, \quad \overline{\left(\frac{z_{1}}{z_{2}}\right)}=\frac{\bar{z}_{1}}{\bar{z}_{2}}, \quad \overline{\bar{z}}=z .
\end{aligned}
$$

$$
z+\bar{z}=(a+i b)+(a-i b)=2 a
$$

$$
z \bar{z}=(a+i b)(a-i b)=a^{2}-i^{2} b^{2}=a^{2}+b^{2}
$$

$$
z-\bar{z}=(a+i b)-(a-i b)=2 i b .
$$

$$
\operatorname{Re}(z)=\frac{z+\bar{z}}{2} \quad \text { and } \quad \operatorname{Im}(z)=\frac{z-\bar{z}}{2 i} .
$$

## Example

Let $z_{1}=4+3 i$ and $z_{2}=2+5 i$.

$$
\begin{aligned}
& \overline{\left(z_{1} z_{2}\right)}=\overline{(4+3 i)(2+5 i)}=\overline{(-7+26 i)}=-7-26 i, \\
& \bar{z}_{1} \bar{z}_{2}=(4-3 i)(2-5 i)=-7-26 i .
\end{aligned}
$$

## Inverses

 In the complex number system, every number $z$ has a unique additive inverse. As in the real number system, the additive inverse of $z=a+i b$ is its negative, $-z$, where $-z=-a-i b$. For any complex number $z$, we have $z+(-z)=0$. Similarly, every nonzero complex number $z$ has a multiplicative inverse. In symbols, for $z \neq 0$ there exists one and only one nonzero complex number $z^{-1}$ such that $z z^{-1}=1$. The multiplicative inverse $z^{-1}$ is the same as the reciprocal $1 / z$.EXAMPLE 3 Reciprocal
Find the reciprocal of $z=2-3 i$.
Solution By the definition of division we obtain

$$
\begin{aligned}
\frac{1}{z}=\frac{1}{2-3 i} & =\frac{1}{2-3 i} \frac{2+3 i}{2+3 i}=\frac{2+3 i}{4+9}=\frac{2+3 i}{13} . \\
\frac{1}{z} & =z^{-1}=\frac{2}{13}+\frac{3}{13} i .
\end{aligned}
$$

You should take a few seconds to verify the multiplication

$$
z z^{-1}=(2-3 i)\left(\frac{2}{13}+\frac{3}{13} i\right)=1 .
$$

## Exercise 1.1

1. Evaluate the following powers of $i$.
(a) $i^{8}$
(b) $i^{11}$
(c) $i^{42}=\left(i^{2}\right)^{21}=(-1)^{21}$
(d) $i^{105}$
2. Write the given number in the form $a+i b$.
(a) $2 i^{3}-3 i^{2}+5 i$
(b) $3 i^{5}-i^{4}+7 i^{3}-10 i^{2}-9$
(c) $\frac{5}{i}+\frac{2}{i^{3}}-\frac{20}{i^{18}}$
(d) $2 i^{6}+\left(\frac{2}{-i}\right)^{3}+5 i^{-5}-12 i$

In Problems 3-20, write the given number in the form $a+i b$.
3. $(5-9 i)+(2-4 i)$
4. $3(4-i)-3(5+2 i)$
5. $i(5+7 i)$
7. $(2-3 i)(4+i)$
6. $i(4-i)+4 i(1+2 i)$
8. $\left(\frac{1}{2}-\frac{1}{4} i\right)\left(\frac{2}{3}+\frac{5}{3} i\right)$
9. $3 i+\frac{1}{2-i}$
11. $\frac{2-4 i}{3+5 i}$
13. $\frac{(3-i)(2+3 i)}{1+i}$
15. $\frac{(5-4 i)-(3+7 i)}{(4+2 i)+(2-3 i)}$
17. $i(1-i)(2-i)(2+6 i)$
19. $(3+6 i)+(4-i)(3+5 i)+\frac{1}{2-i} \quad$ 20. $(2+3 i)\left(\frac{2-i}{1+2 i}\right)^{2}$

