(2) Utility function: $U = xy + 3x + y$, Income constraint: $I = 212$, $Py = 12$, $Px = 8$

$$I = xPx + yPy \Rightarrow 212 = 8x + 12y \Rightarrow 212 - 8x - 12y = 0$$

$$\lambda(212-8x-12y) = 0 \Rightarrow \lambda 212 - \lambda 8x - \lambda 12y = 0$$

$$U = xy + 3x + y + (\lambda 212 - \lambda 8x - \lambda 12y)$$

$$U = xy + 3x + y + \lambda 212 - \lambda 8x - \lambda 12y$$

$$\frac{\partial U}{\partial x} = Ux = y + 3 - \lambda 8 = 0 \ldots\ldots\ldots(1)$$

$$\frac{\partial U}{\partial x} = Uy = x + 1 - \lambda 12 = 0 \ldots\ldots\ldots(2)$$

$$\frac{\partial U}{\partial y} = U\lambda = 212 - 8x - 12y = 0 \ldots\ldots\ldots(3)$$

$$212 - 8x - 12y = 0 \Rightarrow 212 - 8x = 12y \Rightarrow y = \frac{212 - 8x}{12} \Rightarrow$$

$y = 17.66 - 0.66x$. Putting the value of y in (1).

$$y + 3 - \lambda 8 = 0 \Rightarrow 17.66 - 0.66x + 3 - \lambda 8 = 0 \Rightarrow$$

$$20.66 - 0.66x = 8\lambda \Rightarrow 8\lambda = 20.66 - 0.66x \Rightarrow \lambda = \frac{20.66 - 0.66x}{8} \Rightarrow$$

$\boxed{\lambda = 2.582 - 0.0825x}$. While from (2) we see $x + 1 - 12\lambda = 0 \Rightarrow$

$$x + 1 = 12\lambda \Rightarrow 12\lambda = x + 1 \Rightarrow \lambda = \frac{x+1}{12} \Rightarrow \boxed{\lambda = 0.083 + 0.083x}$$

$$\lambda = 2.582 - 0.0825x$$

$$\lambda = 0.083 + 0.083x$$

$$0 = 2.499 - 0.1655x \Rightarrow 0.1655x = 2.499 \Rightarrow x = 15.$$

Putting $x = 15$ in y: $y = \frac{212 - 8(15)}{12} \Rightarrow y = 7.67$, $\lambda = \frac{15+1}{12} = 1.33$

$$212 = 8x + 12y \Rightarrow 212 = 8(15) + 12(7.67) \Rightarrow 212 = 212.$$

$$U = xy + 3x + y,\ Ux = y + 3 = 7.67 + 3 = 10.67$$

$$Uxx = 0,\ Uxy = 1,\ Uy = x + 1 = 15 + 1 = 16,\ Uyy = 0$$

$$g(x,y) = 212 - 8x - 12y,\ gx = \frac{\delta g}{\delta x} = -8,\ gy = \frac{\delta g}{\delta x} = -12$$

$$2Uxy(gx)(gy) - Uxx\,(gy)^2 - Uyy(gx)^2 > 0$$

$$2(1)(-8)(-12) - (0)\,(-12)^2 - (0)\,(-8)^2 = 192 > 0$$

Thus at $x = 15$ and $y = 7.67$, both necessary and sufficient conditions of constraint utility maximization are met.

4. (i) Cost function: $C = 6x^2+10y^2-xy+30$, Production quota constraint: $x+y=34$

$34 - x - y = 0$, $\lambda(34 - x - y) = 0$

$C = 6x^2 + 10y^2 - xy + 30 + \lambda(34 - x - y)$

$$C_x = \frac{\partial C}{\partial x} = 12x - y - \lambda = 0 \quad \text{...........(1)}$$

$$C_y = \frac{\partial C}{\partial y} = 20y - x - \lambda = 0 \quad \text{...........(2)}$$

$$C_\lambda = \frac{\partial C}{\partial \lambda} = 34 - x - y = 0 \quad \text{...........(3)}$$

Equation (1) is solved for λ : $12x - y - \lambda = 0 \Rightarrow$

$12x - y = \lambda$ or $\boxed{\lambda = 12x - y}$. Putting value of λ in (2):

$20y - x - \lambda = 0 \Rightarrow 20y - x = \lambda \Rightarrow$

$20y - x = 12x - y \Rightarrow 20y + y - x - 12x = 0 \Rightarrow \boxed{21y - 13x = 0}$

The equation (3) is brought: $x + y = 34$ or $y + x = 34$.

$$\begin{aligned} 21y - 13x = 0 \quad &\Rightarrow \quad 21y - 13x = 0 \\ y + x = 34 \quad &\Rightarrow \quad 21y + 21x = 714 \\ & \quad\;\; \underline{-\quad\;\; -\quad\;\; -} \\ & \quad\;\; 0 - 34x = -714 \Rightarrow x = 21 \end{aligned}$$

$y + x = 34 \Rightarrow y = 34 - x = 34 - 21 = 13$

$x + y = 34 \Rightarrow 21 + 13 = 34 \Rightarrow 34 = 34$

$\lambda = 20y - x = 20(13) - 21 = 260 - 21 = 239$.

Taking 2^{nd} partial derivatives of cost function and ist partial derivatives of production function:

$C_x = 12x - y - \lambda$, $C_{xx} = 12$, $C_{xy} = -1$, $C_y = 20y - x - \lambda$, $C_{yy} = 20$, $C_{yx} = -1$

$g(x, y) = (34 - x - y)$, $\frac{\partial g}{\partial x} = g_x = -1$, $\frac{\partial g}{\partial y} = g_y = -1$.

$2C_{xy}(g_x)(g_y) - C_{xx}(g_y)^2 - C_{yy}(g_x)^2 < 0$.

$2(-1)(-1)(-1) - (12)(-1)^2 - (20)(-1)^2 = -2 - 12 - 20 = -34 < 0$

Thus it is proved that if firm produces y = 13 and x = 21 its production quota is met, x + y = 34 ⇒ 21 + 13 = 34. Producing such quantities of x and y joint cost are minimized in the light of necessary and sufficient conditions.

(ii) Cost function: $C = f(x, y) = x^2 + 2y^2 - xy$, Production quota constraint:

$g(x, y) = x + y = 8 \Rightarrow 8 - x - y = 0, \lambda(8 - x - y) = 0$

$C = x^2 + 2y^2 - xy + \lambda(8 - x - y)$

$$C_x = \frac{\partial C}{\partial x} = 2x - y - \lambda = 0 \quad \text{..........(1)}$$

$$C_y = \frac{\partial C}{\partial y} = 4y - x - \lambda = 0 \quad \text{..........(2)}$$

$$C_\lambda = \frac{\partial C}{\partial \lambda} = 8 - x - y = 0 \quad \text{..........(3)}$$

Equation (1) is solved for λ: $2x - y - \lambda = 0 \Rightarrow \boxed{2x - y = \lambda}$

Putting value of λ in (2): $4y - x - \lambda = 0 \Rightarrow \boxed{4y - x = \lambda}$

$4y - x = 2x - y \Rightarrow 4y + y - x - 2x = 0 \Rightarrow \boxed{5y - 3x = 0}$

The equation (3) is brought: $\boxed{y + x = 8}$

$$\begin{aligned} 5y - 3x = 0 &\Rightarrow & 5y - 3x &= 0 \\ y + x = 8 &\Rightarrow & 5y + 5x &= 40 \\ & & - \quad - \quad & \quad - \\ \hline & & 0 - 8x = -40 &\Rightarrow x = 5 \end{aligned}$$

$y + x = 8 \Rightarrow y = 8 - x = 8 - 5 = 3.\ \lambda = 2x - y = 2(5) - 3 = 7.$

$C_x = 2x - y - \lambda, C_{xx} = 2, C_{xy} = -1, C_y = 4y - x - \lambda, C_{yy} = 4, C_{yx} = -1$

$g(x, y) = (8 - x - y), \frac{\partial g}{\partial x} = g_x = -1, \frac{\partial g}{\partial y} = g_y = -1.$

$2\,C_{xy}\,(g_x)\,(g_y) - C_{xx}\,(g_y)^2 - C_{yy}\,(g_x)^2 < 0.$

$2(-1)\,(-1)\,(-1) - (2)\,(-1)^2 - (4)\,(-1)^2 = -2 - 2 - 4 = -8 < 0$

Thus it is proved that if firm produces y = 3 and x = 5 its production quota is met, x + y = 8 ⇒ 5 + 3 = 8. Producing such quantities of x and y joint cost are minimized in the light of necessary and sufficient conditions.

5. Profit function: $\pi = 110x - 3x^2 - 2xy - 2y^2 + 140y$.

Fixed output : $\frac{x}{2} = y \Rightarrow x = 2y \Rightarrow 2y - x = 0. \Rightarrow \lambda(2y - x) = 0$

$\pi = 110x - 3x^2 - 2xy - 2y^2 + 140y + \lambda(2y - x)$

$\pi_x = \frac{\partial \pi}{\partial x} = 110 - 6x - 2y - \lambda = 0$ ....(1)

$\pi_y = \frac{\partial \pi}{\partial y} = -2x - 4y + 140 + 2\lambda = 0$ ....(2)

$\pi_\lambda = \frac{\partial \pi}{\partial \lambda} = 2y - x = 0$ .... (3)

with xing ② by 2.

Solving (1) and (2) for 'λ': $110 - 6x - 2y = \lambda,$

$2\lambda = -140 + 2x + 4y.$

$\lambda = 110 - 6x - 2y$ $2\lambda = 220 - 12x - 4y$

$2\lambda = -140 + 2x + 4y \Rightarrow$ $2\lambda = -140 + 2x + 4y$

$\quad - \quad + \quad - \quad -$

$0 = 360 - 14x - 8y$ —— (4)

solving ③ & ④ with xing ③ by 14.

$-14x - 8y + 360 = 0$ $14x + 8y = 360$

$-x + 2y = 0 \Rightarrow$ $14x - 28y = 0$

$\quad - \quad +$

$0 + 36y = 360$

$\Rightarrow y = 10.\ x = 2y = 2(10) = 20.\ \frac{x}{2} = y \Rightarrow \frac{20}{2} = 10$

$2\lambda = -140 + 2(20) + 4(10) \Rightarrow 2\lambda = 60 \Rightarrow \lambda = 30$

Putting value of x and y in π.

$\pi = 110(20) - 3(20)^2 - 2(20)(10) - 2(10)^2 + 140(10) = 1800$

$\pi_x = 110 - 6x - 2y - \lambda,\ \pi_{xx} = -6, \pi_{xy} = -2, \pi_y = -2x - 4y + 140 + 2\lambda, \pi_{yy} = -4,$

$\pi_{yx} = -2.\ g(x, y) = (2y - x), \frac{\partial g}{\partial x} = g_x = -1, \frac{\partial g}{\partial y} = g_y = 2.$

$2\pi_{xy}(g_x)(g_y) - \pi_{xx}(g_y)^2 - \pi_{yy}(g_x)^2 > 0.$

$2(-2)(-1)(2) - (-6)(2)^2 - (-4)(-1)^2 = 8 + 24 + 4 = 36 > 0$

Thus at x = 20 and y = 10, profits are maximized following Ist and 2nd order conditions.

---

6. $P_x = 144 - 2x, P_y = 120 - y$, (Demand function), $C = x^2 + xy + y^2 + 35$ (Cost function), $x + y = 40$ (Quota constraint).

$\pi = (Px)(x) + (Py)(y) - C$

$= (144 - 2x)x + (120 - y)y - (x^2 + xy + y^2 + 35)$

$\pi = 144x - 2x^2 + 120y - y^2 - x^2 - xy - y^2 - 35$

$x + y = 40 \Rightarrow 40 - y - x = 0. \Rightarrow \lambda(40 - y - x) = 0$

$\pi = 144x - 2x^2 + 120y - y^2 - x^2 - xy - y^2 - 35 + (\lambda 40 - \lambda y - \lambda x)$

$\pi_x = \frac{\partial \pi}{\partial x} = 144 - 4x - 2x - y - \lambda = 0$ ....(1)

$\pi_y = \frac{\partial \pi}{\partial y} = 120 - 2y - x - 2y - \lambda = 0$ .... (2)

$\pi_\lambda = \frac{\partial \pi}{\partial \lambda} = 40 - y - x = 0$ .... (3)

With (1) and (2) we get the value of λ.

$144 - 4x - 2x - y = \lambda \quad 120 - 2y - x - 2y = \lambda$

$$\lambda = 144 - 6x - y$$

$$\lambda = 120 - x - 4y$$

$$\underline{\quad - \quad - \quad + \quad + \quad}$$

$$0 = 24 - 5x + 3y \Rightarrow 3y - 5x = -24 \quad \text{——— (4)}$$

Solving (3) & (4) with x'ing (3) by 3.

$$y + x = 40 \Rightarrow$$

$$3y - 5x = -24$$

$$3y + 3x = 120$$

$$\underline{- \quad - \quad -}$$

$-8x = -144 \Rightarrow x = 18.\ y + x = 40 \Rightarrow y = 40 - x = 40 - 18 = 22$

Putting x = 18 and y = 22 in π function we can find its value.

$\pi = 144 - 4x - 2x - y - \lambda,\ \ \pi_{xx} = -4 - 2 = -6,\ \pi_{xy} = -1,$

$\pi_y = 120 - 2y - x - 2y - \lambda,\ \pi_{yy} = -2 - 2 = -4,\ \pi_{yx} = -1.$

$g(x, y) = (40 - y - x), \dfrac{\partial g}{\partial x} = g_x = -1, \dfrac{\partial g}{\partial y} = g_y = -1.$

$2\pi_{xy}(g_x)(g_y) - \pi_{xx}(g_y)^2 - \pi_{yy}(g_x)^2 > 0.$

$2(-1)(-1)(-1) - (-6)(-1)^2 - (-4)(-1)^2$

$= -2 + 6 + 4 = 8 > 0.$

$P_x = 144 - 2(18) = 108,\ P_y = 120 - 22 = 98$

Thus at x = 18 and y = 22, profits are maximized.