Chapter 1

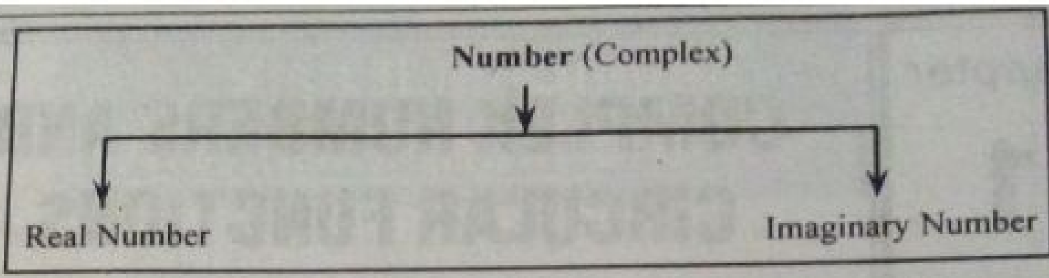
COMPLEX NUMBERS AND CIRCULAR FUNCTIONS

CONCEPT OF IMAGINARY AND COMPLEX NUMBERS

In this respect, first of all we tell about imaginary numbers and complex numbers. According to mathematicians the square root of any positive number can be found out while it is not the case with negative numbers. For example $\sqrt{3}$ =1.732 = 1.73. As Akbar says that he is to get Rs. $\sqrt{3}$ from Aslam, it means Akbar is to get Rs. 1.73 from Aslam. In the same way, Akbar can not say to Aslam that he is to get Rs. $\sqrt{-3}$ from Akbar. As it does not mean anything. This can just be written. It has nothing to do with the transactions. Thus the number which is employed in transactions is called *Real Number*. While the number which does not come under transactions is called Imaginary *Number*. This imaginary number is written as :

$$\sqrt{-3} = \sqrt{3 \times -1} = \sqrt{3} \times \sqrt{-1} = \sqrt{3}\, i.$$

The $\sqrt{-1}$ is represented with i and it is called *Iota*. Again, it is told that the negative numbers written in square–root like $\sqrt{-1}$, when we take their square we get –1. Thus the *square–root of any negative number is called imaginary number*. If we take the number $3 + 2i$ it is also an imaginary number. This number has two components, i.e., 3 and $2i$ where 3 is real number and $2i$ is an imaginary number. The number having one component in the form of real number and other component in the form of imaginary number is called Complex Number. According to this definition any net imaginary number is called complex number. As $0 + 2i = 2i$. Here the real part of number is zero. Again, each real number is also a complex number. As, $3 + 0i = 3$. Here the imaginary part of number is zero. Thus the set of all the real numbers (R) is a sub–set of complex numbers (C). Again the set of all the imaginary numbers (I) is also a sub–set of complex numbers (C). They are written as: $R \subset C, I \subset C$. Moreover, these two sets are disjoint sets, as : $R \cap I = \varnothing$. Whenever, we talk of any number it means the complex number. This may be a net real number where its imaginary part is zero. It may be a net imaginary number where its real part is zero. Thus numbers are as :



Any complex number is shown by z. Its standard form is as: $z = x + iy$ or $z = h + vi$ where h and v are real numbers. It is told that h is horizontal number while v represents vertical number. This is the reason when we show complex number on a graph the values of h are shown on horizontal axis and values of v are shown on vertical axis. The diagram which is used to represent complex numbers is called **Argand diagram**. It is shown in Fig.1.

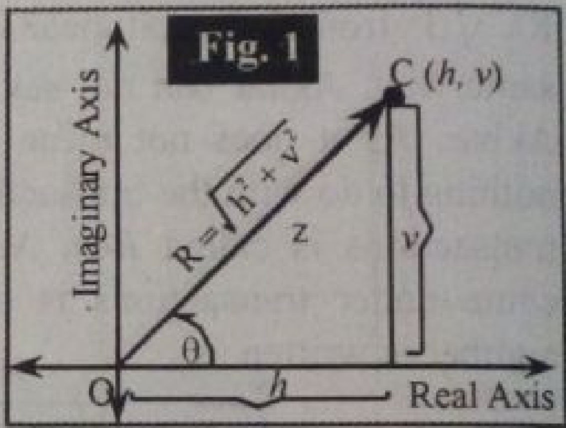


Fig. 1

Here 'h' has been shown horizontally on real axis. While v has been shown vertically on imaginary axis. The complex number $z = h + vi$ has been shown by point C (h, v). Certain algebraic signs are attached with the values of h and v. As if $h < 0$ the point C will lie to the left of origin. Again if the value of v is negative the point C will occur below the horizontal axis.

In the presence of values of h and v we can find the length of OC in the light of Pythagorean Theorem. This theorem states that the square of hypotenuse of a right angle triangle is equal to sum of square of other sides of triangle. If the length of OC is shown by Radius vector (R), then

$$R^2 = h^2 + v^2 \quad \Rightarrow R = \sqrt{h^2 + v^2}$$

where the square–root is always taken positive. Like v and h, R is also a real number. This value of R is sometimes known as *Absolute value* or *Modulus* of complex number $(h + vi)$.

Characteristic Roots

We know that following formula is used to solve second order equation :

$$r_1, r_2 = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} \quad \text{or} \quad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus the solution values of quadratic equation are called as Characteristic Roots. The formula which we wrote above is called the solution of characteristic or Auxiliary equation, $r^2 + b_1 r_1 + b_2 = 0$. There are

ALTERNATIVE REPRESENTATIVE OF COMPLEX NUMBERS

There are three ways to represent complex numbers :

1. Cartesian Form

In connection with complex numbers we told that conjugate complex numbers are represented generally, as : $h \pm vi$. When we demonstrated complex numbers with Argrand diagram (Fig. 1) the 'h' represents Abscissa and 'v' represents Ordinate. Such representation is concerned with Cartesian coordinate system. Accordingly, such representation of conjugate complex numbers is called Cartesian form.

We have discussed above the circular function and Euler relations. Therefore, in their light there are two other methods to represent complex numbers. They are as:

2. Circular Form

When we represent complex numbers into Cartesian form the v and h give rise to Cartesian coordinates. But such coordinates can also be represented into theta (θ) and chord (R). The coordinates so rise are known as polar coordinates. As by converting v and h into sin and cos functions we can represent complex numbers with polar or circular form. As we know that :

$\sin\theta = \dfrac{v}{R} \Rightarrow v = R\sin\theta$	$\cos\theta = \dfrac{h}{R} \Rightarrow h = R\cos\theta$

The standard form of conjugate complex number is $h \pm vi$. Interchanging the above relationships

$$h \pm vi = R\cos\theta \pm Ri\sin\theta \Rightarrow h \pm vi = R(\cos\theta \pm i\sin\theta)$$

$$h + vi = R\cos\theta + Ri\sin\theta \Rightarrow h + vi = R(\cos\theta + i\sin\theta)$$

$$h - vi = R\cos\theta - Ri\sin\theta \Rightarrow h - vi = R(\cos\theta - i\sin\theta)$$

Thus in polar system, the compolex numbers are represented into R and θ, instead of h and v. The right side ofabove expressions are the complex numbers in polar or circular form.

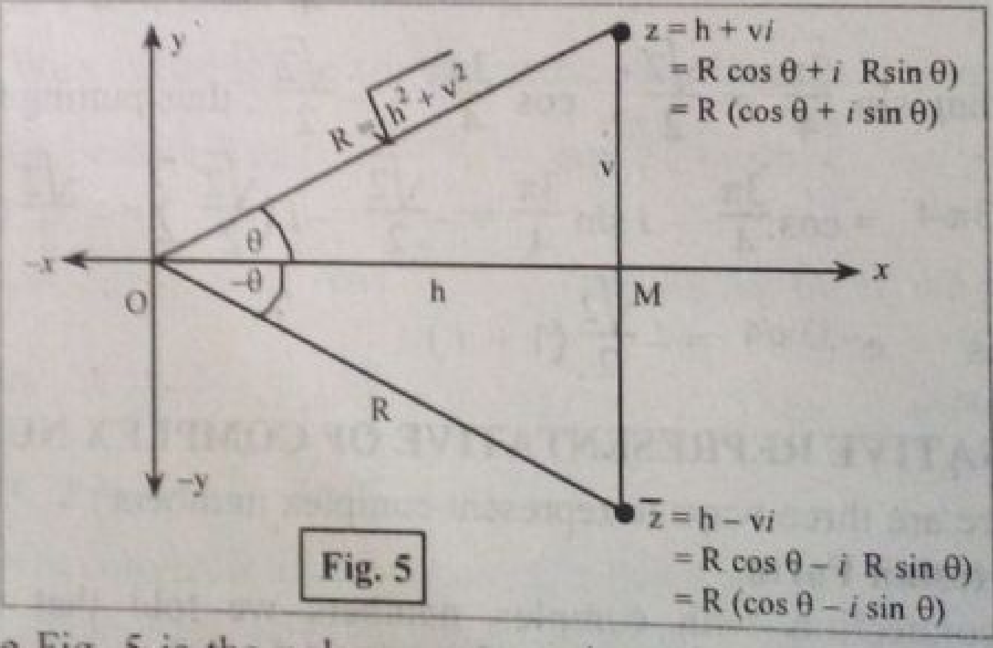


Fig. 5

The Fig. 5 is the polar representation of complex numbers. As in the Cartesian form, in the light of Pythagorean Theorem $R = \sqrt{h^2 + v^2}$. This can be proved with polar system. As the above discussion shows :

$h = R\cos\theta, \quad v = R\sin\theta$

$$\Rightarrow h^2 + v^2 = (R\cos\theta)^2 + (R\sin\theta)^2 = R^2\cos^2\theta + R^2\sin^2\theta$$

$$= R^2(\cos^2\theta + \sin^2\theta) = R^2(1) = R^2 \quad (\text{as } \cos^2\theta + \sin^2\theta = 1)$$

$$\Rightarrow R^2 = h^2 + v^2 \quad \Rightarrow \quad R = \sqrt{h^2 + v^2}$$

3. Exponential Form

In the light Euler Relations the polar form of complex numbers can also be written in exponential form. As in case of Euler relations we know that : $e^{\pm i\theta} = \text{Cos}\,\theta \pm i \sin\theta$

As in polar system the representation of complex numbers is as : R $(\cos\theta \pm i \sin\theta)$. Thus summing these two expression, following relationship exists in case of polar form and exponential form. $R(\cos\theta \pm i \sin\theta) = Re^{\pm i\theta} \Rightarrow Re^{\pm i\theta} = R(\cos\theta \pm i \sin\theta)$

From this discussion we come to this conclusion that to represent complex numbers three methods are adopted (1) Cartesian form (2) circular or polar form and (3) exponential form. They are represented as :

$$h \pm vi = R(\cos\theta \pm i \sin\theta) = Re^{\pm i\theta}$$

Example 1: Write the complex number $5\,e^{-i3\pi/4}$ in Cartesian form.

Solution. Here the complex number is given in exponential form. However, there is π in place of θ. Thus, we have $R\,e^{i\theta} = 5\,e^{-i3\pi/2}$.

Comparing them, we have $R = 5,\ \theta = \frac{3\pi}{2}$.

From our last discussion, $h = R\cos\theta,\ v = R\sin\theta$, therefore $v = 5\sin\frac{3\pi}{2}$, and $h = 5\cos\frac{3\pi}{2}$.

Putting them in standard Cartesian form, we have

$$h + vi = 5\cos\frac{3\pi}{2} + 5i\,5\sin\frac{3\pi}{2} = 5(0) + 5i(-1) = -5i$$

$$\{\text{ as } \cos\frac{3\pi}{2} = 0,\ \sin\frac{3\pi}{2} = -1\ \}\ \text{Thus}\ \boxed{5\,e^{-i3\pi/4} = 0 - 5i}$$

Example 2: Write the complex number $(1 + \sqrt{3}\ i)$ in exponential form.

Solution. Here the complex number is given in Cartesian form. However, here $h = 1,\ v = \sqrt{3}$. Thus, we have $R\,e^{i\theta} = h + vi$.

So $R = \sqrt{u^2 + v^2} = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

Now $\cos\theta = \frac{h}{R} = \frac{1}{2}$ and $\sin\theta = \frac{v}{R} = \frac{\sqrt{3}}{2}$

From the last table, we see that $\theta = \frac{\pi}{3}$ satisfies both of these equations, i.e.,

$\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, Hence $\theta = \frac{\pi}{3}$. Thus we have

$$h + vi = R(\cos\theta + i\sin\theta) = R\,e^{i\theta} \quad \text{i.e.,} \quad 1 + \sqrt{3}\ i$$

$$= R\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\,e^{i\pi/3}$$

This is the required polar form.

Example 3: Write the following complex numbers in Cartesian form:

(a) $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ (b) $4\,e^{-i\pi/3}$

(c) $\sqrt{2}\;e^{-i\pi/4}$

Solution. (a) Here the complex number is given in polar form. However, here $R = 2$, $\theta = \frac{\pi}{6}$. We know that $\cos\theta = \frac{h}{R}$, $\sin\theta = \frac{v}{R}$ i.e.

$$h = R\cos\theta, \quad v = R\sin\theta$$

Putting the above values, using $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{6} = \frac{1}{2}$. $h = R\cos\theta = 2\cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$v = R\sin\theta = 2\sin\frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$

Thus $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = u + vi = \sqrt{3} + i$

(b): $4\,e^{-i\pi/3}$

Solution. Here the complex number is given in exponential form. However, there is $\pi/3$ in place of θ. Thus, we have $R\,e^{i\theta} = 4\,e^{-i\pi/3}$.

Comparing them, we have $R = 4$, $\theta = \frac{\pi}{3}$. From our last discussion, $h = R\cos\theta$, $v = R\sin\theta$, therefore $v = 4\sin\frac{\pi}{3}$, and $h = 4\cos\frac{\pi}{3}$.

Putting them in standard Cartesian form, we have

$$h + vi = 4\cos\frac{\pi}{3} + 4i\sin\frac{\pi}{3} = 4(\frac{1}{2}) + 4i(\frac{\sqrt{3}}{2}) = 2 + i\,2\sqrt{3}$$

$\left(\text{as } \cos\frac{\pi}{3} = \frac{1}{2},\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right)$ Thus $\boxed{4\,e^{-i\pi/3} = 2 + 2\sqrt{3}\,i}$

(c): $\sqrt{2}\;e^{-i\pi/4}$

Solution. Here the complex number is given in exponential form. However, there is $\pi/4$ in place of θ. Thus, we have $R\,e^{i\theta} = \sqrt{2}\;e^{-i\pi/4}$.

Comparing them, we have $R = \sqrt{2}$, $\theta = \frac{\pi}{4}$. From our last discussion, $h = R\cos\theta$, $v = R\sin\theta$, therefore $v = \sqrt{2}\sin\frac{\pi}{4}$, and $h = \sqrt{2}\cos\frac{\pi}{4}$

Putting them in standard Cartesian form, we have

$$h - vi = \sqrt{2}\cdot\cos\frac{\pi}{4} - \sqrt{2}\,i\sin\frac{\pi}{4} = \sqrt{2}(\frac{1}{\sqrt{2}}) - \sqrt{2}\,i(\frac{1}{\sqrt{2}}) = 1 - i$$

$\left(\text{as } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \right)$ Thus $\boxed{\sqrt{2}\, e^{-i\pi/4} = 1 - i}$.

Example 4: Convert the following complex numbers in polar and exponential form:

(a): $\frac{3}{2} + \frac{3\sqrt{3}}{2} i$ **(b) :** $4(\sqrt{3} + i)$

Solution. (a) Here the complex number is given in Cartesian form. Therefore, here

$$h = \frac{3}{2},\ v = \frac{3\sqrt{3}}{2}. \text{ Thus } \frac{3}{2} + \frac{3\sqrt{3}}{2} i = h + vi$$

$$\Rightarrow \quad h = \frac{3}{2},\ v = \frac{3\sqrt{3}}{2}$$

Now we find the values of R and θ.

As $$R = \sqrt{u^2 + v^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = 3$$

$$\text{Sin } \theta = \frac{v}{R} = \frac{3\sqrt{3}}{2} \Rightarrow 3 = \frac{3\sqrt{3}}{2} \times \frac{1}{3} = \frac{\sqrt{3}}{2} \qquad \ldots (1)$$

$$\text{Cos } \theta = \frac{h}{R} = \frac{3}{2} \Rightarrow 3 = \frac{3}{2} \times \frac{1}{3} = \frac{1}{2} \qquad \ldots (2)$$

Now the value of θ which satisfies both (1) and (2) is $\frac{\pi}{3}$. Putting the values $R = 3,\ \theta = \frac{\pi}{3}$, we have $h + vi = R(\cos\theta + i \sin\theta)$

$= 3\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

Thus, $$\boxed{\frac{3}{2} + \frac{3\sqrt{3}}{2} i = 3\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)}$$

To get the exponential form, we have $h + vi = R\, e^{i\theta} = 3\, e^{i\pi/3}$

Thus, $$\boxed{\frac{3}{2} + \frac{3\sqrt{3}}{2} i = 3\, e^{i\pi/3}}$$