Convergence and the Routh Theorem

The Routh theorem provides an easy approach for the determination of convergence or divergence of the time-path of higher order differential equations with the help of determinants formed by the coefficient of the concerned differential equation.

The Routh theorem states: "The real parts of the all the roots of nth degree polynomial equation:

$$a_0 r^n + a_1 r^{n-1} r + \ldots\ldots\ldots\ldots + a^{n-1} r + a_n = 0$$

are negative if and only if the first n of the following sequence of determinants

$$|a1|; \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}; \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}; \begin{vmatrix} a_1 & a_2 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

are all positive.

In applying this theorem it should be kept in view that $| a_1 | = a_1$. Moreover, we should take $a_m = 0$ so that $m > n$.

Example 1. We check the convergence of the time–path of the following differential equation.

$$y^{(4)}_{(t)} + 6y^{(''')}_{(t)} + 14\, y^{('')}_{(t)} + 16y'(t) + 8y_{(t)} = 24.$$

Here $a_1 = 6,\ a_2 = 14,\ a_3 = 16$ and $a4 = 8$.

Thus the required determinants are as: $|a_1| = a_1 = 6$ which is > 0

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 6 & 16 \\ 1 & 14 \end{vmatrix} = 68 > 0.$$

$$\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \left|\begin{array}{ccc:cc} 6 & 16 & 0 & 6 & 16 \\ 1 & 14 & 8 & 1 & 14 \\ 0 & 6 & 16 & 0 & 1 \end{array}\right|$$

$$= (1344 + 0 + 0) - (0 + 288 + 256) = 800 > 0$$

$$\begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} = \begin{vmatrix} 6 & 16 & 0 & 0 \\ 1 & 14 & 8 & 0 \\ 0 & 6 & 16 & 0 \\ 0 & 1 & 14 & 8 \end{vmatrix} = 8 \begin{vmatrix} 6 & 16 & 0 \\ 1 & 14 & 8 \\ 0 & 6 & 16 \end{vmatrix}$$

$$= 8\,(800) = 6400 > 0.$$

As all above determinants are positive. Therefore, we can conclude that the timepath is convergent.

Example 2. $\mathbf{y'''(t) - 2y''(t) - y'(t) + 2y = 4.}$ Here $a_1 = -2,\ a_2 = -1$ and $a_3 = 2$.

The determinants are as: $|a_1| = a_1 = -2 < 0$

$$|a_2| = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = 0.$$

$$|a_3| = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \left|\begin{array}{ccc:cc} -2 & 2 & 0 & 2 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & -2 & 2 & -2 & 2 \end{array}\right| = 4 > 0$$

As all the determinants are not positive, therefore, the time path is divergent.

Example 3. $\mathbf{y'''(t) - 10y''(t) + 27y'(t) - 18y = 3}$ **where** $\mathbf{a_1 = -10,}$ $\mathbf{a_2 = 27}$ **and** $\mathbf{a_3 = -18.}$

The determinants are: $|a_1| = a_1 = -10 < 0$

$$|a_2| = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} -10 & -18 \\ 1 & 27 \end{vmatrix} = -288 < 0.$$

$$|a_3| = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} -10 & -18 & 0 \\ 1 & 27 & 0 \\ 0 & -10 & -18 \end{vmatrix}$$

$$= \begin{vmatrix} -10 & -18 & 0 & -10 & -18 \\ 1 & 27 & 0 & 1 & 27 \\ 0 & -10 & -18 & 0 & -10 \end{vmatrix} = 4860 - 324 = 4536 > 0.$$

As all the determinants are not positive. Therefore, the time path is divergent.

Example 4. $\mathbf{y'''(t) + 4y''(t) + 5y'(t) - 2y_{(t)} = -2.}$ Here

$a_1 = 4,\ a_2 = 5,\ a_3 = -2.$

The required determinants are as: $|a_1| = a_1 = -10 < 0$

$$|a_2| = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ 1 & 5 \end{vmatrix} = 22 > 0.$$

$$|a_3| = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 4 & -2 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & -2 \end{vmatrix} \begin{vmatrix} 4 & -2 \\ 1 & 5 \\ 0 & 4 \end{vmatrix} = -48 < 0$$

Since all the determinants are not positive. Thus, the time path will be divergent.